School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Work, Energy and Linear Momentum
Due Friday March 6th
45 POINTS MAX

1. The Rydberg potential is a simple model of atomic interactions. It specifies the potential energy of a bond between two atoms in the form

$$V = -D(1+a\rho)\exp(-a\rho) \qquad \rho = (r/R)-1$$

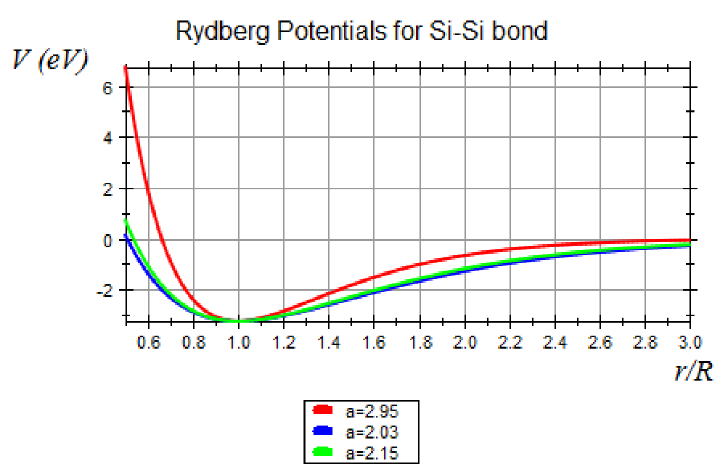
where r is the distance between atoms; R is the separation between atoms at minimum energy, and D and a are constants. The table below (from Molecular Physics, 106, 2008, p.753) gives values for D, a and R for several bonds.

Table 1. Continued.

Diatoms	D(eV)	R (Å)	a_1(Å)	a_2(Å^{-2})	a_3(Å^{-3})	a by Method 1	a by Method 2	a by Method 3
NN	9.905	1.0977	5.396	7.328	4.988	5.396	3.802738	4.006181
NO	6.614	1.1508	5.398	7.041	4.823	5.398	3.880258	4.0476
NP	6.443	1.4909	4.491	5.165	2.882	4.491	3.136731	3.322883
NS	4.875	1.4940	4.926	6.677	4.539	4.926	3.303252	3.546221
NSi	5.701	1.5718	3.732	2.975	1.460	3.732	2.824504	2.875488
OO	5.213	1.2075	6.080	11.477	11.003	6.08	3.743314	4.171992
OP	6.226	1.4759	4.275	4.399	2.717	4.275	3.078575	3.178364
OS	5.430	1.4811	4.748	6.504	5.228	4.748	3.087961	3.308623
OSi	8.337	1.5097	3.208	1.685	1.217	3.208	2.63083	2.517422
PP	5.081	1.8934	3.920	4.266	2.246	3.92	2.614269	2.821203
SiSi	3.242	2.2460	2.957	2.300	0.962	2.957	2.035645	2.151074
SS	4.414	1.8892	3.954	4.312	2.332	3.954	2.647662	2.846695
SSi	6.466	1.9293	2.773	1.462	0.647	2.773	2.183009	2.16445

1.1 Plot the variation of V with r/R for the Si-Si bond (compare the curves for the three values of a in the paper). Use eV for the units of energy, and $0.5<r/R<3$

The plot is shown below (see overleaf for the Mupad script)



```
reset()
V := -3.24*(1+a*(rr-1))*exp(-a*(rr-1)):
plot(plot::Function2d( subs(V,a=2.95),rr=0.5..3,
      Header = "Rydberg Potentials for Si-Si bond",
      GridVisible=TRUE,
      AxesTitles = ["r/R","V (eV)"],
      AxesTitleFont = ["Times",14,Italic],
      Axes = Boxed,
      LineWidth=0.5,
      LegendText = "a=2.95",
      Color = RGB::Red),
   plot::Function2d(subs(V,a=2.03),rr=0.5..3,
      LineWidth=0.5,
      LegendText = "a=2.03",
      Color = RGB::Blue),
   plot::Function2d(subs(V,a=2.15),rr=0.5..3,
      LineWidth=0.5,
      LegendText = "a=2.15",
      Color = RGB::Green),LegendVisible)
```

[3 POINTS]

1.2 Find a formula for the force of attraction between the atoms, in terms of D, a, R and ρ (if you use Mupad to do the derivative be careful not to use capital D as a variable – D means a derivative to mupad).

We find $F = -\dfrac{dV}{dr} = \dfrac{Da^2}{R}\rho\exp(-a\rho)$

[1 POINT]

1.3 Hence, calculate a formula for the force required to break the bond (i.e. the maximum value of F)

We need to maximize F - differentiating F with respect to r shows the maximum value occurs for $\rho = 1/a$, and so the maximum force is

$$F = -\frac{dV}{dr} = \frac{Da}{R}\exp(-1)$$

[2 POINTS]

1.4 Find a value for the strength of the Si-Si bond for the three possible choices of the value of a. You can use units of $eV/\text{Å}$ for the units of force – this is a funny unit but often used in atomistic calculations.

The three values of bond force are 1.57, 1.08 and 1.14 $eV/\text{Å}$. The bond strength is rather sensitive to the detailed shape of the function used to fit the potential.

[1 POINT]

These simple potentials are helpful as illustrative examples but they are not usually accurate enough for modern atomistic simulations. A number of organizations are currently working to organize and catalog more realistic potentials – the NIST interatomic potentials repository project is one example. Most potentials contain huge numbers of parameters and you need a computer to evaluate them.

2. The Zero SR ZF12.5 electric motorcycle has the following specifications:

- Acceleration from 0 to 60mph in 3.3 sec.
- Curb weight of 188kg.
- Nominal Battery capacity (total energy stored in the battery): 11kWh (kilo-watt-hours)
- Maximum engine power 50kW
- Range at 55mph (89 km/hr) 151 km
- Height 56"; width 77"

Assume that air resistance can be calculated from the formula

$$F_D = cv^2$$

with c a constant.

2.1 Assuming that air resistance is the dominant contribution to energy consumption during steady cruise, use the given range and battery capacity to calculate c. (find the total work done against air drag in terms of c, and set this equal to the battery capacity).

The work done against air resistance is the range * drag force. This is equal to the battery capacity. The units all have to be converted to SI – 1 kWhr = 3600 000 Joules, and 89 km/hr = 89000/3600 m/s.

This gives $c\left(89000/3600\right)^2 \times 151000 = 11000 \times 3600 \Rightarrow c = 0.43 Ns^2/m^2$

[2 POINTS]

2.2 Assume that the power produced by the electric motor is related to the vehicle's speed by $P = 4P_{\max}(1 - v/v_0)(v/v_0)$. Assuming that all the power developed by the motor is available to increase its kinetic energy during acceleration (neglect drag), show that the acceleration satisfies

$$a = \frac{4P_{\max}}{mv_0}(1 - v/v_0)$$

Hence calculate a formula for the speed as a function of time, and use the given power and time to reach 60mph to find a value for v_0. Assume a rider weight of 70 kg. Use Mupad to solve the equation.

We have that

$$P = \frac{d}{dt}(KE) = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dt}$$

$$\Rightarrow 4P_{\max}\left(1 - \frac{v}{v_0}\right)\frac{v}{v_0} = mva$$

This gives the answer stated.

[1 POINT]

We can integrate the acceleration to find velocity as a function of time

$$\int_0^v \frac{dv}{(1 - v/v_0)} = \frac{4P_{\max}}{mv_0}t \Rightarrow -v_0 \log\left(1 - v/v_0\right) = \frac{4P_{\max}}{mv_0}t$$

$$\Rightarrow v = v_0\left(1 - \exp(-\frac{4P_{\max}}{mv_0^2}t)\right)$$

[2 POINTS]

60 mph is 26.8 m/s. With the given numbers, v_0 must satisfy

$$26.8 = v_0 \left(1 - \exp\left[\frac{-4 \times 50000 \times 3.3}{(188+70)v_0^2}\right]\right) \Rightarrow v_0 = 77.9 m/s$$

Here's the Mupad

```
26.8=x*(1- exp(-4*50000*3.3/(188+70)/x^2))
```

$$26.8 = -x \left(e^{-\frac{2558.139535}{x^2}} - 1\right)$$

```
solve(%,x)
```

$\{77.87770556\}$

[2 POINTS]

2.3 Use the solution to 2.1 and 2.2 to estimate the maximum possible speed of the motorcycle.

At maximum speed the engine power is equal to the work done against drag. Thus

$$4P_{\max}\left(1 - \frac{v}{v_0}\right)\frac{v}{v_0} = cv^2 \times v$$

Solving the equation (Mupad) gives the max speed as 48 m/s (107 mph). For comparison, the website gives the max speed as 102 mph.

[1 POINT]

3. The Orion launch vehicle is placed in an initial orbit whose perigee (closest to earth's surface) and apogee (furthest) have altitudes 185 km and 888 km above the earth's surface, respectively. The velocity of the satellite at perigee is 7.989 km/s. Take the earth's radius as 6370 km

3.1 Calculate the speed of the vehicle at the apogee of this orbit

Energy conservation gives $T + V = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}$

We can calculate the constant GM from the condition that

$GM / R_e^2 = g \Rightarrow GM = 3.98 \times 10^{14} m^3/s^2 = 3.98 \times 10^5 km^3/s^2$

The velocity at apogee follows as $v_a = \sqrt{v_p^2 - \frac{2GM}{r_p} + \frac{2GM}{r_a}} = 7.21 km/s$

We can also use angular momentum conservation $v_a = v_p r_p / r_a$ which gives the same answer.

[1 POINT]

3.2 When the vehicle reaches its perigee a rocket burn changes the orbit to -23km x 5800km which has a 4.74 km/s velocity at apogee. Calculate the speed of the vehicle just after the rocket burn.

The same calculation gives

$$v = \sqrt{v_a^2 - \frac{2GM}{r_p} + \frac{2GM}{r}} = 8.86 km/s$$

[1 POINT]

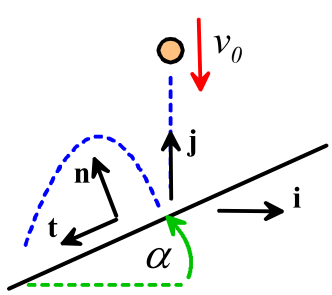
3.3 Assuming that the vehicle has a constant mass of 50 000 kg, calculate the impulse exerted by the rocket burn

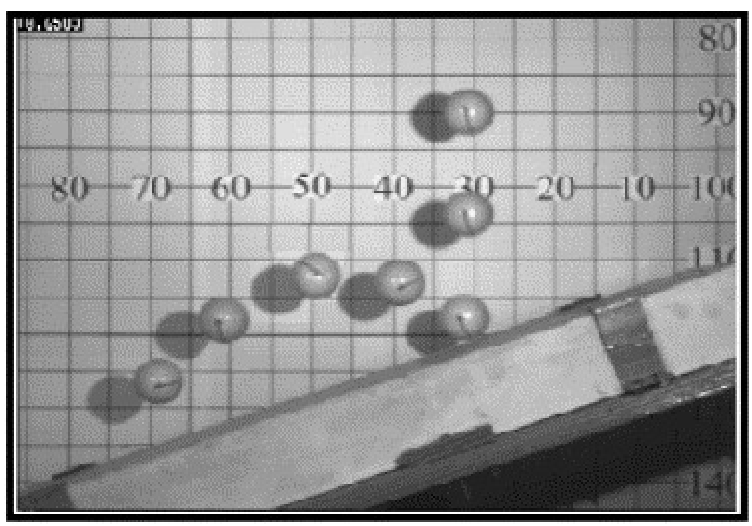
The impulse is $m\Delta v = 50000 \times (8.86 - 7.989) \times 1000 = 44 \times 10^6 Ns$

[1 POINT]

4. The figure shows a sequence of images recording the position of a spherical piece of rock as it bounces off an inclined plane. The image (from this paper) is part of an experiment to measure the restitution coefficient of collisions between rocks, which is of interest to geologists and civil engineers studying rock-slides). The scale is in cm, the rock has mass 204.33g and the time interval between frames is 0.04s.

[GRADERS – numbers in this problem will vary because it is hard to estimate the distances accurately. If the method used to do the calculations is explained clearly and is correct it should get full credit regardless of numbers. Also don't deduct points for errors propagating from one part of the problem to the next – any section that uses the correct method should get credit. But if the working is impossible to follow deduct points….





4.1 Estimate the angle of the slope α

Counting squares, the slope is approximately $\tan^{-1}(4/11) = 19.98^o$

[1 POINT]

4.2 Suppose the specimen is dropped from some (unknown) initial height *h* at time *t*=0, and impacts the slope at time t_1 Write down a formula for the height of the specimen *y* above the impact point as a function of time, for $t < t_1$.

During freefall the position of the specimen satisfies $y = h - \frac{1}{2}gt^2$

[1 POINT]

4.3 Assume that the first two images are taken at times $t = t_0$ and $t = t_0 + \Delta t$, where $\Delta t = 0.04\ s$. Use your solution to 3.2 and the figure to estimate t_0. Hence use the height of the first image above the ramp to find *h*

We have that

$$y_0 = h - \frac{1}{2}gt_0^2$$

$$y_1 = h - \frac{1}{2}g(t_0 + \Delta t)^2$$

Thus $y_0 - y_1 = \frac{1}{2}g(t_0 + \Delta t)^2 - \frac{1}{2}gt_0^2 = gt_0\Delta t + \frac{1}{2}g\Delta t^2 \Rightarrow t_0 = \frac{(y_0 - y_1)}{g\Delta t} - \Delta t / 2$

The figure suggests that $y_1 - y_0 \approx 13cm$ which gives $t_0 = \frac{0.13}{9.81 \times 0.04} - 0.03 = 0.301s$

The specimen is about 27.5cm above the slope in the first image which gives

$y_0 + \frac{1}{2}gt_0^2 = h = 0.72m$

[3 POINTS]

4.4 Use energy conservation to determine the speed of the specimen just before it impacts the ramp.

Energy conservation gives $\frac{1}{2}mv^2 = mgh$ so the velocity just before impact is $v = \sqrt{2gh} = 3.76$ m/s.

[1 POINT]

4.5 Assume that the 5th frame gives the maximum height of the rebound. Use its position to estimate the horizontal and vertical components of velocity after the impact.

There are several ways to do this problem, which give slightly different answers.

1. You can use the trajectory formulas - the change in height from the rebound point to the max height is about 5cm. The vertical velocity is zero at the top of the bounce so the constant acceleration formula gives $\Delta h = 5cm \Rightarrow gt^2 / 2 = 5cm \Rightarrow t = 0.1s$ The vertical velocity after impact is thus $v_y = gt$ = 1 m/s
2. You can assume that 3rd frame is at the time of impact, which gives the time between impact and the 5th frame as 0.08s. The constant acceleration formula would then give $v_y = gt = 0.8m / s$ (it is not clear that the time intervals between the frames is constant for the 4th and 5th frames however since the horizontal distances don't look equal between the 3rd and 4th frames and the 4th and 5th frames).

3. You can use energy conservation between the point just after impact and the peak of the trajectory which gives $m\left(v_x^2+v_y^2\right)/2 = mv_x^2/2 + mg\Delta h \Rightarrow v_y = \sqrt{2g\Delta h}$, which gives the same answer as (1).

   The horizontal velocity is constant, and the projectile travels about 17cm horizontally after the impact. The horizontal velocity is thus about 1.7 m/s (assuming a 0.1s time between the impact and the 5th frame; if you assume 0.08s you get a slightly larger number….)

   (The velocity vector after impact is thus $\mathbf{v} = -1.7\mathbf{i} + 1.0\mathbf{j}$

[2 POINTS]

4.6 Calculate the normal and tangential components of impulse exerted on the specimen during the impact.

The normal and tangential components of velocity before impact are
$v_n = 3.76\cos(19.98) = 3.53\,\mathrm{m/s} \qquad v_t = 3.76\sin(19.98) = 1.28\,\mathrm{m/s}$

After impact,
$v_n = 1\cos(19.98) + 1.7\sin(19.98) = 1.52\,\mathrm{m/s} \qquad v_t = 1.7\cos(19.98) - 1\sin(19.98) = 1.26\,\mathrm{m/s}$

(to see the conversion to **n-t** draw the velocity triangles resolving the **i** and **j** components of velocity into **n** and **t** components)

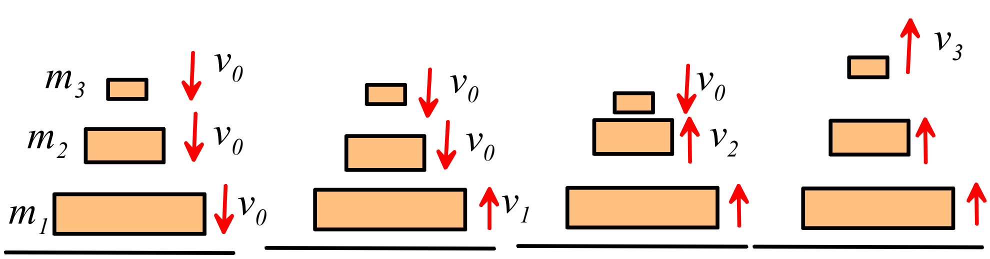
The normal and tangential impulses are thus
$\mathbf{I} = m\Delta\mathbf{v} = 0.204\times\left(1.52\mathbf{n} + 1.26\mathbf{t} - (-3.53\mathbf{n}) - 1.28\mathbf{t}\right) = \left(1.0\mathbf{n} - 0.004\mathbf{t}\right)Ns$

[2 POINTS]

3.4. Calculate the restitution coefficient for normal impact

The restitution coefficient is $-v_{n1}/v_{n0} = 1.52/3.53 = 0.43$

[1 POINT]



5. The figure shows an approximate model of an EN40 project 1 mass launcher with three masses. The bottom and top masses m_1, m_3 are fixed. The goal of this problem is to determine the value of m_2 that will maximize the launch velocity. Instead of considering the springs directly, we approximate the interaction between masses (and the lowest mass with the ground) as perfectly elastic (*e=1*) rigid body collisions. The collisions occur in sequence: first, mass 1 hits the ground; then collides with mass 2; and finally mass 2 collides with 3 to launch it. The stack is dropped with mass 1 a height *h* above the ground. The distance between the masses is very small compared to *h*.

4.1 Calculate the velocity of the masses just before the first mass hits the ground, in terms of *g* and *h*.

Free fall, the usual formula gives $v_0 = \sqrt{2gh}$

[1 POINT]

4.2 Write down the velocities of each mass after mass m1 has rebounded, but has not yet collided with mass *m2*.

The restitution formula with *e*=1 shows that impact with a rigid elastic surface reverses the velocity while keeping the magnitude fixed. Mass *m1* now therefore has velocity $v_1 = \sqrt{2gh}$ upwards, and the other two have velocity $v_0 = \sqrt{2gh}$ downwards.

[1 POINT]

4.3 Calculate the velocity of mass 2 after its collision with mass 1, but before its collision with mass 3, in terms of *g*, *h* and m_1, m_2

Let v_1^* denote the velocity of mass 1 (upwards) after impact, and take upwards velocities positive.

Impact is elastic, so the restitution formula gives $(v_2 - v_1^*) = (v_1 + v_0) = 2\sqrt{2gh}$, and

Momentum is conserved, so $m_1 v_1 - m_2 v_0 = (m_1 - m_2)\sqrt{2gh} = m_1 v_1^* + m_2 v_2$

Solving these equations for v_2 gives $v_2 = \dfrac{(3m_1 - m_2)}{m_1 + m_2}\sqrt{2gh}$

[2POINTS]

4.4 Calculate the velocity of mass 3 after its collision with mass 2, in terms of *g*, *h* and the masses.

Let v_2^* denote the velocity of mass 2 (upwards) after impact. The same approach as 4.3 gives

$(v_3 - v_2^*) = (v_2 + v_0) = \dfrac{4m_1 + m_2}{m_1 + m_2}\sqrt{2gh}$, and $m_2 v_2 - m_3 v_0 = \left(m_2 \dfrac{(3m_1 - m_2)}{m_1 + m_2} - m_3 \right)\sqrt{2gh} = m_2 v_2^* + m_3 v_3$

Solving these equations gives

$$v_3 = \sqrt{2gh}\,\frac{-m_2^2 + (7m_1 - m_3)m_2 - m_1 m_3}{(m_1 + m_2)(m_2 + m_3)}$$

[2 POINTS]

4.5 Hence, find a formula for the value of mass m_2 that will maximize the launch velocity, in terms of m_1, m_3. Compare the values with the predictions of the MATLAB code from your project.

To maximize, we need to set the derivative of this expression with respect to m_2 to zero, and solve for m_2. This gives

$$\sqrt{2gh}\frac{8m_1(m_1m_3-m_2^2)}{(m_1+m_2)^2(m_2+m_3)^2}=0 \Rightarrow m_2=\sqrt{m_1m_2}$$

The comparison will depend on the algorithm. My optimizer gives
$m_1=6lb,\quad m_3=0.11lb,\quad m_2=0.816lb$. The rigid body impact calculation gives $m_2=0.8124lb$

The algebra in this problem can also be done with mupad

[2 POINTS]

```
eq1 := v2-v1star=2*v0:
eq2 := (m1-m2)*v0=m1*v1star+m2*v2:
eq3 := v3-v2star = v2+v0:
eq4 := m2*v2-m3*v0=m2*v2star+m3*v3:
solve({eq1,eq2,eq3,eq4},{v3,v2,v1star,v2star},IgnoreSpecialCases)
```

$$\left\{\left[v2=\frac{3\,m1\,v0-m2\,v0}{m1+m2},\ v3=-\frac{m2^2\,v0-7\,m1\,m2\,v0+m1\,m3\,v0+m2\,m3\,v0}{\sigma_1},\ v1star=\frac{m1\,v0-3\,m2\,v0}{m1+m2},\ v2star=-\frac{m2^2\,v0-3\,m1\,m2\,v0+5\,m1\,m3\,v0+m2\,m3\,v0}{\sigma_1}\right]\right\}$$

where

$$\sigma_1=(m1+m2)\,(m2+m3)$$

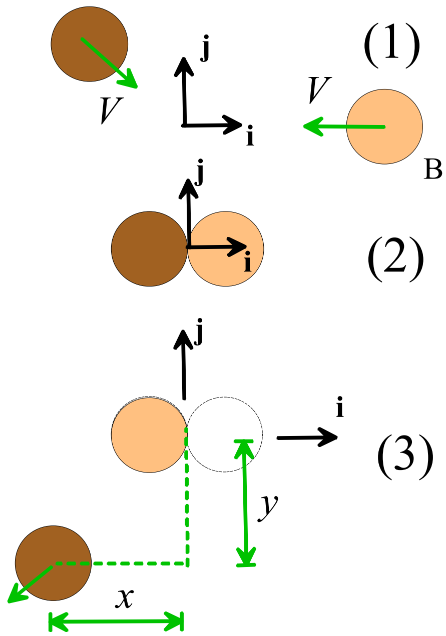
```
VV2 := subs(v3,%[1])
```

$$-\frac{m2^2\,v0-7\,m1\,m2\,v0+m1\,m3\,v0+m2\,m3\,v0}{(m1+m2)\,(m2+m3)}$$

```
simplify(diff(VV2,m2))
```

$$\frac{8\,m1\,v0\,(m1\,m3-m2^2)}{(m1+m2)^2\,(m2+m3)^2}$$

6. The figure shows a frictionless collision between two identical spheres with mass m and radius R. The restitution coefficient for the collision is e. The numbers (1), (2), (3) show the sequence of the pic tures - (1) is before impact; (2) is impact, and (3) is after impact. At the instant (1) particle A has velocity $\mathbf{v} = V(\mathbf{i}-\mathbf{j})/\sqrt{2}$ and particle B has velocity $\mathbf{v}_B = -V\mathbf{i}$



6.1 Write down the total linear momentum of the system before the impact, in {**i**,**j**} components.

The momentum is $\mathbf{p} = mV(\mathbf{i}-\mathbf{j})/\sqrt{2} - mV\mathbf{i}$

[1 POINT]

6.2 Write down the total linear momentum of the system after impact, in terms of V

Momentum is conserved, so $\mathbf{p} = mV(\mathbf{i}-\mathbf{j})/\sqrt{2} - mV\mathbf{i}$

[1 POINT]

6.3 Explain why sphere B must continue to move parallel to the **i** direction after impact

The contact is frictionless so no impulse is exerted in the **j** direction on either sphere during impact. The momentum (and hence velocity) of each must be constant in the **j** direction (this means zero for sphere B).

[1 POINT]

6.4 Write down **j** component of velocity of sphere A after impact.

The **j** velocity is not changed, so $-V\mathbf{j}/\sqrt{2}$

[1 POINT]

6.5 Use momentum conservation and the restitution formula parallel to the **i** direction to find the velocities of the two spheres after impact

We can treat the collision in the **i** direction as a 1-D collision. The restitution formula gives

$$\frac{v_{Ax}^1 - v_{Bx}^1}{v_{Bx}^0 - v_{Ax}^0} = e \Rightarrow v_{Ax}^1 - v_{Bx}^1 = -eV\left(\frac{1}{\sqrt{2}}+1\right)$$

Momentum conservation gives

$$v_{Ax}^1 + v_{Bx}^1 = V\left(\frac{1}{\sqrt{2}}-1\right)$$

Solving these equations (add/subtract the equations from one another)

$$v_{Bx}^{1} = \frac{V}{2}\left(\frac{1+e}{\sqrt{2}} + e - 1\right) \qquad v_{Ax}^{1} = \frac{V}{2}\left(\frac{1-e}{\sqrt{2}} - e - 1\right)$$

Thus $\mathbf{v}_B = \frac{V}{2}\left(\frac{1+e}{\sqrt{2}} + e - 1\right)\mathbf{i} \qquad \mathbf{v}_A = \frac{V}{2}\left(\frac{1-e}{\sqrt{2}} - e - 1\right)\mathbf{i} - \frac{V}{\sqrt{2}}\mathbf{j}$

[2 POINTS]

6.6 Assuming that $e < (\sqrt{2}-1)/(\sqrt{2}+1)$, find the distances *x,y* at the instant (3) (at this instant particle B is located at the point occupied by particle A at the instant of collision).

The time taken for B to reach the point indicated is $-2R / v_{Bx}$ so

$$t = \frac{4\sqrt{2}R}{V\left(\sqrt{2} - 1 - e(1+\sqrt{2})\right)}$$

The position of A at this time is

$$\mathbf{r}_A = -R\mathbf{i} - 2R\frac{\left(\sqrt{2} - 1 + e(1+\sqrt{2})\right)}{\left(\sqrt{2} - 1 - e(1+\sqrt{2})\right)}\mathbf{i} - \frac{4R}{\left(\sqrt{2} - 1 - e(1+\sqrt{2})\right)}\mathbf{j}$$

[2 POINTS]