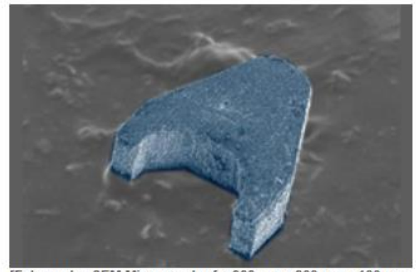
School of Engineering
Brown University

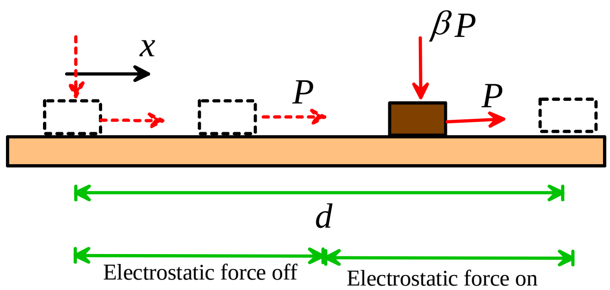
EN40: Dynamics and Vibrations

Homework 2: Kinematics and Dynamics of Particles
Due Friday Feb 7, 2014
Max Score 45 Points + 8 Extra Credit

1. An experimental 'micro-robot' (see a description here – most people would call the robot a speck of dirt…) is moved around on a surface by subjecting it to a constant magnetic force P that acts tangent to the surface. In addition, an electrostatic force with magnitude βP can be applied to attract the micro-robot to the surface. Friction with coefficient μ acts between the robot and the surface, so the particle can be moved around by turning the electrostatic force on and off. Gravity may be neglected.


[False-color SEM Micrograph of a 300 μm x 300 μm x 100 μm Mag-μBot with a claw.]

Suppose that the micro-robot has mass m, and is at rest at $x=0$ at time $t=0$, with the electrostatic force turned on. To move the robot through a distance d, the electrostatic force is turned off for a time $0<t<t_1$, and then turned back on again. The goal of this problem is to derive a formula relating t_1 to d.



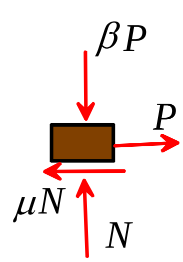
1.1 Use Newton's law to find the acceleration of the micro-robot during the time when the electrostatic force is switched off $0<t<t_1$, and hence find formulas that give the speed and position of the micro-robot as functions of time for $0<t<t_1$, in terms of P and m.

During this phase P is the only force acting on the robot. Its acceleration is constant and has magnitude P/m. The straight-line motion formulas give

$$v=(P/m)t \qquad x=(P/m)t^2/2$$

[2 POINTS]

1.2 Similarly, find formulas that predict the speed and position of the micro-robot for time $t>t_1$



A free body diagram is shown in the figure to the right. Newton's law shows that $(P-\mu N)\mathbf{i}+(N-\beta P)\mathbf{j}=ma\mathbf{i}$

The $\mathbf{i}$ and $\mathbf{j}$ components show that $N=\beta P \quad a=-(\mu\beta-1)P/m$

At time t_1 the position and velocity of the particle are

$$v = (P/m)t_1 \qquad x = (P/m)t_1^2/2$$

The straight-line motion formulas then give

$$\begin{aligned} v &= v_0 + a(t - t_0) \\ &= (P/m)t_1 - (\mu\beta - 1)P(t - t_1)/m \\ x &= x_0 + v_0(t - t_0) + a(t - t_0)^2/2 \\ &= (P/m)t_1^2/2 + (P/m)t_1(t - t_1) - (\mu\beta - 1)P(t - t_1)^2/2m \end{aligned}$$

[3 POINTS]

1.3 Use the conditions that the robot comes to rest at position *d* to determine t_1 in terms of *P*, *d* μ, β and *m*. Check the dimensions (units) of your solution.

When motion stops we know that

$$v = 0 \quad \Rightarrow (t - t_1) = t_1/(\mu\beta - 1)$$

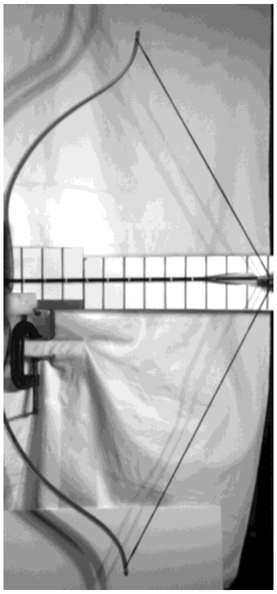
$$\Rightarrow d = (P/2m)\left[t_1^2 + \frac{2t_1^2}{(\mu\beta - 1)} - (\mu\beta - 1)\frac{t_1^2}{(\mu\beta - 1)^2} \right]$$

$$d = \frac{Pt_1^2}{2m}\left[\frac{(\mu\beta - 1) + 1}{\mu\beta - 1} \right]$$

$$\Rightarrow t_1 = \sqrt{\frac{2md(\mu\beta - 1)}{\mu\beta P}}$$

[2 POINTS]

2. This webpage shows some high speed videos of three different bow designs as they fire an arrow (Courtesy of Ahyoung Choe, 15). Position-v-time data for the arrow has been extracted from the images, using the MATLAB image processing toolbox. Your mission is to write a MATLAB script that will analyze this data to plot graphs showing the velocity and acceleration of the arrow.



- The position-v-time data are stored in three .csv ('comma separated value') files that can be downloaded from this webpage.
- You can read the data files directly into MATLAB using the 'csvread' command – for example

```
compoundbow_data = csvread('compound_data.csv');
```

  The variable 'compoundbow_data' is a matrix, with the first column (compoundbow_data(:,1)) containing time values, in seconds, and the second column (compoundbow_data(:,2)) containing the distance moved by the arrow, in m. You can also open the files with excel or any text editor.
- The easiest approach to calculating velocity and acceleration is to numerically differentiate the data. For example, let the *N* time and

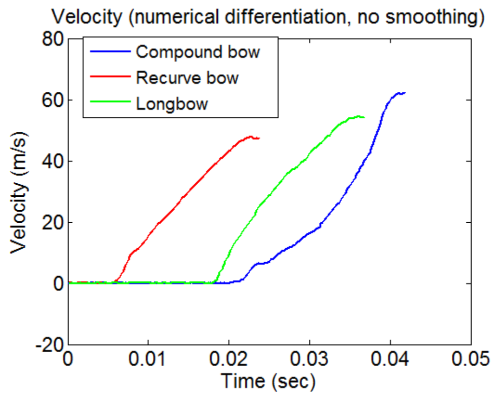
position values be t_i, x_i with $i=1 \ldots N$. You can estimate the velocity at the midpoint of a time interval between two successive data points as $v = \Delta x / \Delta t$, giving

$$\bar{t}_i = (t_i + t_{i+1})/2 \qquad v_i = (x_{i+1} - x_i)/(t_{i+1} - t_i) \quad i = 1 \ldots N-1$$

You can calculate vectors of time and velocity using these formulas and a loop, and then plot them. You can then calculate the accelerations by differentiating the velocity $a = \Delta v / \Delta \bar{t}$. The acceleration will be quite noisy. If you would like to try something more sophisticated you can explore MATLAB's 'Curve Fitting Toolbox,' which has a lot of capabilities for curve-fitting, smoothing, and computing numerical derivatives.

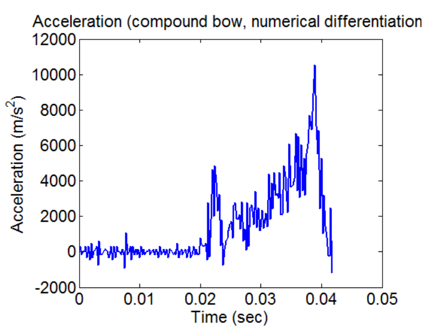
1.4 Submit graphs that show the velocity and acceleration of at least **ONE** bow as a function of time (you can plot them all if you are curious but this is not required). Be sure to specify which bow you analyzed. Also include a brief description that will help the grader understand how you processed the data (numerical differentiation, curve fitting, etc). You do not need to submit MATLAB code.

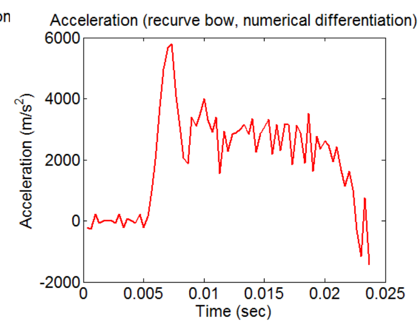
Velocity-v-time is shown below

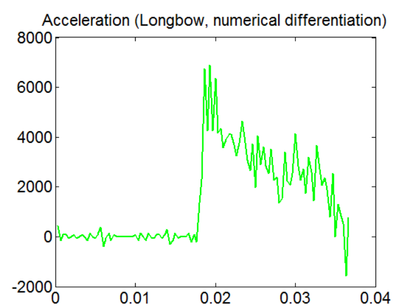


[3 POINTS]

Acceleration results computed by direct numerical differentiation are shown below

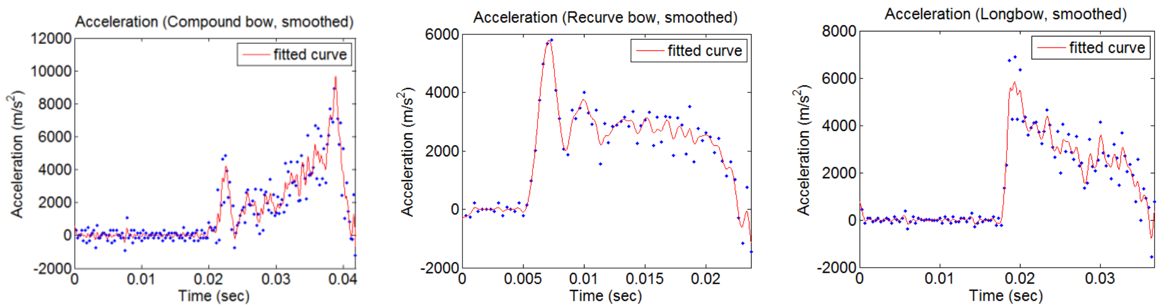






Acceleration data smoothed using the MATLAB cfit function (the dots show the raw data) are shown below

[3 POINTS – any approach is fine]



1.5 Estimate the maximum force exerted by the bow on the arrow as it is fired. For comparison, the *static* force-v-draw curves (i.e. the force required to draw the bow through a distance *d*) for the three bows are shown on the webpage. Why do you think the dynamic force differs from the static force? Would you expect the average dynamic force as the arrow is fired to be greater, or less than the static force?

The maximum dynamic forces are the arrow mass multiplied by the maximum acceleration:

- Longbow – approx. 250N
- Compound bow – approx. 400N
- Recurve bow – approx. 150N

[1 POINT]

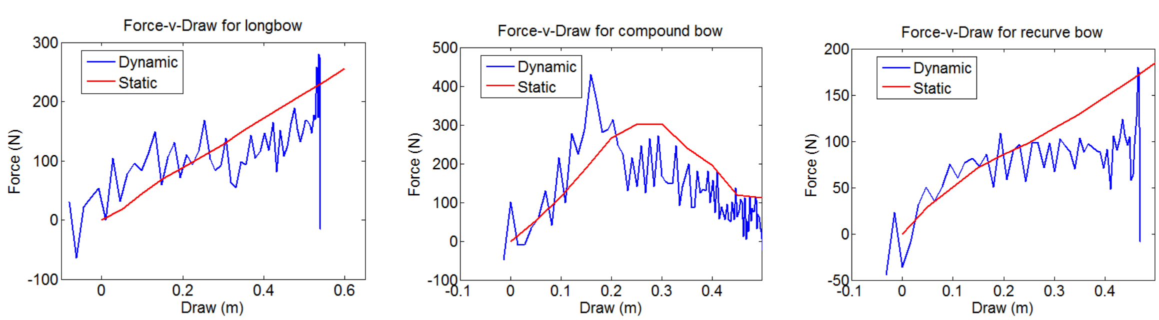
The dynamic force (roughly speaking) differs from the static force because under static loading the internal force in the bow balances the external force exerted on the bow to draw it $F_{\text{int}} = F_{ext}$. Under dynamic conditions the internal forces in the bow must accelerate the bow $F_{\text{int}} - F_{ext} = m_{bow} a_{bow}$ (it's not clear quite what a_{bow} and m_{bow} represent because the bow is not a particle).

[1 POINT]

The average dynamic force must be less than the static force.

[1 POINT]

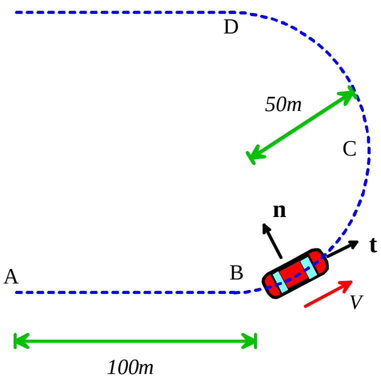
1.6 **Optional – for extra credit** Plot graphs that compare the static force exerted by the bow as a function of draw distance *d* to the actual, dynamic force exerted on the arrow while it is fired (plot draw distance on the horizontal axis and the force on the vertical axis)

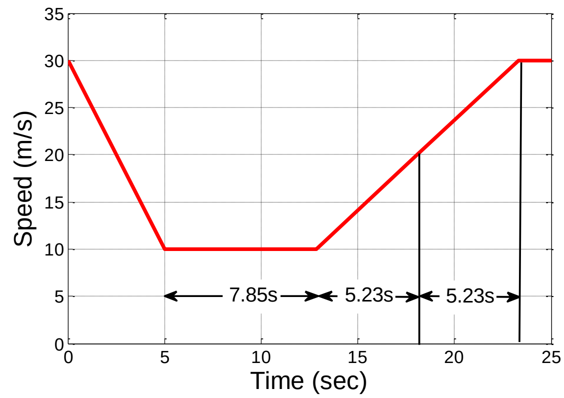


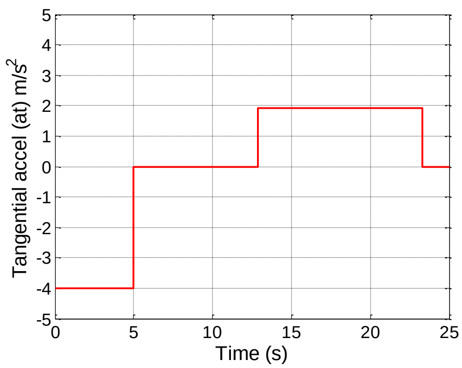
[5 POINTS]

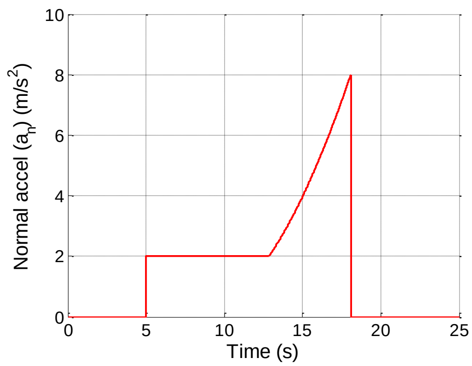
3. The path and speed of a vehicle driving around a sharp bend is shown in the figure below (the vehicle is at point A at time *t=0*).

   3.1 Sketch graphs of the normal and tangential acceleration of the vehicle. Explain briefly how you calculated relevant quantities.









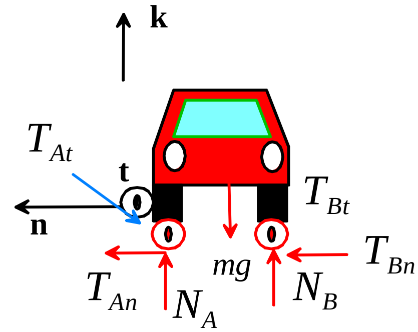
[1 POINT] **[1 POINT]**

This problem involves applying and interpreting the formula for acceleration in normal-tangential coordinates

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

- During the first 5 sec the car travels with average speed 20 m/s, and therefore travels a total distance 100m and just reaches B. **[1 POINT]**
- During the first 5 sec the tangential acceleration is -20/5=-4m/s^2 **[1 POINT]**
- The distance traveled during the subsequent 7.85 sec is 78.5m – the angle BOC is therefore 78.5/50=1.57 radians (approximately $\pi/2$). The car reaches C at the end of this time. **[1 POINT]**
- During this period the tangential acceleration is zero (speed is constant). The normal acceleration is 100/50=2m/s^2 **[1 POINT]**
- During the subsequent 5.23sec, the average speed is 15m/s, and the car therefore travels 78.5m and reaches D at the end of the time period. **[1 POINT]**
- The tangential acceleration is equal to 10/5.23 = 1.91m/s^2 (for 10.46 sec) **[1 POINT]**
- The speed during this phase varies linearly with time – the normal acceleration varies quadratically. At D, the speed is 20 m/s, so the normal acceleration is 400/50=8 m/s^2. **[1 POINT]**
- Thereafter, the normal acceleration drops to zero (the radius of curvature is infinity). **[1 POINT]**

3.2 **Optional – extra credit** Calculate the minimum friction coefficient required to prevent the vehicle from slipping (Draw a FBD; use Newton's laws to calculate the reactions; and use the friction law to find μ)



A FBD is shown. Newton's law gives

$$(T_{At}+T_{Bt})\mathbf{t}+(T_{An}+T_{Bn})\mathbf{n}+(N_A+N_B-mg)\mathbf{k}=ma_t\mathbf{t}+m\frac{V^2}{R}\mathbf{n}$$

Thus $T_{At}+T_{Bt}=a_t \qquad T_{An}+T_{Bn}=\dfrac{V^2}{R} \qquad N_A+N_B=mg$

For no slip

$$\sqrt{(T_{At}+T_{Bt})^2+(T_{An}+T_{Bn})^2}<\mu(N_A+N_B)$$

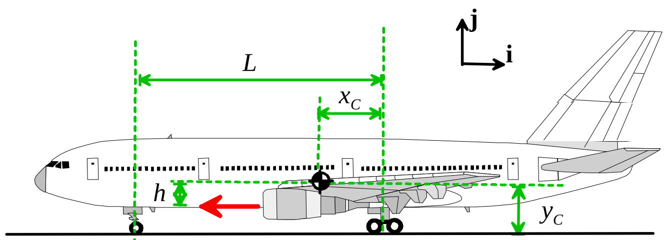
$$\Rightarrow\sqrt{a_t^2+\left(\frac{V^2}{R}\right)^2}<\mu g$$

To find the critical friction coefficient we need to identify the maximum value of $\sqrt{a_t^2+\left(\frac{V^2}{R}\right)^2}$ - from the graphs it is clear that the max value occurs at point D, where $a_t=1.9$ and $V^2/R=8.22$. The friction coefficient must exceed

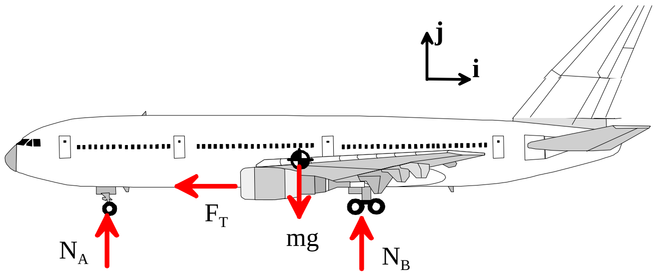
$$\mu>\frac{8.22}{9.81}=0.83$$

[3 POINTS EXTRA CREDIT]

4. The figure shows an aircraft just starting its take-off roll. The engines provide a total thrust F_T that act a height h below the center of mass, producing an acceleration $a_x=-(g/2)\mathbf{i}$. Since the aircraft is not yet moving lift and drag forces are zero.



4.1 Draw a free body diagram showing the forces acting on the aircraft.



[2 POINTS]

4.2 Write down Newton's law of motion and the equation of rotational motion for the aircraft (assume straight line motion without rotation)

$$\mathbf{F} = m\mathbf{a} \Rightarrow -F_T\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = -\frac{g}{2}\mathbf{i}$$

$$\mathbf{M}_C = \left[-N_A(L - x_c) - F_T h + x_c N_B\right]\mathbf{k} = \mathbf{0}$$

[2 POINTS]

4.3 Hence, find formulas for the reaction forces on the wheels (for the rear wheels, calculate the total force).

The previous problem gives three equations, which can be solved for N_A, N_B, F_T with the results

$$F_T = mg/2 \qquad N_A = mg\left(\frac{x_c}{L} - \frac{h}{2L}\right) \qquad N_B = mg\left(1 + \frac{h}{2L} - \frac{x_c}{L}\right)$$

[2 POINTS]

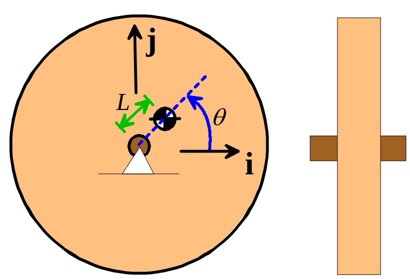
4.4 Hence, show that the front wheel will lose contact with the ground if *h* exceeds a critical value, and find a formula for this critical value of *h*.

$$\text{Notice that } N_A > 0 \Leftrightarrow \left(\frac{x_c}{L} - \frac{h}{2L}\right) > 0 \Rightarrow h < 2x_c$$

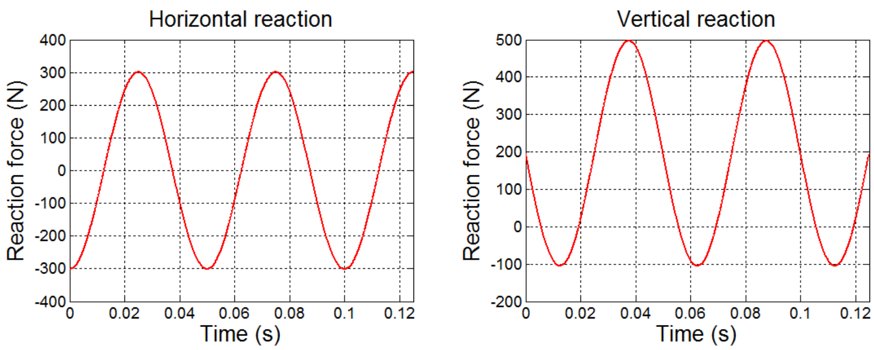
The critical value of *h* is $2x_c$

[2 POINTS]

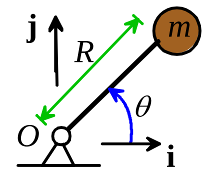
5. The figure shows an unbalanced rotor that spins at constant angular speed $d\theta / dt = \omega = 40\pi$ rad/s. The center of mass of the rotor is a distance *L* from the axle. As a result, large fluctuating reaction forces develop at the axle. The horizontal and vertical reaction forces acting on the wheel are plotted in the graphs shown (the forces act in the positive **i** and **j** directions). At time *t=0* the angle θ is zero.



A 100 gram mass is to be added to the disk to balance it. The goal of this problem is to calculate the position.



5.1 As a preliminary step, consider a mass m that rotates at constant angular rate $\frac{d\theta}{dt} = \omega$ at the end of a massless link with radius R (see the figure). Find the acceleration vector for the mass, expressing your answer as components in the **i,j** basis.



The position vector is easy to write down and can then be differentiated. Or you can just use the standard formulas for circular motion if you prefer.

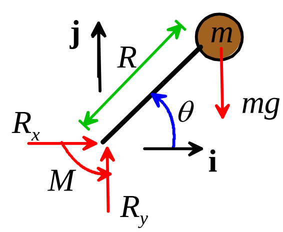
$$\mathbf{r} = R(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{v} = R\frac{d\theta}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

$$\mathbf{a} = -R\left(\frac{d\theta}{dt}\right)^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -R\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

[2 POINTS]

5.2 Draw a free body diagram showing the forces and moments acting on the link and mass together (include reaction forces/moments at O and gravity)



[3 POINTS]

5.3 Use Newton's laws to calculate a formula for the reaction forces acting on the link, in terms of ω, θ, R, m, g .

$$\mathbf{F} = R_x\mathbf{i} + R_y\mathbf{j} - mg\mathbf{j} = m\mathbf{a} = -mR\omega^2(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\Rightarrow R_x = -mR\omega^2\cos\theta \qquad R_y = -mR\omega^2\sin\theta + mg$$

[2 POINTS]

5.4 Hence, find the distance R for which the 100 gram mass will produce the same horizontal reaction force as the disk (see the figure) when rotating at the same speed as the disk.

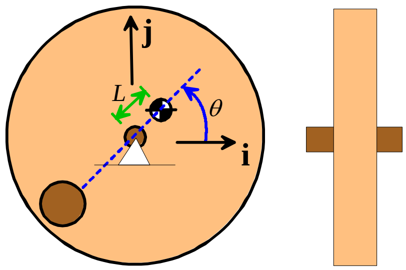
The out-of-balance disk has a reaction force that varies harmonically with time, with frequency equal to the angular speed of the disk (see the figure – the period is 0.05 sec, which is 40π rad/s

Notice that the mass produces a harmonic reaction force – exactly the same as what is measured on the out-of-balance rotor. We can make the mass produce a reaction force that is equal and opposite to what is measured on the disk by choosing

$$mR\omega^2 = 300N \Rightarrow R = 300/(0.1 \times (40\pi)^2) = 0.19m \text{ N}$$

[2 POINTS]

5.5 Draw a sketch showing where the mass *m* should be located on the disk (show the location of the center of mass of the disk, together with the added mass).



(any sketch with the added mass diametrically opposite the COM is fine).

[1 POINT]

5.6 What are the reaction forces acting on the disk after the disk has been balanced?

The fluctuating (harmonic) forces cancel, but the average vertical reactions add. Notice that the average vertical reaction force on the unbalanced disk is 200 N so after adding the mass the total vertical reaction is $R_y = 200 + 0.1g = 201N$

[1 POINT]