School of Engineering
Brown University

EN40: Dynamics and Vibrations

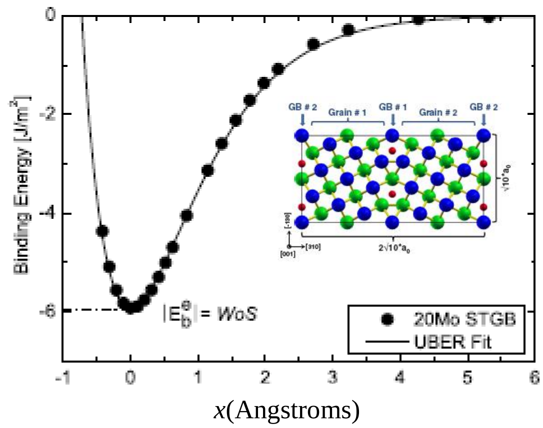
Homework 4: Work, Energy and Linear Momentum
Due Friday March 1st

[Max score: 5 POINTS + 4 extra credit]

1. The figure shows the energy per unit area required to displace two atomic planes of Molybdynum by a distance x from their lowest energy positions (from this publication). The authors report that:

$$W(x \to \infty) = 6.23 Jm^{-2}$$

$$\frac{d^2W}{dx^2}(x=0) = 43.9 \times 10^{19} N/m^3$$



1.1 Calculate the values of the constants E_0, d in the Universal Binding Energy relation

$$W = E_0 - E_0\left(1 + \frac{x}{d}\right)\exp(-x/d)$$

that will fit this data. Hence, estimate the force per unit area that will cause the planes to separate (i.e. the max force of attraction between the planes).

The formula gives

$$W(x \to \infty) = E_0$$

$$\frac{d^2W}{dx^2}(x=0) = \frac{E_0}{d^2}$$

Therefore $E_0 = 6.23$ $\quad d = \sqrt{\frac{E_0}{W''(0)}} = 1.19 \times 10^{-10} m = 1.19 \overset{o}{A}$

The force of attraction between the planes is related to W by

$$F = \frac{dW}{dx} = \frac{E_0}{d}\frac{x}{d}\exp(-x/d)$$

which has maximum value $E_0/\left[\exp(1)d\right]$ where $x=d$. With numbers F_{max} =1.923x10^{10}N/m^2

[2 POINTS]

1.2 Instead of the UBER, the authors decided to fit their calculated work of separation using the more elaborate function (their eq. (9)) of the form

$$W = E_0 - E_0\left(1 + \frac{x}{d} - 0.079\frac{x^3}{d^3} + 0.0063\frac{x^4}{d^4} + 0.00002\frac{x^5}{d^5}\right)\exp(-x/d)$$

use Mupad to plot the attractive force as a function of x for both the universal binding energy function and the modified formula (use the values of E_0, d from part 1.1. Why is this correct?). Calculate the maximum force predicted by the new formula.

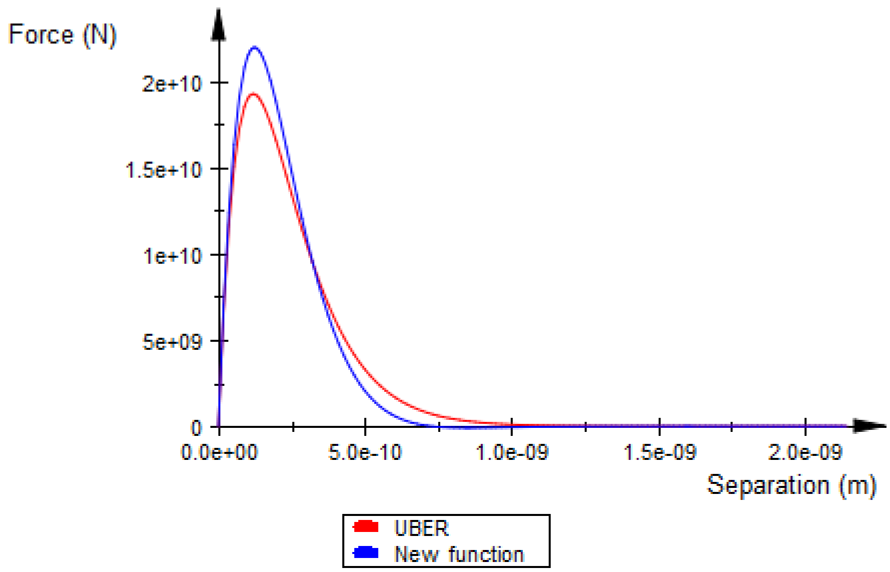
The same values of E_0, d can be used for the new fit because the formulas for

$$W(x \to \infty) = 6.23 Jm^{-2}$$

$$\frac{d^2W}{dx^2}(x=0) = 43.9 \times 10^{19} N/m^3$$

remain unchanged.

The force $F = \frac{dW}{dx}$ can be found with Mupad



We can find the peak of the new function by differentiating F, setting the result to zero and solving for x with mupad. We find five roots – the relevant one (from the graph) occurs at $x=1.037d$. Substituting back into the force function shows that the maximum force is

$$F_{\max} = 0.419 \frac{E_0}{d} = 2.19 \times 10^{10} N/m^2$$

[3 POINTS]

2. The Tesla Model S electric vehicle has the following specifications:

- Acceleration from 0 to 60mph in 4.2sec.
- Curb weight of 4647.3 lbs.
- Battery capacity: 60kWh
- Range at 55mph 244 miles
- Height 56”; width 77”

Assume that air resistance can be calculated from the formula

$$F_D = \frac{1}{2}\rho C_D A v^2$$

with drag coefficient C_D, air density $\rho = 1.2 kgm^{-3}$ and projected frontal area *A* and *v* the speed

2.1 Assuming that air resistance is the dominant contribution to energy consumption during steady cruise, use the given range and battery capacity to calculate the drag coefficient.

We have to do some preliminary unit conversions:
60 kWh is 216 MJoules
55mph is 24.5 m/s
Height is 1.4m, width 1.95m, frontal area 2.73m^2
244 miles is 393 km
Weight 4647.3 lb is 2108 kg

Assume that *W*=60kWh capacity is fully drained after distance *L*=393km. The work done against air resistance is $W = F_D L$, therefore the drag force is

$$F_D = \frac{216\times 10^6}{393\times 10^3} = 550N$$

The drag coefficient follows as

$$C_D = 2\frac{F_D}{\rho A v^2} = 2\frac{550}{1.2\times 2.73\times (24.5)^2} = 0.56$$

This is a bit higher than one might guess, but there are other energy losses besides air resistance so we would expect to overestimate the drag coefficient.

[3 POINTS]

2.2 Calculate the range of the vehicle at 70 mph.

The drag force is proportional to the square of the velocity and so increases to $F_D = 550\times(70/55)^2$. The range therefore decreases to $L = 244\times(55/70)^2 = 150$ miles

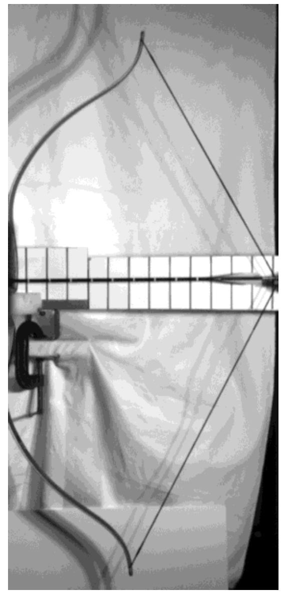
[2 POINTS]

2.3 Assuming that the propulsion system produces a constant power (i.e. independent of velocity) estimate the power necessary to accelerate the vehicle to 60mph in 4.2 sec. You can neglect air resistance to keep the calculation simple (recall that in class we showed that air resistance had a very small effect on the acceleration).

The total work done by the propulsion system must equal the change in kinetic energy

$$P\Delta t = \Delta KE = \frac{1}{2} 2107 \times (26.8)^2$$
$$\Rightarrow P = \frac{1}{2 \times 4.2} 2107 \times (26.8)^2 = 180 kW = 241 hp$$

[2 POINTS]

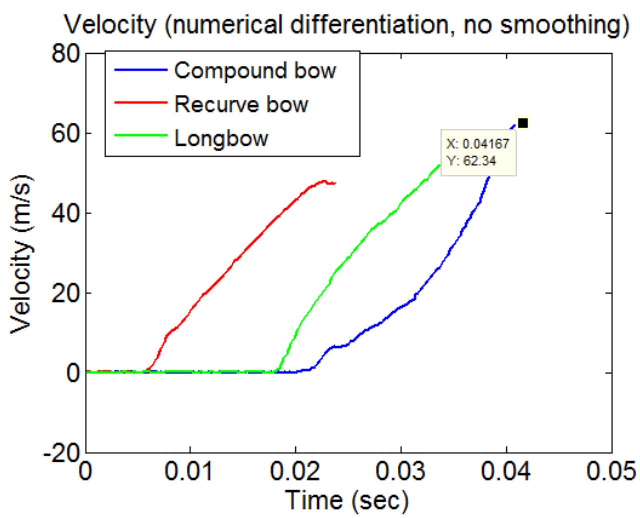


3. The static force-v-displacement measurements for the three bows that you analyzed in Homework 2 can be downloaded from this webpage.

3.1 Use the MATLAB code that you wrote in Homework 2 to calculate the kinetic energy of one (or more) of the arrows just after they leave the bow. Be sure to state which bow!

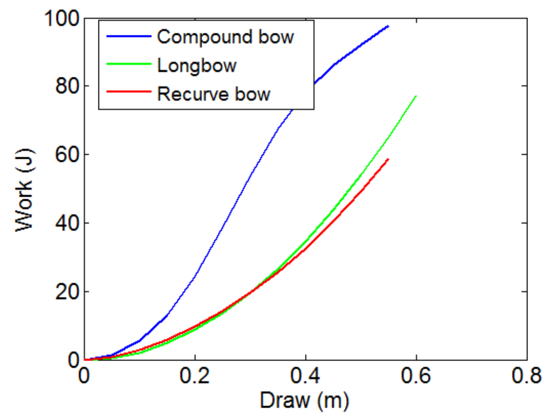
The velocities just after release can be read off the graphs that you plotted in HW2. The results are:

- Longbow v=55m/s m=0.0391kg KE = 59J
- Recurve bow v=47m/s m=0.031kg KE= 34J
- Compound bow v=62 m/s m=0.0374kg KE=72J



[2 POINTS]

3.2 The static force-v-draw data for each bow are available as .csv files on the webpage. By integrating the force-v-draw curve, plot a graph of the work done in drawing the bow(s) considered in 2.1 as a function of draw distance *d*. You can use the MATLAB 'cum trapz' function to do the integral, or a method of your own design. There is no need to submit a copy of your MATLAB code.



The work done is plotted as a function of draw distance in the figure.

[4 POINTS]

3.3 Hence, calculate the *dynamic efficiency* of each bow (the ratio of the kinetic energy of the arrow to the work done in drawing the bow. Be sure to use the *actual* draw distance corresponding to each shot.

We can read off the energy stored in each bow at the maximum draw used in the experiment from the graph. The results are

- Longbow E=64J
- Recurve bow E=45J
- Compound bow E=93J

The dynamic efficiencies follow as

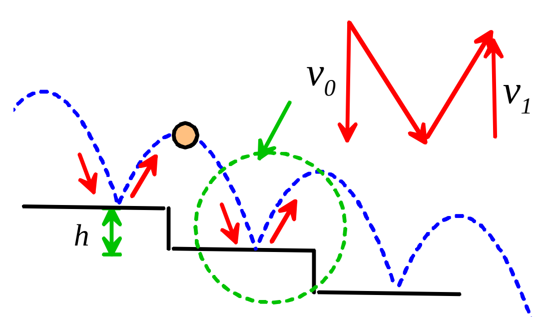
- Longbow = 59/64=92%
- Recurve bow = 34/45=76%
- Compound bow 72/93=77%

Graders – numbers will vary depending on how the KE/work were determined – anything in the right range should get credit.

The longbow data gives a suspiciously high efficiency… the arrow mass may have been recorded incorrectly.

[2 POINTS]

4. The figure shows a ball bouncing down a flight of stairs. Assume that steady state conditions hold, so that the ball lands on each successive step with the same (downwards) vertical velocity v_0 . The goal of this problem is to calculate this special velocity, in terms of the restitution coefficient *e* and the height of the step *h*.



4.1 Write down the (upwards) vertical velocity v_1 just after the bounce, in terms of v_0 and *e*.

The restitution coefficient formula gives $v_1 = ev_0$

[1 POINT]

4.2 Use energy conservation to find a formula relating v_1 to v_0 and *h*.

During free flight energy is conserved (PE+KE=constant). Thus, if the ball has tangential velocity v_t

$$\frac{1}{2}m(v_1^2+v_t^2)+mgh=\frac{1}{2}m(v_0^2+v_t^2)\Rightarrow v_0^2=v_1^2+2gh$$

(any algebraically equivalent form is fine), where we have noted that the ball has vertical velocity v_0 just before it lands on the next step.

[2 POINTS]

4.3 Hence, show that $v_0=\sqrt{2gh/(1-e^2)}$

Substituting the result of 3.1 into the result of 3.2 and rearranging gives this formula.

[1 POINT]

4.4 Calculate the average vertical velocity of the ball (it is easiest to do this by finding the time between two successive bounces).

We can use the trajectory formula – the y component shows that the time of flight satisfies

$$y=ev_0t-\frac{1}{2}gt^2=-h$$

$t=\dfrac{ev_0+\sqrt{e^2v_0^2+2gh}}{g}$. You can also find the time from the velocity

$$-v_0=v_1-gt\Rightarrow t=(v_1+v_0)/g=(1+e)v_0/g$$

Substituting for v_0 and simplifying gives

$$t=(1+e)\sqrt{\frac{2h}{g(1-e^2)}}=\sqrt{\frac{2h(1+e)}{g(1-e)}}$$

The average vertical velocity is thus

$\dfrac{h}{t}=\sqrt{\dfrac{gh(1-e)}{2(1+e)}}$ **(Graders – there are other algebraically equivalent formulas – check carefully!)**

[3 POINTS]

4.5 **Optional (and quite hard) – for extra credit:** Suppose that the ball is launched at the top of a flight of stairs with vertical velocity v_0 . Show that it will land on the *nth* step with vertical velocity

$$v_n=\sqrt{e^{2(n-1)}v_0^2+2gh\frac{1-e^{2n}}{1-e^2}}$$

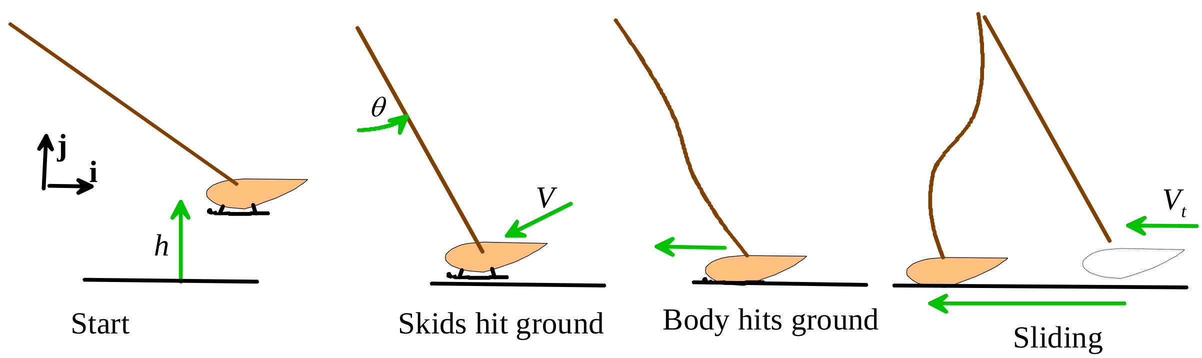
If we calculate the velocity for the first few steps, we start to see the pattern emerge

$$
\begin{aligned}
&v_1^2 = v_0^2 + 2gh \\
&v_2^2 = e^2 v_1^2 + 2gh = e^2(v_0^2 + 2gh) + 2gh \\
&v_3^2 = e^2 v_2^2 + 2gh = e^4(v_0^2 + 2gh) + e^2 2gh + 2gh \\
&v_4^2 = e^2 v_3^2 + 2gh = e^6(v_0^2 + 2gh) + e^4 2gh + e^2 2gh + 2gh \\
&\Rightarrow v_n^2 = e^{2(n-1)} v_0^2 + 2gh \sum_{i=0}^{n-1} e^{2i} \\
&\Rightarrow v_n = \sqrt{e^{2(n-1)} v_0^2 + 2gh \frac{1-e^{2n}}{1-e^2}}
\end{aligned}
$$

(the series was summed using mupad, but you guys probably know how to do it by hand….) As a check, if we let $n \to \infty$, we see we recover the solution to 3.3.

[4 POINTS]

5. The figure shows a schematic diagram of a helicopter impact test used by NASA. If you watch the movie, you will see that the impact takes place in two stages: (i) the skids hit the ground, and get crushed; and (ii) the main body of the helicopter hits the ground. The goal of this problem is to estimate how much force the skids exert on the helicopter body as they are crushed.



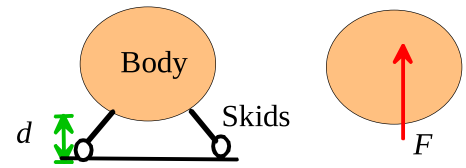
5.1 The helicopter starts at a height h above the ground. Use energy methods to find a formula for its speed V just before the skids hit the ground. Hence, calculate the **i,j** components of velocity just before the skids hit the ground, in terms of the angle θ.

Energy conservation gives $\frac{1}{2}mV^2 = mgh \Rightarrow V = \sqrt{2gh}$

Simple geometry shows that $\mathbf{v} = -V\cos\theta\mathbf{i} - V\sin\theta\mathbf{j} = -\sqrt{2gh}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

[2 POINTS]

5.2 As the skids are crushed, they exert a constant vertical force F on the helicopter body (horizontal force can be neglected). The cable exerts no force on the helicopter during this period. Use Newton's laws to calculate the **i,j** components of velocity of the helicopter body just before it impacts the ground, in terms of F, the helicopter mass m, g, h, θ and the height d of the body above the base of the skids.



F=m**a** and the constant acceleration formulas give the acceleration, velocity and position as

$$\mathbf{a} = (F/m - g)\mathbf{j} \Rightarrow \mathbf{v} = -\sqrt{2gh}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + (F/m - g)t\mathbf{j}$$

$$\mathbf{r} = d\mathbf{j} - \sqrt{2gh}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})t + \frac{1}{2}(F/m - g)t^2\mathbf{j}$$

When the body hits the ground the **j** component of position is zero, so

$$0 = d - \sqrt{2gh}\sin\theta t + \frac{1}{2}(F/m - g)t^2$$

$$t = \frac{\sqrt{2}\left(\sqrt{gh}\sin\theta - \sqrt{g(h\sin^2\theta + d) - Fd/m}\right)}{(F/m - g)}$$

Hence

$$\mathbf{v} = -\sqrt{2gh}\cos\theta\mathbf{i} - \sqrt{2}\left(\sqrt{g(h\sin^2\theta + d) - Fd/m}\right)\mathbf{j}$$

[3 POINTS]

5.3 During the impact with the ground, the helicopter body is subjected to an impulse $\mu I_N\mathbf{i} + I_N\mathbf{j}$ where μ is the coefficient of friction. Find a formula for the horizontal velocity of the helicopter body just after impact, in terms of

The impulse-momentum equations yield

$$-mV_t\mathbf{i} - m\mathbf{v} = \mu I_N\mathbf{i} + I_N\mathbf{j}$$

$$\Rightarrow \mu I_N\mathbf{i} + I_N\mathbf{j} = m\left(-V_t + \sqrt{2gh}\cos\theta\right)\mathbf{i} + m\sqrt{2}\left(\sqrt{g(h\sin^2\theta + d) - Fd/m}\right)\mathbf{j}$$

We can eliminate I_N from the **i**, **j** components of this equation

$$mV_t = m\sqrt{2gh}\cos\theta - \mu I_N$$

$$I_N = m\left[\sqrt{2}\sqrt{g(h\sin^2\theta + d) - Fd/m}\right]$$

$$\Rightarrow V_t = \sqrt{2gh}\cos\theta - \mu\left[\sqrt{2}\sqrt{g(h\sin^2\theta + d) - Fd/m}\right]$$

[3 POINTS]

5.4 After impact, the helicopter slides over the ground for a time t before coming to rest. Use impulse-momentum to find a formula for the speed of the helicopter V_t at the start of the skid, in terms of the friction coefficient μ

The net horizontal impulse exerted on the helicopter during skidding is $I = \mu mgt$. The change in momentum is $\Delta\mathbf{p} = 0\mathbf{i} - m(-V_t)\mathbf{i}$. Therefore $V_t = \mu gt$.

[2 POINTS]

5.5 Use the results of (3) and (4) to calculate a formula for *F*.

$$\mu gt = \sqrt{2gh}\cos\theta - \mu\left[\sqrt{2}\sqrt{g(h\sin^2\theta + d) - Fd/m}\right]$$

$$\Rightarrow \sqrt{2}\sqrt{g(h\sin^2\theta + d) - Fd/m} = (\sqrt{2gh}\cos\theta/\mu) - gt$$

$$\Rightarrow Fd/m = g\left(h\sin^2\theta + d\right) - \left((\sqrt{gh}\cos\theta/\mu) - gt/(\sqrt{2})\right)^2$$

$$\Rightarrow F = mg(1 + h\sin^2\theta/d) - (mgh/d)\left(\cos\theta/\mu - t\sqrt{g/2h}\right)^2$$

[3 POINTS]

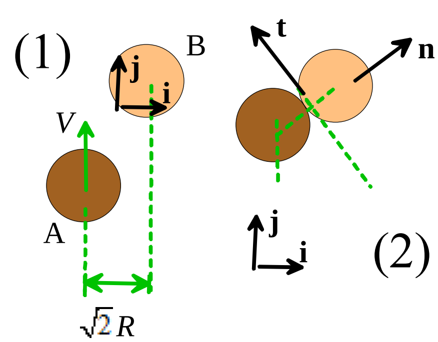
5.6 Calculate the force for the following parameters:

- Slip time t=2sec
- Friction coefficient $\mu = 0.5$
- Drop height h 12m
- Skid height d 1m
- Mass 1300kg
- Cable angle at impact $\theta = 30^o$

Substituting the numbers gives F= 19 kN. The acceleration produced by this force is about 15g.

[2 POINTS]

6. The figure shows a collision between two identical spheres with mass *m* and radius *R*. The restitution coefficient for the collision is *e*.



6.1 Write down the total linear momentum of the system before the impact, in {**i,j**} components.

$$\mathbf{p} = mV\mathbf{j}$$

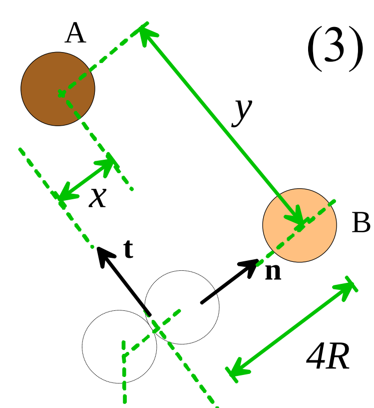
[1 POINT]

6.2 Find the velocity of sphere A before impact in **n,t** components, and hence write down the total linear momentum of the system in **n,t** coordinates.

$$\mathbf{v} = V(\mathbf{n} + \mathbf{t})/\sqrt{2}$$

$$\mathbf{p} = mV(\mathbf{n} + \mathbf{t})/\sqrt{2}$$

[1 POINT]



6.3 Which of the following are conserved during the impact:

(1) The total linear momentum of the system in the **i** direction
(2) The total linear momentum of the system in the **j** direction
(3) The total linear momentum of the system in the **n** direction

(4) The total linear momentum of the system in the **t** direction
(5) The linear momentum of sphere A in the **i** direction? **j** direction? **n** direction? **t** direction?
(6) The linear momentum of sphere B in the **i** direction? **j** direction? **n** direction? **t** direction?

(1-4) all conserved, (5,6) momentum of A and B conserved only in the **t** direction.

[3 POINTS]

6.4 Write down the velocities of the two spheres in the **t** direction after impact (you don't need to do any calculations!)

Since momentum in the **t** direction is conserved the **t** component of velocity is constant.
Therefore Sphere A : $V/\sqrt{2}$; Sphere B 0.

[1 POINT]

6.5 Find a formula for the velocities of the two spheres in the **n** direction, in terms of V and the restitution coefficient *e*.

The normal component of velocity is governed by the 1-D restitution coefficient formula:

$$v_n^{B1} - v_n^{A1} = -e\left(v_n^{B0} - v_n^{A0}\right) = eV/\sqrt{2}$$

Also total momentum is conserved in the **n** direction so

$$mv_n^{B1} + mv_n^{A1} = mV/\sqrt{2}$$

Solving these equations gives

$$v_n^{B1} = (1+e)V/(2\sqrt{2})$$
$$v_n^{A1} = (1-e)V/(2\sqrt{2})$$

[2 POINTS]

6.6 Suppose that sphere *B* travels a distance 4*R*. Calculate the coordinates *x*,*y* of sphere *A* relative to the impact point in the {**n,t**} basis at the same instant, in terms of *e* and *R*.

We know that it takes sphere B a time $t = 4R/v_n^B$ to travel a distance 4R.

The distance travelled by sphere A during this time follows

$$d_t = \frac{v_t^A}{v_n^B}4R = \frac{8R}{(1+e)}$$
$$d_n = \frac{v_n^A}{v_n^B}4R = \frac{(1-e)}{(1+e)}4R$$

The problem asks for the coordinates of sphere *A* relative to the contact point, which are

$$y = d_t \frac{8R}{(1+e)}$$
$$x = d_n - R = \frac{(1-e)}{(1+e)}4R - R$$

(Graders – don't deduct points for forgetting to shift the origin of the coord system – that's a bit mean. Formulas for the distance traveled by the sphere are fine. [2 POINTS]