EN40: Dynamics and Vibrations

Homework 3: Solving equations of motion for particles
Due Friday Feb 14th – happy Valentine's day!

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1. In class, we calculated formulas that describe the trajectory of a projectile moving near the earth's surface, ignoring drag. The goal of this problem is to find the influence of viscous drag forces on the trajectory. For simplicity assume that the particle moves in the *x,y* plane and has instantaneous velocity $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. Gravity acts in the negative $\mathbf{j}$ direction and the drag force is proportional and opposite to the velocity vector $\mathbf{F}_D = -c\mathbf{v}$, where c is a constant. The particle has mass m.

    1.1 Use Newton's law to show that the coordinates and velocities satisfy

 $$\frac{d^2x}{dt^2} = \frac{dv_x}{dt} = -\frac{cv_x}{m} \qquad \frac{d^2y}{dt^2} = \frac{dv_y}{dt} = -\frac{cv_y}{m} - g$$

    1.2 Assume that at time *t=0* the velocity components are (V_{x0}, V_{y0}). Find formulas for v_x, v_y as functions of time (you can separate variables and integrate, or use mupad).

    1.3 Assume that at time *t=0* $x = y = 0$. Use the solution to 1.2 to calculate *x* and *y* as functions of time.

    1.4 Plot graphs showing the trajectory (*y-v-x*) for $V_{x0} = V_{y0} = 1m/s$, $m = 0.5kg$, for $0 < t < 2.5s$, and with $c = 0.001$, $c = 1$, $c = 10$ Ns/m (there is no need to submit Mupad or MATLAB code)

2. The figure shows the forces acting on a helicopter (mass *m*) in flight. For simplicity, assume that the helicopter flies in the *x,y* plane and has instantaneous velocity $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. In addition
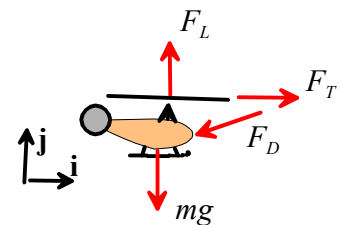
*   The lift and thrust $F_L(t), F_T(t)$ act vertically and horizontally, respectively (and can be controlled by the pilot).
*   The magnitude of the lift force is given (very roughly) by $F_L = P(t) / \left((v_y/2) + \sqrt{mg/(\pi\rho L^2)} \right)$ where L is the length of the rotor blades, $\rho = 1.2$ kg/m³ is air density and *P(t)* is the engine power
*   The magnitude of the drag force $F_D = cV^2$ where V is the speed of the helicopter and c is a constant, and the drag acts opposite to the direction of motion of the helicopter.

2.1 Write down Newton's laws of motion for the helicopter. Rearrange the equations of motion to show that the position and velocity satisfy

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ (F_T - cVv_x)/m \\ (F_L - cVv_y)/m - g \end{bmatrix}$$

2.2 Write a MATLAB code that will calculate and plot the trajectory of the helicopter for a given history of $F_L(t), F_T(t)$ (there is no need to submit Mupad or MATLAB code)

2.3 Calculate and plot the trajectory for the following parameters
- Helicopter mass m=6000kg
- Rotor length L=5.2m
- Drag coefficient c=0.8Ns^2/m^2
- Power $P(t) = mg\sqrt{\dfrac{mg}{\pi\rho L^2}}\left(1 + 0.05\sin\left[\dfrac{2\pi t}{T}\right]\right)$ with T=150 sec
- Thrust $F_T = 200\sin(1.8\pi t/T)$ N
- Initial conditions $x=y=0$ $v_x = v_y = 0$

Hand in plots of the trajectory (x-v-y) and the velocity components v_x, v_y as functions of time.

You can download a script that will animate the trajectory from this webpage (store the file in the same directory as your homework, and call the function using `animate_helicopter(sol);` where sol is the vector of positions and velocities returned by ode45.

3 A 'Dynamic Kingdon trap' is a device for storing charged particles (see e.g. this recent patent). It consists of a wire that runs down the axis of a cylinder.   A time varying electric field is applied to the wire, which subjects the particle to a radial force

$$\mathbf{F} = q\left(V_0 + \Delta V \cos\Omega t\right)\frac{(x\mathbf{i} + y\mathbf{j})}{2\pi\varepsilon_0 r^2}$$

Here, $V_0, \Delta V$ are a static and fluctuating voltage applied between the wire and cylinder; Ω is its frequency, q is the charge of the trapped ion, and ε_0 is the permittivity of free space.
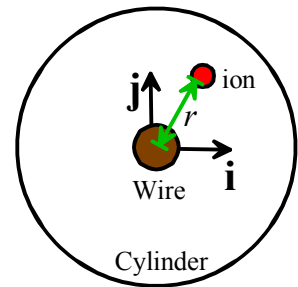
The goal of this problem is to analyze the motion of an ion in the trap. We will assume that the particle moves in the x,y plane, and so must solve for $[x, y, v_x, v_y]$

3.1 Write down Newton's laws for the particle and show that they can be re-arranged into the form

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ (\lambda + \mu\cos\Omega t)x/r^2 \\ (\lambda + \mu\cos\Omega t)y/r^2 \end{bmatrix}$$

and find formulas for λ, μ in terms of $V_0, \Delta V, \varepsilon_0, q, m$.

3.2 Write a MATLAB code that will calculate $[x, y, v_x, v_y]$. (there is no need to submit code)

3.3 Calculate trajectories (x-v-y) for an ion with the following parameter values, for a time period $0 < t < 400$ time units.

(a) $\lambda = -1, \mu = 0, \Omega = 2$ (this is a static trap), with initial conditions $x = 1, y = 0, v_x = 0, v_y = 1$.

For extra credit – calculate a formula for the velocity of a particle following a circular orbit in a static trap, in terms of λ. Does the velocity depend on the radius of the orbit? How does this differ from a particle following a circular orbit in a gravitational field?

(b) $\lambda = -1, \mu = 0, \Omega = 2$ with initial conditions $x = 0.8, y = 0, v_x = 0, v_y = 0.25$. The static trap is stable – a particle remains trapped, as long as nothing slows it down.

(c) $\lambda = -1, \mu = 8, \Omega = 2$ with initial conditions $x = 2.5, y = 0, v_x = 0, v_y = 0.25$. Particles with the right radius and velocity are trapped.

(d) $\lambda = -1, \mu = 8, \Omega = 2$ with initial conditions $x = 1.5, y = 0, v_x = 0, v_y = 0.25$. (this one should go on a separate figure) A dynamic trap will expel particles that are too fast or slow, and are too close or too far from the wire

4 The swing and rotation of a gymnast on a high-bar, or a child on a swing are somewhat mysterious – what force acts to cause the swinging motion? This problem provides a partial explanation. A gymnast swinging on a high horizontal bar can be idealized as a pendulum shown in the figure, with a point mass at B and pin joint at A. The length of member AB varies with time according to the equation $L(t) = L_0 + \Delta L \sin \Omega t$, were, L_0 and ΔL are constants, and Ω is the (constant) frequency at which the athlete 'pumps' to start swinging.
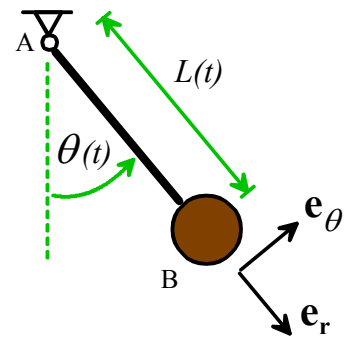
4.1 Find the acceleration vector of the mass at B, in terms of the angle θ and its time derivatives, L_0, ΔL and Ω. Express your answer using the polar coordinate basis vectors shown in the figure.

4.2 Draw a free body diagram showing the force acting on the mass at B.

4.3 Using Newton's law, show that the equation of motion for the angle θ is

$$\left(L_0 + \Delta L \sin \Omega t\right)\frac{d^2\theta}{dt^2} + 2\Delta L\Omega \cos \Omega t \frac{d\theta}{dt} + g \sin \theta = 0$$

4.4 Rearrange the equation of motion into a form that MATLAB can solve.

4.5 Write a MATLAB script that will solve the equations derived in part 4.4.

4.6 Plot graphs showing the variation of the angle θ with time, for a time interval $0<t<50$ sec, initial conditions $\theta = 0.1$ $d\theta / dt = 0$ and the following parameters

4.6.1 $L_0 = 1.2m$ $\Delta L = 0.2m$ $\Omega = 0.1\sqrt{g / L_0}$

4.6.2 $L_0 = 1.2m$ $\Delta L = 0.2m$ $\Omega = 1.9\sqrt{g / L_0}$

To visualize the meaning of the graphs, you can download a script that will animate the motion of the pendulum from the homework page of the website. To use the script, download it into the same directory as your homework script, and then use

```
animate_pendulum(time,sol,L0,dL,Omega)
```
where time and sol are the time values and solution values returned by ode45.