School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 4th

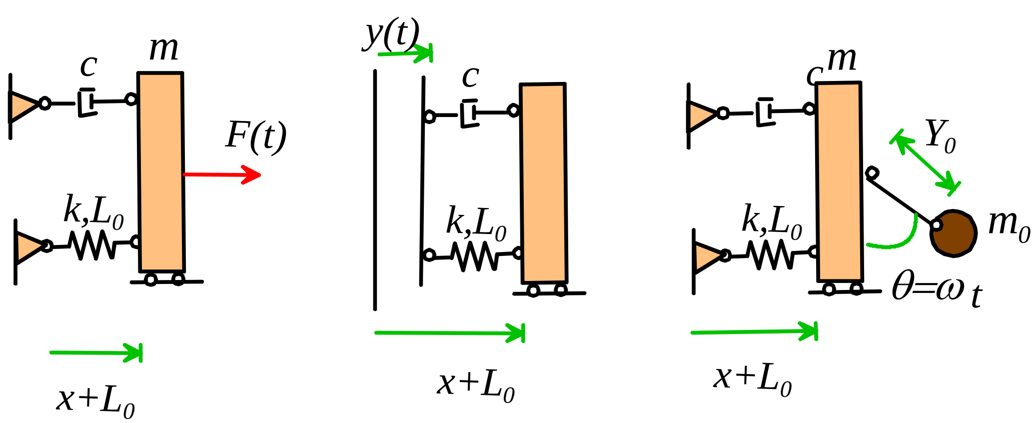
1. Solve the differential equation

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} + 16y = 16\sin 2t \qquad y = 9 \quad \frac{dy}{dt} = 0 \quad t = 0$$

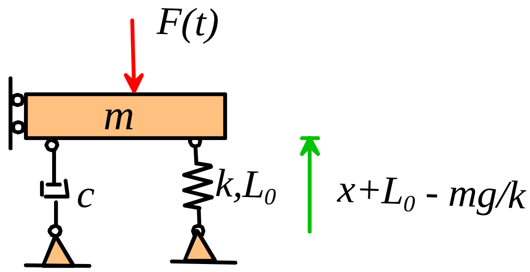
Identify the transient part of the solution and the steady-state part. What is the amplitude of the steady-state solution? What is the phase of the steady-state solution?

2. Determine the steady-state amplitude of vibration for the spring-mass systems shown in the figure (you don't need to derive the equations of motion – these are standard textbook systems and you can just use the standard formulas). In each case the mass *m=10*kg, the stiffness *k=1000N/m c=20Ns/m*.

The force is $F(t) = 20\sin 20t$; the base excitation is $y(t) = \sin 20t$, the length of the rotor is 0.25m; the eccentric mass $m_0 = 4kg$ and the angular velocity of the rotor is $\omega = 20$ rad/s.



3. The specifications for a force plate designed for jump performance analysis can be found at this website. A force-plate is essentially a spring-mass system – the force acting on the plate is determined by measuring the change in length of the spring x (the spring actually a piezoelectric transducer). The force is then estimated by multiplying the length change by the spring stiffness kx. This works perfectly for a static force, but if the force applied to the plate varies with time, then $kx(t) \neq F(t)$. The goal of this problem is to work out how to design the system to minimize the error in measuring a dynamic force, and to calculate the maximum frequency of a fluctuating force that can be measured accurately. Suppose that:



(i) The natural frequency of the force-plate is 150 Hz

(ii) The system is critically damped (why is this a good choice for measuring a static force?)

(iii) The force applied to the plate is $F(t) = F_0 \sin \omega t$.

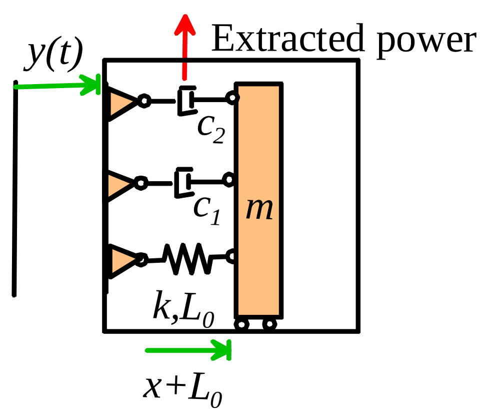
(iv) The amplitude of the force reading on the force-plate is $F_0^* = kX_0$ where X_0 is the vibration amplitude.

3.1 Derive the equation of motion for x. Note that the length of the spring is $x + L_0 - mg/k$ - the reason for this definition should become clear as you do the calculation. Arrange the equation into the 'standard form' and substitute the known values of ζ, ω_n. Leave the spring stiffness on the right hand side of the equation as an unknown.

3.2 Find a formula for F_0^* in terms of ω. Hence, determine the frequency range for which $\left| F_0^* / F_0 - 1 \right| < 0.05$ (i.e. 5% error in the force reading)

3.3 This frequency range can be improved by reducing the damping (to see this, plot the graph in 3.1 for ζ a bit less than 1. You don't have to submit this graph). Find the damping coefficient ζ that will maximize the frequency range for which $\left| F_0^* / F_0 - 1 \right| < 0.05$ and determine the corresponding frequency range.

4. The figure shows a simplified idealization of a micro-scale energy harvesting device intended to scavenge energy from the motion of an insect (from Aktakka *et al* 2011). The damper c_1 represents 'parasitic damping' (e.g. from friction or air resistance); the damper c_2 represents (e.g.) the effects of a magnet moving through a coil, generating an electrical current that can do useful work. The goal of this problem is to derive the equations that are used to design an optimized energy harvesting device.



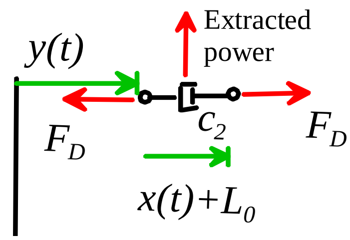
4.1 Use Newton's law of motion to show that the equation of motion for the length of the spring/dampers x can be arranged into the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2(\zeta_1+\zeta_2)}{\omega_n}\frac{dx}{dt}+x=-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

Give formulas for $\omega_n, \zeta_1, \zeta_2, K$.

4.2 Assume that the device vibrates harmonically, so that $y(t)=Y_0 \sin \omega t$. Write down the formula for the vibration amplitude X_0 in terms of $\omega_n, \zeta_1, \zeta_2, K$ and Y_0

4.3 The power extracted from the electromagnetic coil is equal to the rate of work done by the forces acting on the damper c_2. Show that the power can be expressed as



$$P(t)=F_D\frac{dx}{dt}=c_2\left(\frac{dx}{dt}\right)^2$$

4.4 Hence, show that instantaneous power generated is $P(t)=c_2 X_0^2 \omega^2 \cos^2(\omega t+\phi)$

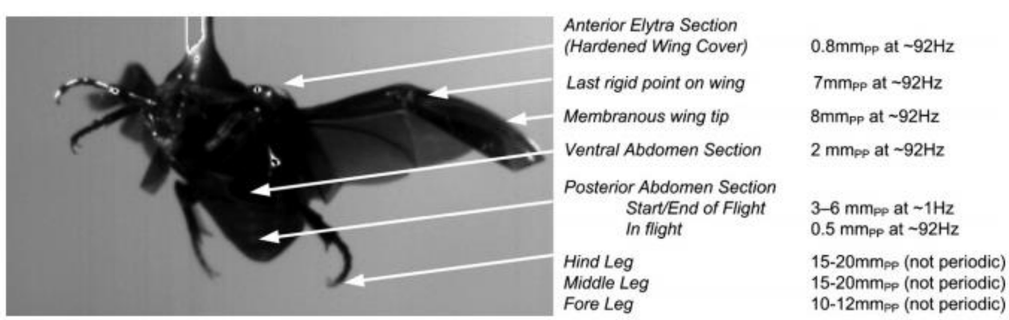
4.5 The average power generated by the device can be computed by averaging the instantaneous power over one period of vibration, i.e.

$$\bar{P}=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} P(t)dt$$

Use this expression and the results of 4.4, 4.2 to show that

$$\bar{P}=mY_0^2\omega^3\frac{\zeta_2\left(\frac{\omega}{\omega_n}\right)^3}{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2(\zeta_1+\zeta_2)\frac{\omega}{\omega_n}\right)^2}$$

4.6 For a fixed excitation frequency ω (e.g. the frequency of the insect's wing beat), the power is maximized (for small damping) by tuning the natural frequency of the spring-mass system to the same frequency ($\omega=\omega_n$) . With this choice, show that the power is maximized when $\zeta_2=\zeta_1$, and determine an expression for the optimal power, in terms of m, Y_0, ζ_1, ω .



4.7 The figure shows measured frequencies and amplitudes of vibration from a Green June Beetle. Estimate the power that could be harvested from the bug assuming a vibration amplitude of 1mm, frequency of 92Hz, mass of 0.13gram (10% of the beetle's mass), and damping $\zeta=0.1$