EN40: Dynamics and Vibrations

Homework 2: Kinematics and Dynamics of Particles
Due Friday Feb 6, 2015
45 POINTS MAX

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1. The specifications for the recent Orion launch can be found here.
Among other data, they state that
> (a) Orion reaches the speed of sound after 1 min 25 sec
> (b) The liftoff weight is 750000kg
> (c) The booster rockets generate 9MN of thrust at sea level

The speed of sound is approximately 300 m/s above an altitude of 10km.

1.1 Assuming constant acceleration straight-line motion (in a vertical direction), and neglecting changes in the vehicle mass, use the given trajectory data to calculate (a) the acceleration, and (b) the altitude of the vehicle at the instant that it reaches the speed of sound. Compare the average acceleration with the value calculated from the given mass and thrust at sea level.

The straight line motion formulas give the acceleration as
$a = \Delta v / \Delta t = 300 / 85 = 3.53 m / s^2$

The altitude follows as $h = \frac{1}{2} a \Delta t^2 = 12.8 km$

Newton gives the acceleration as $T - mg = ma \Rightarrow a = \frac{T}{m} - g = 2.19 ms^2$

[3 POINTS]

1.2 Suppose that the motor generates a constant thrust T, has initial mass m_0 and expends propellant mass at a constant rate $\dot{\mu}$. It can be shown that (see the course notes Sect 4.2.6) the acceleration of the rocket satisfies $(m_0 - \dot{\mu}t)a = T - (m_0 - \dot{\mu}t)g$. Find a formula for the speed of the rocket (you will need to do the integral)

Now

$$(m_0 - \dot{\mu}t)\frac{dv}{dt} = T - (m_0 - \dot{\mu}t)g$$

$$\Rightarrow \frac{dv}{dt} = \frac{T}{(m_0 - \dot{\mu}t)} - g$$

$$\Rightarrow v = \int_0^t \left(\frac{T}{(m_0 - \dot{\mu}t)} - g \right) dt = -\frac{T}{\dot{\mu}} \log \frac{m_0 - \dot{\mu}t}{m_0} - gt$$

[3 POINTS]

1.3 Use the data provided and your solution to 1.2 to estimate $\dot{\mu}$ and hence obtain a revised estimate for the altitude at the instant that the vehicle reaches the speed of sound. You can use Mupad to solve the equation and do the integral to find the height.

The mupad to solve the equation is shown below

```
reset():
T := 9000000: m0 := 750000: g:=9.81:
v := -T/mu*log(1-mu*t/m0)-g*t
```
$$-9.81\,t - \frac{9000000\,\ln\left(1 - \frac{mu\,t}{750000}\right)}{mu}$$
```
solve(subs(v,t=85)=300,mu)
{1710.552882}
```

Thus $\dot{\mu} = 1710kg/s$.

[1 POINT]

The altitude must be found by integrating the velocity.
$$v = \frac{dh}{dt} = -\frac{T}{\mu}\log\frac{m_0 - \dot{\mu}t}{m_0} - gt \Rightarrow h = \int_t \left(-\frac{T}{\mu}\log\frac{m_0 - \dot{\mu}t}{m_0} - gt \right) dt$$

Here's the mupad

```
reset():
T := 9000000: m0 := 750000: g:=9.81:
v := -T/mu*log(1-mu*t/m0)-g*t
```
$$-9.81\,t - \frac{9000000\,\ln\left(1 - \frac{mu\,t}{750000}\right)}{mu}$$
```
solve(subs(v,t=85)=300,mu)
{1710.552882}
mu := %[1]:
int(v,t=0..85)
11020.55412
```

So $h=11$km

[2 POINTS]

2. The homework website provides two datafiles containing measurements of the acceleration and orientation of a small 'crazyflie' quadcopter as a function of time.

The file called calibration.txt measures acceleration while the copter is stationary (so it senses only gravity). The purpose of this file is to calibrate the accelerometers. The file called flight_data.txt contains accelerations measured during a short flight.
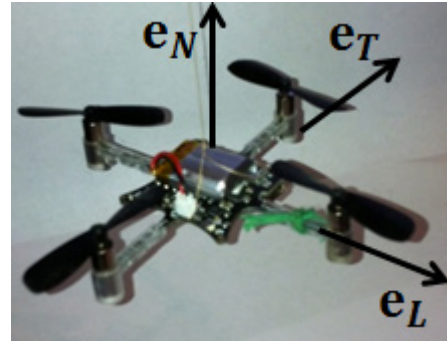
The seven columns in each file specify the time, the longitudinal, transverse and normal accelerations a_L, a_T, a_N , and the roll, pitch and yaw angles θ, ϕ, ψ recorded by the on-board inertial measurement unit.

You can convert the measured accelerations from components in $\mathbf{e}_L, \mathbf{e}_T \mathbf{e}_N$ to components in a fixed $\{\mathbf{i,j,k}\}$ basis (with $\mathbf{i}$ pointing North and $\mathbf{k}$ vertically upwards) with the transformation

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\phi & -\sin\psi\cos\theta - \sin\theta\sin\phi\cos\psi & \sin\theta\sin\psi - \cos\theta\sin\phi\cos\psi \\ \sin\psi\cos\phi & \cos\psi\cos\theta - \sin\theta\sin\phi\sin\psi & -\cos\theta\sin\phi\sin\psi - \sin\theta\cos\psi \\ \sin\phi & \sin\theta\cos\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} a_L \\ a_T \\ a_N \end{bmatrix}$$

Write a MATLAB code that will do the following:
  (i)   Read the datafile using the MATLAB csvread() command **(1/2 point)**
  (ii) Correct the time data so that it starts at zero, and convert it to seconds (the raw data is in milliseconds and it is using the internal clock on the quadcopter so it starts at some random time). Convert the accelerations to m/s^2 (they are recorded in units of g, the gravitational acceleration, 1 g is about 9.8 m/s^2) **(1/2 point)**
  (iii) Convert the accelerations to $\{\mathbf{i,j,k}\}$ components a_x, a_y, a_z (loop over the datapoints one at a time and store the accelerations in a new vector). Don't forget to convert the angles to radians. **(2 points)**
  (iv) Correct the vertical acceleration to remove gravity. **(1 point)**
  (v)   Integrate the accelerations using the MATLAB 'cumtrapz' function to find the velocity, and plot the three velocity components as a function of time **(2 points)**
  (vi) Integrate the velocities to calculate the position of the aircraft, and plot its path as a 3D trajectory. **(2 points)**

A matlab solution is provided in a separate file
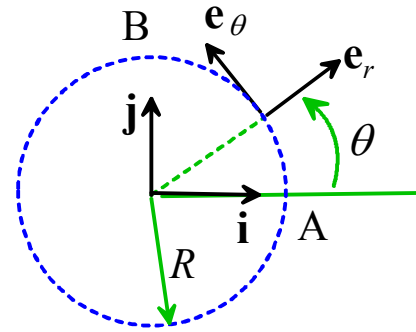
[8 POINTS]

3. A particle travels around a circular path with radius R. The particle starts at rest at A and travels with constant tangential acceleration a. Find formulas in terms of a,R and time t for:

(i) The angle θ

The tangential acceleration $a_t = R\dfrac{d^2\theta}{dt^2} \Rightarrow \dfrac{d^2\theta}{dt^2} = \dfrac{a}{R}$

We can use the constant acceleration formula to find θ

$$\theta = \frac{1}{2}\frac{a}{R}t^2$$

[1 POINT]

(ii) The time taken to reach point B (at $\theta = \pi/2$)

We can simply set $\theta = \pi/2$ in the preceding solution and solve for t $t = \sqrt{\pi R/a}$

[1 POINT]

(iii) The velocity vector at point B, in $\{\mathbf{i},\mathbf{j}\}$ components, and $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ components

The velocity is tangential to the path, which is always in the $\mathbf{e}_\theta$ direction, and at B is in the negative $\mathbf{i}$ direction. The magnitude of the velocity is $R\dfrac{d\theta}{dt} = at = \sqrt{\pi a R}$. Thus

$\mathbf{v} = \sqrt{\pi a R}\mathbf{e}_\theta = -\sqrt{\pi a R}\mathbf{i}$

[2 POINTS]

(iv) The acceleration vector at point B, in $\{\mathbf{i},\mathbf{j}\}$ components, and $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ components

We know the tangential acceleration is a, and the normal acceleration (pointing towards the center of the circle) is $\dfrac{v^2}{R} = \pi a$. The total acceleration is therefore $\mathbf{a} = a\mathbf{e}_\theta - \pi a\mathbf{e}_r = -a\mathbf{i} - \pi a\mathbf{j}$
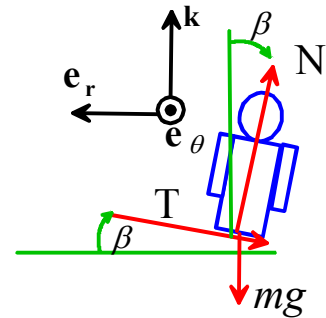
[2 POINTS]

4. The detailed specifications for a Japanese design for a 'tilting train' can be found here. Among other information, they specify that (i) a control system starts the train tilting before it enters a curve; and (ii) the maximum tilt is 5 degrees. The purpose of this problem is to estimate how much the tilting mechanism increases the train speed.
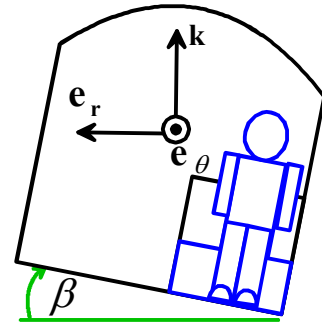
You can quantify the effects of tilting while traveling around a curve on passengers in various ways, but here we will consider the sideways forces acting on a passenger. The figure shows a tilted carriage, which travels around a curve with radius R at constant speed V. In the picture shown, the train is traveling towards you and turning towards the right.

4.1 Idealize the passenger as a particle, which is subjected to normal and tangential forces by the seat, and gravity. Draw a free body diagram showing these forces, and identify their directions with respect to the $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}\}$ directions by showing the angle β on the sketch.

The FBD is shown.

[2 POINTS]

4.2 Write down the acceleration vector of the passenger in terms of V and R

$$\mathbf{a} = -\frac{V^2}{R}\mathbf{e}_r$$

[1 POINT]

4.3 Hence, calculate a formula for the reaction forces acting on the passenger in terms of V, R and β and the passenger's mass m.

F=ma gives $\quad -(N\sin\beta + T\cos\beta)\mathbf{e}_r + (N\cos\beta - T\sin\beta - mg)\mathbf{k} = -m\frac{V^2}{R}\mathbf{e}_r$

The components of this expression give

$$N\cos\beta - T\sin\beta - mg = 0$$

$$N\sin\beta + T\cos\beta = \frac{mV^2}{R}$$

Solving for N,T gives

$$N = mg\cos\beta + \frac{mV^2}{R}\sin\beta$$

$$T = \frac{mV^2}{R}\cos\beta - mg\sin\beta$$

[2 POINTS]

4.4 The specifications suggest that the train tilts by about 1.5 degrees while still traveling along straight track. Find a formula for the tangential reaction force induced by this tilt in terms of *mg*. We will assume that this tangential force is the maximum allowable force before passengers start to sense the turn.

For straight track R is infinite, so $T = -mg\sin 1.5^o$

[1 POINT]

4.5 Hence, find a formula for the maximum allowable train speed for a non-tilting train, in terms of R and g (i.e. find a formula in terms of g and R for the speed that will cause the same tangential force magnitude as in part 4.4).

For a non-tilting train to generate the same force, we have $m\dfrac{V^2}{R} = mg\sin 1.5 \Rightarrow V = \sqrt{gR\sin 1.5}$

[1 POINT]

4.6 Find a formula for the maximum allowable train speed that will generate the same tangential force in a train with 5 degree tilt, in terms of R and g.

With a 5 degree tilt we have that

$$\frac{mV^2}{R}\cos 5^0 - mg\sin 5^0 = mg\sin 1.5^0 \Rightarrow V = \sqrt{gR(\sin 5^0 + \sin 1.5^0)/\cos 5^0}$$

[2 POINTS]
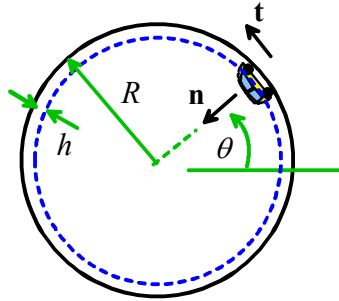
4.7 Hence, determine the percentage increase in speed that is achieved by tilting the train.

The percentage increase is

$$\sqrt{gR(\sin 5^0 + \sin 1.5^0)/\cos 5^0} / \sqrt{gR\sin 1.5^0} = \sqrt{((\sin 5^0/\sin 1.5^0)+1)/\cos 5^0}$$

$$= 208\%$$

[1 POINT]

5. The figure shows a picture of a car inside the 'Dunlop Death Loop.' The goal of this problem is to calculate a formula for the minimum speed at which the car can drive around the track.

5.1 Assuming the car drives at constant speed V and its center of mass is a height h above the road, write down a formula for the acceleration of the vehicle in terms of V, the radius R of the track and h. Express your answer as components in the normal-tangential basis shown in the figure.

$$\mathbf{a} = \frac{V^2}{R-h}\mathbf{n}$$

[1 POINT]

5.2 Draw a free body diagram showing the forces acting on the vehicle. Assume that the car has rear wheel drive, and that the front wheels (B) roll freely.
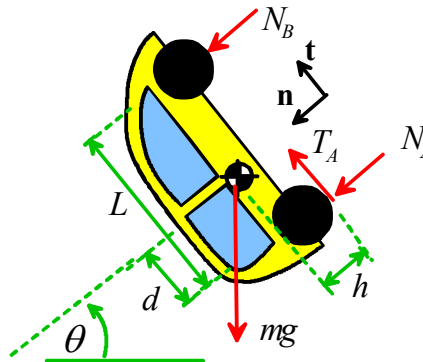
[2 POINTS]

5.3 Write down Newton's law of motion and the equation for rotational motion for the vehicle. Use the geometric variables d,L,h shown in the figure for 5.2. Express your answer in normal-tangential components.

$$\mathbf{F} = (N_A + N_B + mg\sin\theta)\mathbf{n} + (T - mg\cos\theta)\mathbf{t} = m\frac{V^2}{(R-h)}\mathbf{n}$$

$$Th + N_B(L-d) - N_A d = 0$$

[2 POINTS]

5.4 Hence calculate the reaction forces acting on both wheels of the vehicle

The vector equation shows that

$$T = mg\cos\theta$$

$$N_A + N_B = m\frac{V^2}{(R-h)} - mg\sin\theta$$

Substituting for T in the moment balance equation

$$N_A d - N_B(L-d) = mg\cos\theta h$$

Multiplying the second equation by (L-d) and adding the third shows that

$$N_A L = m\frac{V^2}{R-h}(L-d) + mg\left[-(L-d)\sin\theta + h\cos\theta\right]$$

Multiplying the second equation by d and subtracting the third gives

$$LN_B = m\frac{V^2}{(R-h)}d - mg\left(d\sin\theta + h\cos\theta\right)$$

Hence

$$N_A = mg\left[\frac{V^2}{g(R-h)}(1-\frac{d}{L}) - (1-\frac{d}{L})\sin\theta + \frac{h}{L}\cos\theta\right]$$

$$N_B = mg\left[\frac{V^2}{(R-h)g}\frac{d}{L} - \left(\frac{d}{L}\sin\theta + \frac{h}{L}\cos\theta\right)\right]$$

[3 POINTS]

5.5 Hence, calculate a formula for the minimum speed required to ensure that the front wheels remain in contact with the track, in terms of h, L, g, d and R.

To remain in contact with the road the reactions must both be positive for all values of θ. For the front wheel

$$\frac{V^2}{(R-h)}d \geq g\left(h\cos\theta + d\sin\theta\right)$$

One can maximize the right hand side – differentiate wrt θ and set to zero – this shows that

$$-h\sin\theta + d\cos\theta = 0 \Rightarrow \theta = \tan^{-1}\frac{d}{h} \Rightarrow \cos\theta = \frac{h}{\sqrt{d^2+h^2}} \quad \sin\theta = \frac{d}{\sqrt{d^2+h^2}}$$

$$\Rightarrow \left(h\cos\theta + d\sin\theta\right)_{max} = \sqrt{d^2+h^2}$$

,or (more quickly) note that we can write

$$g\left(h\cos\theta + d\sin\theta\right) = g\sqrt{h^2+d^2}\left(\frac{h}{\sqrt{h^2+d^2}}\cos\theta + \frac{d}{\sqrt{h^2+d^2}}\sin\theta\right) = g\sqrt{h^2+d^2}\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\right)$$

$$= g\sqrt{h^2+d^2}\cos(\theta - \alpha) \qquad \alpha = \tan^{-1}\frac{d}{h}$$

and the maximum value of $\cos(\theta - \alpha)$ is 1.

Therefore $N_B > 0 \Rightarrow V > \sqrt{(R-h)g\sqrt{1+\frac{h}{d^2}}^2}$

The same approach for N_A shows that

$$N_A > 0 \Rightarrow V > \sqrt{(R-h)g\sqrt{1+\frac{h^2}{(L-d)^2}}}$$

The critical velocity is given by the larger of the two.

[4 POINTS]