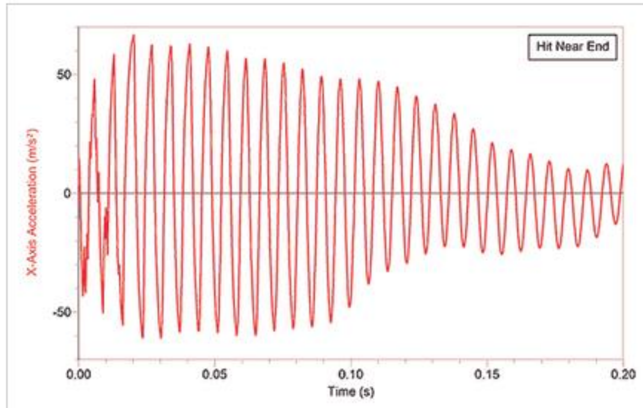
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Free Vibrations
Due Friday March 21st

1. The figure shows a vibration measurement from an accelerometer attached to a baseball bat (from this website). For the time interval between 0.02sec and 0.06sec, estimate:



1.1 The amplitude and frequency of the acceleration

The acceleration amplitude is about 60 m/s^2. There are about 7 cycles in 0.05 sec, so the frequency is about $\frac{2\pi\times 7}{0.05} = 880$ rad/s.

[2 POINTS]

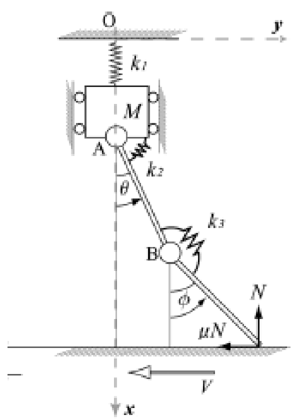
1.2 The amplitude of the displacement of the bat (at the point where the accelerometer is attached).

For harmonic vibrations

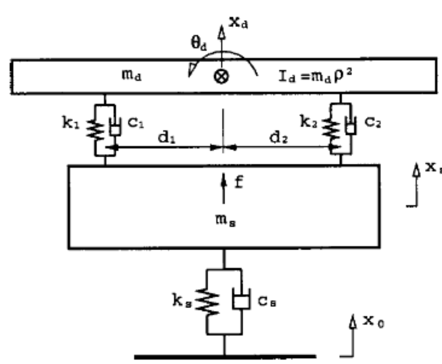
$a(t) = A_0 \sin\omega t = -X_0\omega^2 \sin\omega t$ so the displacement amplitude is $60/880^2 = 0.077mm$

[2 POINTS]

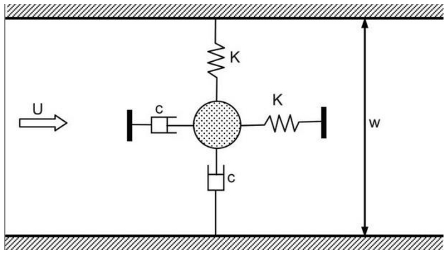
2. State the number of degrees of freedom and the number of natural frequencies of vibration for each of the systems shown below (do not include rigid body modes when counting natural frequencies) (The articles linked give some practical examples of vibration). Note that the first two show the coordinates so this is sort of trivial)!



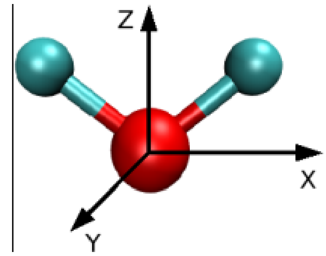
(a) 2D Laser printer wiper



(b) 2D Model of a tuned vibration absorber
The top mass is a rigid body,
the bottom mass is a particle



(c) Model of a vibrating pipe in a fluid flow



(d) Water molecule

(a-b) are easy because the figures show the coordinates and have no rigid body modes. So (a) – 2 DOF, 2 modes; (b) 3DOF, 3 modes;

(c) – the pipe moves in the plane horizontally and vertically, 2DOF . (if the cylinder were a rigid body there would also be a rotational DOF but no dimensions are given for the cylinder so presumably it is idealized as a particle). There are 2 modes.

(d) There are 3 atoms (which are particles) with 3 DOF each – total of 9 DOF. The assembly of atoms has 6 rigid body modes – 3 translation, 3 rotation. There must be 3 vibration modes

[6 POINTS]

3. Solve the following differential equations (use the Solutions to Differential Equations)

3.1 $4\frac{d^2y}{dt^2}+16y=16 \qquad y=1 \quad \frac{dy}{dt}=2 \quad t=0$

Rearrange: $\frac{1}{4}\frac{d^2y}{dt^2}+y=1 \qquad y=1 \quad \frac{dy}{dt}=2 \quad t=0$

This is case I $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C$ with solution $x(t)=C+(x_0-C)\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t$

So the solution is $y=1+\sin 2t$

3.2 $4\frac{d^2y}{dt^2}+4\frac{dy}{dt}+16y=0 \qquad y=0.1 \quad \frac{dy}{dt}=0 \quad t=0$

Rearrange $\frac{1}{4}\frac{d^2y}{dt^2}+\frac{2}{2}\frac{1}{4}\frac{dy}{dt}+y=0 \qquad y=0.1 \quad \frac{dy}{dt}=0 \quad t=0$

This is case III $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C$ The damping ratio $\zeta=1/4$ so this is an underdamped system.

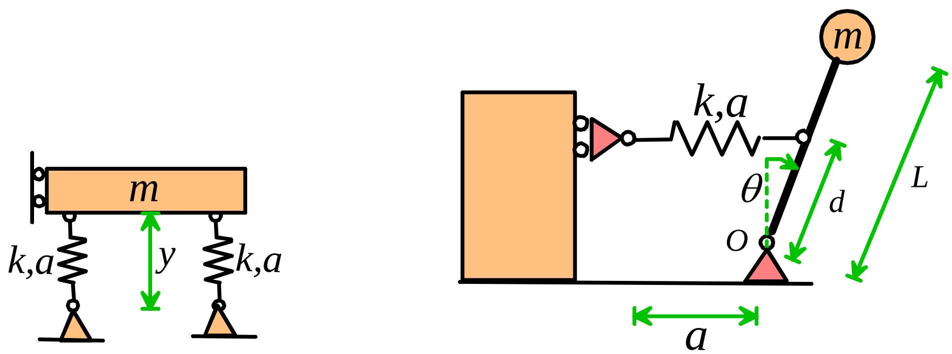
The solution is $x(t) = C + \exp(-\varsigma\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \varsigma\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$

with $\omega_d = \omega_n\sqrt{1-\varsigma^2}$

and so substituting numbers gives $\omega_d = \sqrt{15}/2$ and

$$y = 0.1\exp(-t/2)\left\{\cos(\sqrt{15}t/2) + \frac{1}{\sqrt{15}}\sin(\sqrt{15}t/2)\right\}$$

[4 POINTS – 2 POINTS EACH]



4. For the two conservative single-degree of freedom systems shown in the figure:

4.1 Derive the equation of motion (use energy methods). State whether the equation of motion is linear or nonlinear

For the first system the total potential + kinetic energy is

$$T + V = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + k(y-a)^2 + mgy$$

Take the time derivative (which must vanish because this is a conservative system)

$$T + V = m\left(\frac{dy}{dt}\right)\frac{d^2y}{dt^2} + 2k(y-a)\frac{dy}{dt} + mg\frac{dy}{dt} = 0$$

$$\Rightarrow m\frac{d^2y}{dt^2} + 2k(y-a) - mg = 0$$

The equation is linear.

For the second system, note that the speed of the mass is $L\frac{d\theta}{dt}$ so

$$T+V=\frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2+\frac{1}{2}k(d\sin\theta)^2+mgL\cos\theta$$

$$\Rightarrow m\left(L^2\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2}+kd\sin\theta\cos\theta\frac{d\theta}{dt}-mgL\sin\theta\frac{d\theta}{dt}=0$$

$$\Rightarrow mL^2\frac{d^2\theta}{dt^2}+kd\sin\theta\cos\theta-mgL\sin\theta=0$$

This equation is nonlinear (because of the trig terms)

[4 POINTS]

4.2 If appropriate, linearize the equation of motion for small amplitude vibrations

The first equation is already linear.

For the second system we Taylor expand the trig terms $\sin\theta\approx\theta \quad \cos\theta\approx 1$ so

$$mL^2\frac{d^2\theta}{dt^2}+kd\sin\theta\cos\theta-mgL\sin\theta=0$$

$$\Rightarrow mL^2\frac{d^2\theta}{dt^2}+kd\theta-mgL\theta=0$$

[2 POINTS]

4.3 Arrange the (linearized) equation of motion into standard form, and find an expression for the natural frequency of vibration. Identify conditions where the natural frequency cannot be calculated and explain what the system will do if disturbed from $\theta=0$ in this case.

For the first system

$$m\frac{d^2y}{dt^2}+2k(y-a)-mg=0$$

$$\Rightarrow \frac{m}{2k}\frac{d^2y}{dt^2}+y=a-\frac{mg}{2k}$$

This has the form

$$\Rightarrow \frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+y=C\Rightarrow \omega_n=\sqrt{\frac{2k}{m}}$$

[2 POINTS]

For the second system

$$mL^2\frac{d^2\theta}{dt^2}+kd\theta-mgL\theta=0$$

$$\Rightarrow \frac{mL^2}{kd-mgL}\frac{d^2\theta}{dt^2}+\theta=0$$

The two standard forms for the undamped free vibration problem is

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+y=C$$

The natural frequency is thus $\omega_n=\sqrt{\frac{kd-mgL}{mL^2}}$

The natural frequency becomes complex if $mgL>kd$ and so can't be found (at least in the conventional sense – a complex number for the natural frequency means that the sinusoidal oscillations transform to an exponentially increasing solution. To see what to do in this case rearrange the equation as

$$\frac{mL^2}{mgL-kd}\frac{d^2\theta}{dt^2}-\theta=0$$

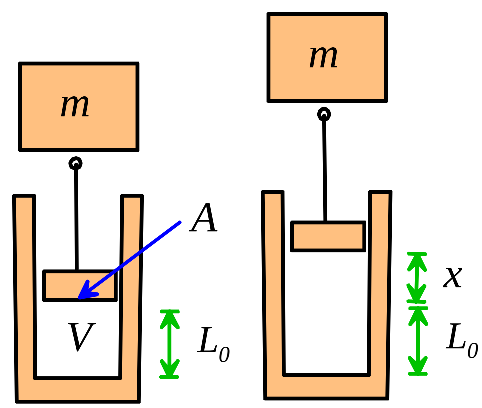
And now the table of solutions shows that we get an exponentially increasing function for θ. This means the pendulum topples over – it is unstable.

[2 POINTS]

5. A pneumatic vibration isolator works by using a compressed air column as a spring. The air pressure in the piston obeys (approximately) the adiabatic ideal gas law

$$\frac{p}{p_0}\left(\frac{V}{V_0}\right)^{\gamma}=1$$

Where p_0, V_0 are the air pressure and volume in the cylinder when it is fully extended (constants), and $\gamma\approx 7/5$ for air. The goal of this problem is to derive a formula for the natural frequency of vibration of a mass supported by the isolator (the specification sheet gives formulas, but it is not easy to see where they come from).



5.1 Show that the force exerted by the isolator (air pressure x cylinder area) is related to the air cylinder area *A* and length *L* by

$$F=\frac{p_0V_0^{\gamma}}{A^{\gamma-1}L^{\gamma}}$$

This is simply geometry – the cylinder volume is $V=AL$ and the force is

$$F=pA=Ap_0\left(\frac{V_0}{V}\right)^{\gamma}=Ap_0\left(\frac{V_0}{AL}\right)^{\gamma}$$

[1 POINT]

5.2 Suppose that the isolator supports an object with mass *m* and is in static equilibrium (no motion or vibration). Show that the length L_0 of the actuator satisfies the equation

$$\frac{p_0V_0^{\gamma}}{A^{\gamma-1}}=mgL_0^{\gamma}$$

This is statics – in equilibrium the force *F* balances the weight

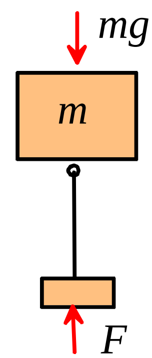
$$Ap_0\left(\frac{V_0}{AL_0}\right)^{\gamma} = mg$$

and this can be rearranged into the required result

[1 POINT]

5.3 Suppose that as the mass vibrates, its length changes to $L = L_0 + x$. Use Newton's law of motion for the mass, and the results of 5.1 and 5.2 to show that x must satisfy the equation of motion

$$m\frac{d^2x}{dt^2} = mg\frac{L_0^{\gamma}}{(L_0+x)^{\gamma}} - mg$$



A FBD is shown. Newton's law gives

$$m\frac{d^2}{dt^2}(L_0+x) = F - mg$$

$$\Rightarrow m\frac{d^2x}{dt^2} = p_0\left(\frac{V_0}{A(L_0+x)}\right)^{\gamma} - mg$$

Note that

$$p_0\left(\frac{V_0}{AL_0}\right)^{\gamma} = mg \Rightarrow p_0\left(\frac{V_0}{A(L_0+x)}\right)^{\gamma} = mg\left(\frac{L_0}{L_0+x}\right)^{\gamma}$$

which then yields the result.

[2 POINTS]

5.4 Linearize the equation of motion for $x << L_0$, and hence show that the natural frequency of vibration is

$$\omega_n = \sqrt{\frac{\gamma g}{L_0}}$$

In many applications the actuator is pressurized so as to maintain a constant static length L_0 by an automatic leveling system. If this is done, the natural frequency of the system is independent of the mass *m*.

We have to do a Taylor expansion of the nonlinear term (oh, the horror!)

$$mg\frac{L_0^{\gamma}}{(L_0+x)^{\gamma}} \approx mgL_0^{\gamma}\left(\frac{1}{(L_0)^{\gamma}} + \left[\frac{d}{dx}\frac{1}{(L_0+x)^{\gamma}}\right]_{x=0} x + \ldots\right) = mgL_0^{\gamma}\left(\frac{1}{(L_0)^{\gamma}} - \frac{\gamma x}{L^{\gamma+1}}\right) = mg - \frac{\gamma x}{L_0}$$

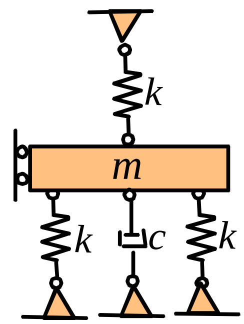
Thus

$$m\frac{d^2x}{dt^2} = mg\left(1 - \frac{\gamma x}{L_0}\right) - mg \Rightarrow \frac{1}{\gamma g / L_0}\frac{d^2x}{dt^2} + x = 0$$

Comparing with the standard form gives the expression for the natural frequency.

[2 POINTS]

6. Replace the system shown in the figure with an equivalent spring-mass system consisting of a mass with only one spring and dashpot. Hence, determine a formula for the undamped natural frequency and the damping factor for the system.



The springs are in parallel so the effective stiffness is $3k$.

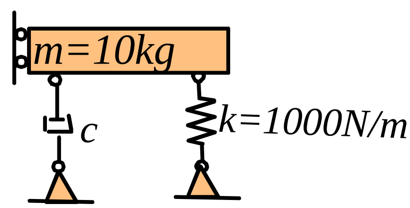
From the standard results for a spring-mass system we see that

$$\omega_n = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{3k}{m}}$$

$$\zeta = \frac{c}{2\sqrt{k_{eff}m}} = \frac{c}{2\sqrt{3km}}$$

[2 POINTS]

7. The spring-mass system shown in the figure is at rest for time $t<0$. The mass is then displaced from its equilibrium position vertically by a distance $x_0 = 10mm$ and released (from rest). Find formulas for the subsequent motion of the mass $x(t)$ for each of the following cases (you don't need to re-derive the equations of motion, since this is a standard system. Note also that since x is the displacement from equilibrium the constant C in the standard equations of motion is zero).



7.1 $c = 20Ns/m$
7.2 $c = 200Ns/m$

This is a standard spring-mass system with undamped natural frequency $\omega_n = \sqrt{k/m} = 10rad/s$
We can simply write down the known solution.

The damping factor for the three cases is

7.1 $\zeta = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{10000}} = 0.1$ (underdamped)

7.2 $\zeta = \frac{c}{2\sqrt{km}} = \frac{200}{2\sqrt{10000}} = 1$ (critically damped)

The damped natural frequencies follow as

7.1 $\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-\frac{1}{100}} = \sqrt{99}$

7.2 N/A

We can get the solution from the list of solutions:

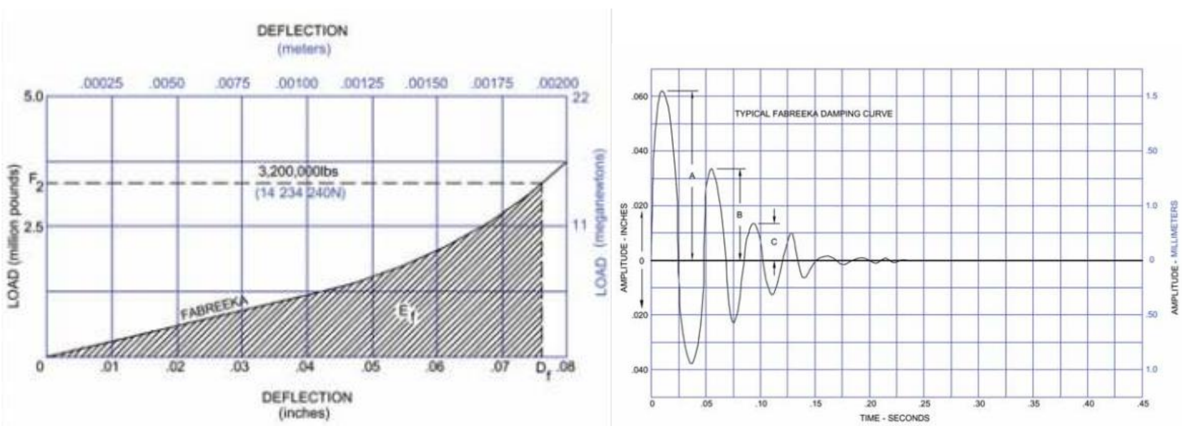
7.1
$$x(t) = C + \exp(-\varsigma\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \varsigma\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$$
$$= \exp(-t)\left\{10\cos\sqrt{99}t + \frac{10}{\sqrt{99}}\sin\sqrt{99}t\right\}$$

(x is given in mm)

7.2
$$x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n(x_0 - C)\right]t\right\}\exp(-\omega_n t)$$
$$= [10 + 100t]\exp(-10t)$$

(x is given in mm)

[4 POINTS – 2 EACH]



8. The figures show the results of (a) a static test and (b) a dynamic test on a shock absorbing pad (details here). In the dynamic test, a mass (with unknown value) was placed on the pad and struck to start the mass vibrating on the pad, and the displacement of the mass was measured.

8.1 Use the results of the static test to estimate the stiffness of the pad (you can assume that the deflection is less than 1mm).

The stiffness is the slope of the curve – approximately 5.6MN/0.001mm = 5600 MN/m

[1 POINT]

8.2 Use the dynamic test to determine the log decrement, and hence determine the natural frequency ω_n and damping factor ζ

We can get the log decrement off the graph – using the first and third peaks (doesn't matter whether you use inches or mm for the displacement since the log decrement is a ratio)

$$\delta \approx \frac{1}{2}\log\left(\frac{0.061}{0.013}\right) \approx 0.77$$

The period can be estimated by noting that 2.5 cycles occur in about 0.08 sec, which gives *T=0.032*.

The formulas for damping factor and natural frequency then give

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.122 \qquad \omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T} = 198\,\text{rad/s}$$

[2 POINTS]

8.3 Hence, calculate the value of the mass used in the test, and determine a value for the dashpot coefficient *c* that would model the energy dissipation.

The standard formulas for natural frequency and damping factor then give

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow m = k / \omega_n^2 = \frac{5600 \times 10^6}{198^2} = 143 \times 10^3 kg$$

$$\zeta = \frac{c}{2\sqrt{km}} \Rightarrow c = 2\zeta\sqrt{km} = 5.7 MNs / m$$

[2 POINTS]

8.4 What value of mass on the pad would lead to the system being critically damped?

For critical damping $\zeta = 1 \Rightarrow m = \frac{c^2}{4k} = 1.4 \times 10^3 kg$

[1 POINT]