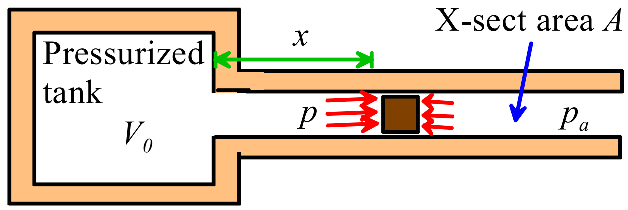
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Kinematics and Dynamics of Particles
Due Friday Feb 12, 2016
SOLUTIONS [53 POINTS TOTAL]

1. The figure illustrates an idealized model of a gas gun (used, e.g. in Professor Clifton's high-stain rate impact experiments at Brown)). The goal of this problem is to derive an approximate formula for the exit velocity of the projectile.



The projectile is propelled by a pressurized gas chamber. If the gas expands isothermally the pressure p in the tank (together with the portion of the barrel behind the projectile) is related to its volume V by Boyle's law $pV = \text{constant}$. The external air has pressure p_a. Friction can be neglected.

1.1. Show that the force acting on the projectile is related to its position by

$$F = \frac{Ap_0V_0}{V_0 + Ax} - p_a A$$

where p_0 is the initial pressure in the tank when the projectile is at position $x=0$.

- The pressure exerts a horizontal force pA
- Note that the volume of gas in the tank and barrel when the projectile is at position x is $V = V_0 + Ax$.
- Boyle's law gives $pV = p_0V_0 \Rightarrow p(V_0 + Ax) = p_0V_0$
- The resultant force is thus

$$F = A(p - p_a) = \frac{Ap_0V_0}{V_0 + Ax} - p_a A$$

[3 POINTS]

1.2 Hence, show that the velocity of the projectile is

$$v = \sqrt{\left[2p_0V_0\log\left(1 + Ax/V_0\right) - 2p_a Ax\right]/m}$$

(you will need to use $a = vdv/dx$ and separate variables to do the integral. Be careful when substituting the limits of integration)

- Newton's law gives $ma = mv\dfrac{dv}{dx} = F = \dfrac{Ap_0V_0}{V_0 + Ax}$
- Separate variables and integrate

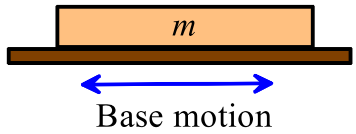
$$\int_0^v mvdv = \int_0^x \left(\frac{Ap_0V_0}{V_0 + Ax} - Ap_a\right)dx$$

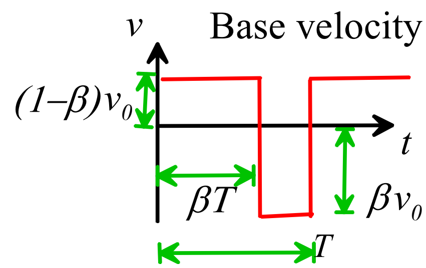
$$\Rightarrow \frac{1}{2}mv^2 = p_0V_0\left[\log(V_0 + Ax)\right]_0^x - Ap_a x = p_0V_0\left(\log(V_0 + Ax) - \log(V_0)\right) - Ap_a x$$

$$\Rightarrow \frac{1}{2}mv^2 = p_0V_0\log(1 + Ax/V_0) - Ap_a x \Rightarrow v = \sqrt{\left[2p_0V_0\log\left(1 + Ax/V_0\right) - 2p_a Ax\right]/m}$$

[3 POINTS]

2. Piezoelectric inertia drives are used as high-precision actuators and as a propulsion mechanism in micro-robots. They are based on the physics that makes the 'table cloth' trick demonstrated in class possible.





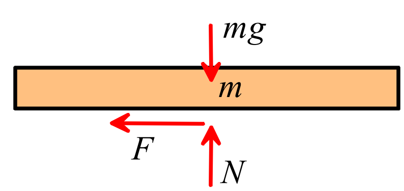
The figure illustrates a simple idealization that can be used to illustrate how an inertia drive works. It consists of a mass *m* on a vibrating base (the mass represents the platform and payload, while the base is the piezoelectric actuator). The contact between the mass and the base has friction coefficient μ . The base oscillates left and right with a velocity

$$v(t) = \begin{cases} v_0(1-\beta) & 0 < t < \beta T \\ -v_0\beta & \beta T < t < T \end{cases}$$

where v_0 and $0.5 < \beta < 1$ are constants, and T is the time for one cycle. The base moves faster while moving to the left than while it is moving to the right. The slip/stick behavior at the contact will cause the mass to gradually move to the right (you can demonstrate this by putting something on a piece of paper and pulling the paper back-and-forth). The goal of this problem is to derive an expression for the average velocity of the mass in terms of v_0, β . To keep things simple, assume steady-state conditions, so the platform moves at the same speed at time $t=0$ and time T. The speed will vary between $0<t<T$, and our goal is to find this speed using Newton's laws of motion and straight-line motion formulas.

2.1 Start by considering the motion of the platform during the time interval $\beta T < t < T$. During this phase of the cycle the mass is moving to the right, and the base is moving left, so slip must occur at the contact between them. Draw a free body diagram showing the forces acting on the mass during this phase of the motion, and use Newton's law to write down a formula for its acceleration, in terms of μ and g.

- **F**=m**a** gives $-F\mathbf{i} + (N - mg)\mathbf{j} = ma_x\mathbf{i}$
- Hence $N = mg \qquad a_x = -F/m$
- The friction law gives $F = \mu N$ so $a_x = -\mu g$



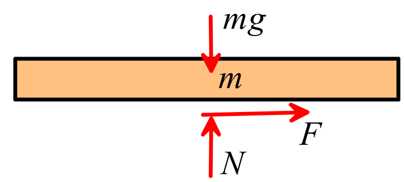
[2 POINTS]

2.2 Assume that that at time $t=\beta T$ the mass has the same velocity as the base, i.e. $v = v_0(1-\beta)$. Hence, use the result of (2.1) to calculate its speed at time *T*.

- Use the straight-line motion formula $v = v_0(1-\beta) - \mu g(t - \beta T)$.
- At time T $v = (1-\beta)(v_0 - \mu g T)$

2.3 Next, consider the motion of the platform during the time interval $0 < t < \beta T$. During this phase of the motion, both the mass and the base move to the right. At time $t=0$ the mass is moving more slowly than the base, so slip must occur at the contact between them (slip will stop when the mass reaches the same speed as the base). Draw a free body diagram showing the forces acting on the platform during the period of slip, and use Newton's law to write down a formula for its acceleration.

- **F**=m**a** gives $F\mathbf{i} + (N - mg)\mathbf{j} = ma_x\mathbf{i}$
- Hence $N = mg \qquad a_x = F / m$
- The friction law gives $F = \mu N$ so $a_x = \mu g$



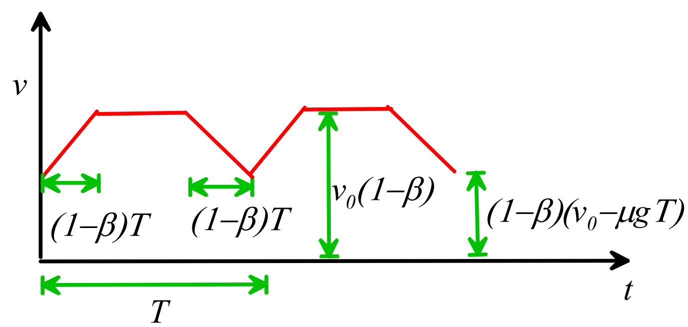
[2 POINTS]

2.4. Using the results of 2.2 and 2.3, calculate how long it takes the mass to reach the same speed as the base (recall that we assume that during steady-state slip-stick the speed of the mass at time *t=0* and *t=T* are equal, so you can find the speed at time *t=0* using the solution to part 2.2).

- Use the straight-line motion formula during slip $v = (1-\beta)(v_0 - \mu gT) + \mu gt$.
- When slip stops $v = v_0(1-\beta)$, substitute this into the previous result and solve $t = (1-\beta)T$

[2 POINTS]

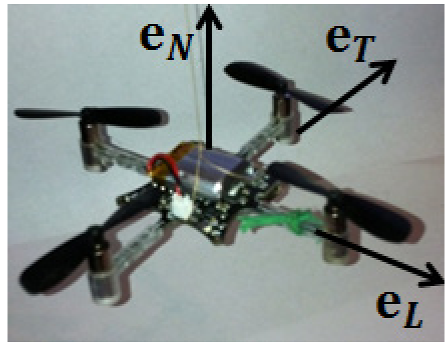
2.5 Hence, sketch a graph showing the speed of the platform as a function of time. Use the graph to deduce the distance traveled by the mass during one cycle, and hence find a formula for its average speed.



- The displacement is the area under the velocity graph:
  $\Delta x = (1-\beta)T[(1-\beta)v_0 + (1-\beta)(v_0 - \mu gT] + (1-\beta)v_0[T - 2(1-\beta)T]$
  $= (1-\beta)T(v_0 - (1-\beta)\mu gT)$
- The average speed follows as $\Delta x / T = (1-\beta)(v_0 - (1-\beta)\mu gT)$

[4 POINTS]

3. The homework website provides a datafile that contains measurements of the position and acceleration of a quadcopter during flight.

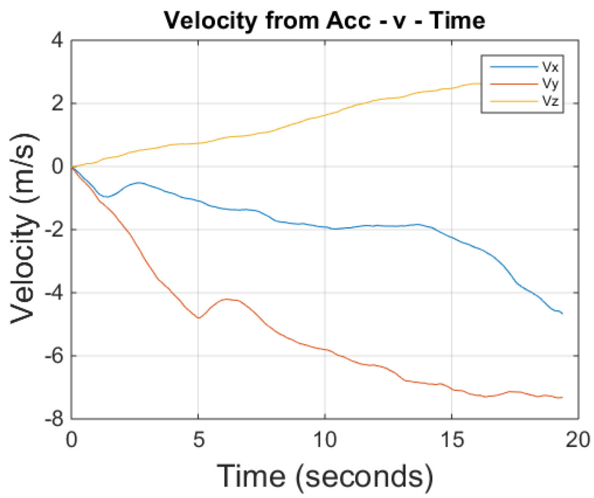


The seven columns in the file specify the time, three components of position *(x,y,z)* (measured by the Kinect sensor) and the accelerations (a_x, a_y, a_z) determined from the accelerometers and gyros on board the quadcopter. The coordinate system is such that x is the perpendicular distance of the quadcopter from the Kinect sensor, z is vertically upwards, and y is lateral. The data is in SI units (m and s).

Write a MATLAB code that will do the following:

(i) Read the datafile using the MATLAB data=csvread('*filename*') command. Note that the variable data will be a matrix: the first column of the matrix will be time values, the second column the x component of position, and so on.

(ii) Estimate the velocity by integrating the accelerations using the MATLAB 'cumtrapz' function, and plot the three velocity components as a function of time. Note that the accelerometers sense gravity, in addition to acceleration. You will need to subtract the gravitational acceleration (9.8m/s^2) from a_z .



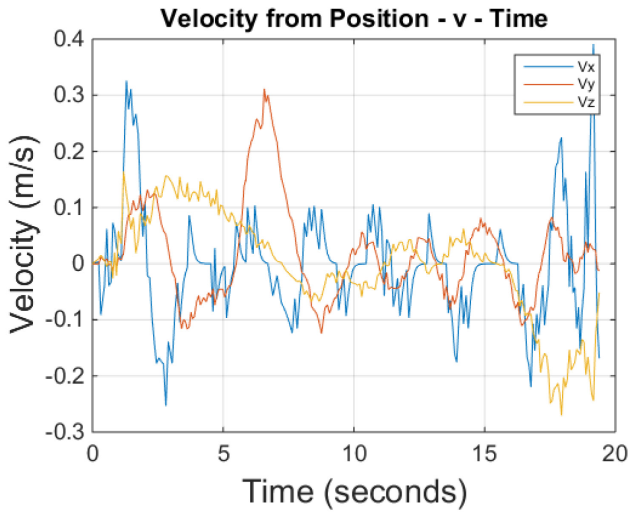
(iii) Estimate the velocity by differentiating the position vector, and plot this estimate as a function of time. You can do this by calculating the change in position between two successive readings, and dividing by the time difference between them, e.g. if $x(i)$ denotes the ith value of x, then

$$v_x(i) = \big(x(i) - x(i-1)\big) / (t(i) - t(i-1))$$

You can construct the vector v_x using a loop in MATLAB, e.g. to find the x component of velocity

```
vx=zeros(length(time),1)
for i=2:length(time)
    vx(i) = (x(i)-x(i-1))/(time(i)-time(i-1));
end
```

Here x is a vector containing the x component of position.



(iv) Why do the two estimates of velocity differ? Which is likely to be more accurate?

- The velocity obtained from the accelerometers is clearly wrong – it shows that the quadcopter has a large velocity at the end of the test, which is inconsistent with the measured position. The data from the Kinect is noisy, but is consistent with what happened.

[1 POINT]

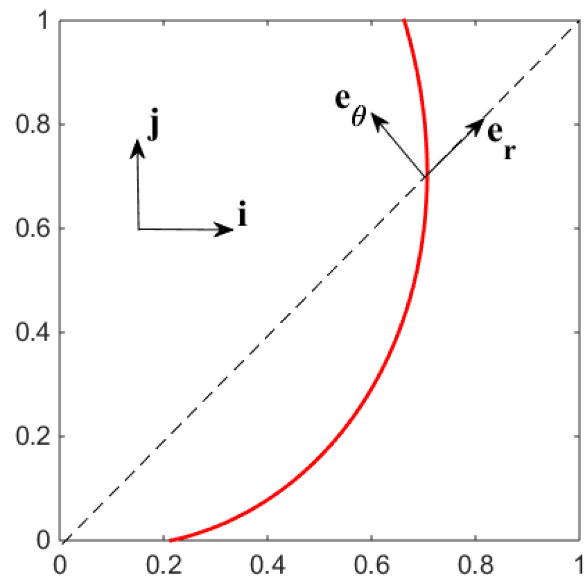
- There are various problems with using the accelerometers to calculate velocity – the biggest one is that the accelerations we are measuring are small compared to gravity. If there is a small error in the gyro readings giving the orientation of the quadcopter, or the calibration of the accelerometers, the data (after correcting for gravity) will show a constant acceleration even if the quadcopter is sitting on the table. No matter what we do, the integral of this error eventually predicts a very large constant velocity superimposed on top of whatever is actually happening. This is indeed what we see on the data.

[3 POINTS]

Please submit your MATLAB code on Canvas. Please include a separate printed copy of your graphs with your homework

[6 POINTS FOR MATLAB CODE PLUS CORRECT FIGURES]

4. The motion of a particle is described using polar coordinates as $r = 1 + t - (\pi / 4) \qquad \theta = t$ where t is time. The figure shows the resulting trajectory. At the instant when $\theta = \pi / 4$ calculate:



4.1 The position vector of the particle in the polar coordinate basis

- Note that $r = 1$ and recall position vector in polar coordinates is always $\mathbf{r} = r\mathbf{e}_r$ so $\mathbf{r} = \mathbf{e}_r$

[1 POINT]

4.2 The position vector of the particle in the **i,j** basis

- $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = (\mathbf{i} + \mathbf{j}) / \sqrt{2}$

[1 POINT]

4.3 The velocity vector of the particle in the polar coordinate basis

- Use the polar coordinate formula for velocity $\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = \mathbf{e}_r + \mathbf{e}_\theta$

[1 POINT]

4.4 The velocity vector of the particle in the **i,j** basis

- The brute force method is to take the time derivative of **r**, which gives
$$\mathbf{v} = \left(\frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta\right)\mathbf{i} + \left(\frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta\right)\mathbf{j} = \sqrt{2}\mathbf{j}$$
Alternatively you can note that when $\theta = \pi / 4 \quad \mathbf{e}_r = (\mathbf{i} + \mathbf{j}) / \sqrt{2} \quad \mathbf{e}_\theta = (-\mathbf{i} + \mathbf{j}) / \sqrt{2}$ and use 4.3.

[2 POINTS]

4.5 A unit vector parallel to the direction of motion of the particle in the **i,j** basis

- From 4.4: **t=j**

[1 POINT]

4.6 A unit vector in the perpendicular to the direction of the motion of the particle, in **i,j** basis

- In general for planar motion the operation $\pm\mathbf{k} \times \mathbf{v}$ always gives a vector perpendicular to **v**, but in this case it is obvious that $\pm\mathbf{i}$ is perpendicular to **j**.

[1 POINT]

4.7 The velocity vector in normal-tangential coordinates

- In normal-tangential coordinates the velocity vector is always $V\mathbf{t}$ where V is the speed (the magnitude of the velocity); so $\mathbf{v} = \sqrt{2}\mathbf{t}$

[1 POINT]

4.8 The acceleration vector in polar coordinates

- Use the formula $\mathbf{a} = \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right)\mathbf{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \right)\mathbf{e}_\theta = -\mathbf{e}_r + 2\mathbf{e}_\theta$

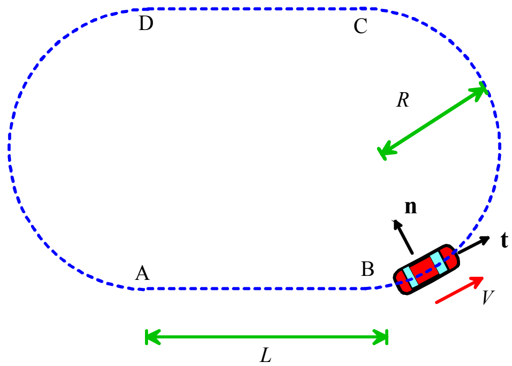
[2 POINTS]

4.9 The acceleration vector in **n,t** coordinates (this is quite tricky – if you can work this out from 4.8 and 4.3 without a lot of calculus you really understand vectors and the meaning of acceleration and velocity components!)

- To find the tangential component of acceleration, note we can find the tangent vector **t** in polar coordinates since **t** is parallel to **v**. Creating a unit vector from **v** in 4.3 gives $\mathbf{t} = (\mathbf{e}_r + \mathbf{e}_\theta)/\sqrt{2}$.
- Now the tangential component of **a** is simply $a_t = \mathbf{a}\cdot\mathbf{t} = (-\mathbf{e}_r + 2\mathbf{e}_\theta)\cdot(\mathbf{e}_r + \mathbf{e}_\theta)/\sqrt{2} = 1/\sqrt{2}$.
- Note that since $\mathbf{a} = a_t\mathbf{t} + a_n\mathbf{n} \Rightarrow a_n\mathbf{n} = \mathbf{a} - a_t\mathbf{t} = -\mathbf{e}_r + 2\mathbf{e}_\theta - (\mathbf{e}_r + \mathbf{e}_\theta)/2 = (-3\mathbf{e}_r + 3\mathbf{e}_\theta)/2$.
- Taking the magnitude of this vector gives $a_n = \sqrt{18}/2 = 3/\sqrt{2}$.
- Hence $\mathbf{a} = a_t\mathbf{t} + a_n\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{t} + \frac{3}{\sqrt{2}}\mathbf{n}$

[4 POINTS]

5. A race-car travels around the track shown in the figure. The magnitude of the acceleration of the vehicle cannot exceed μg (μ is the friction coefficient and g is the gravitational acceleration – if $|\mathbf{a}| > \mu g$ the car will skid). The goal of this problem is to find a formula for the shortest possible time for the vehicle to complete a lap around the course, in terms of μ, g, R, L.



5.1 Assume the vehicle travels at constant speed from B to C, and hence write down a formula for the acceleration vector in **n,t** coordinates in terms of speed and *R*. Hence find a formula for the shortest time to travel from B to C.

- $\mathbf{a} = \dfrac{V^2}{R}\mathbf{n}$
- Hence $V = \sqrt{\mu g R}$
- The distance traveled is πR so $t = \pi R / \sqrt{\mu g R} = \pi\sqrt{R/\mu g}$

[2 POINTS]

5.2 Assume that while traveling from C to D the car first speeds up with the maximum possible acceleration and then slows down again (with maximum deceleration). Using 5.1, find a formula for the time to travel from C to D.

- By symmetry, the car must reach its maximum speed midway between C and D.
- Let T denote the time to travel from C to the mid-point
- The straight-line motion formula for distance gives $L/2 = \sqrt{\mu g R}\,T + \dfrac{1}{2}\mu g T^2$
- Solve for $T = \dfrac{-\sqrt{\mu g R} + \sqrt{\mu g R + L\mu g}}{\mu g} = \sqrt{\dfrac{L+R}{\mu g}} - \sqrt{\dfrac{R}{\mu g}}$. The total time is twice this value.

Graders – there are many other ways to solve this problem….

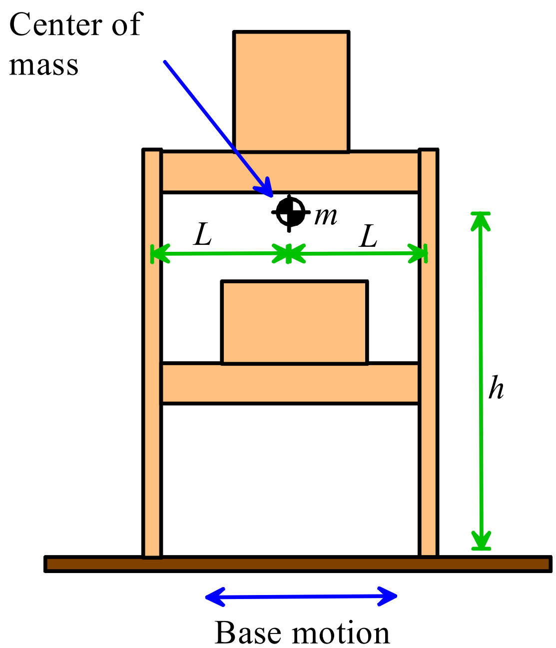
[2 POINTS]

5.3 Hence, calculate the total time for one lap of the course.

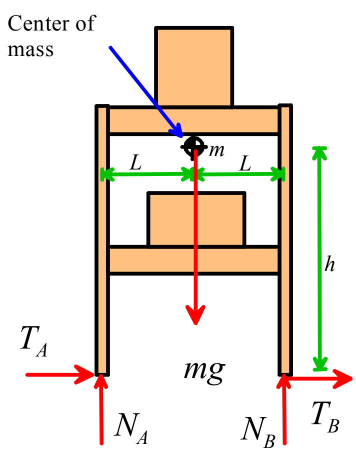
- Add 5.2 and 5.3 and double $t = 4\sqrt{\dfrac{L+R}{\mu g}} + 2(\pi - 2)\sqrt{\dfrac{R}{\mu g}}$

[1 POINT]

6. The figure shows a shelving rack during an earthquake. The unit has width $2L$ and the center of mass of the shelf and its contents is a height h above the ground. Assume that the ground vibrates with a horizontal motion $x(t) = X_0 \sin 2\pi t / T$ where X_0 is the amplitude and T is the time for one cycle. The goal of this problem is to find a formula for the amplitude at which the rack will tip over.



6.1 Draw a free body diagram showing the forces acting on the shelf and its contents together.



[3 POINTS]

6.2 Use Newton's law **F**=m**a** and the equation of rotational motion **M**=**0** to calculate formulas for the vertical reaction forces at the base. Assume there is enough friction to prevent slip at the contacts

- $(T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = -mX_0 \dfrac{4\pi^2}{T^2} \sin(2\pi t / T)\mathbf{i}$
- Take moments about COM $[(T_A + T_B)h + (N_B - N_A)/L]\mathbf{k} = \mathbf{0}$
- Solve the three equations to see that

$$(T_A + T_B) = -mX_0 \frac{4\pi^2}{T^2} \sin(2\pi t / T)$$

$$N_B = \frac{mg}{2} + m\frac{h}{2L} X_0 \frac{4\pi^2}{T^2} \sin(2\pi t / T)$$

$$N_A = \frac{mg}{2} - m\frac{h}{2L} X_0 \frac{4\pi^2}{T^2} \sin(2\pi t / T)$$

[3 POINTS]

6.3 Hence, find a formula for the critical value of X_0 that will cause the shelf to start to tip.

The shelf will start to tip if either of the reaction forces goes to zero. This gives

$$\frac{mg}{2} + m\frac{h}{2L}X_0\frac{4\pi^2}{T^2}\sin(2\pi t/T) = 0$$
$$\Rightarrow X_0 = \frac{gT^2}{4\pi^2}\frac{L}{h}$$

[2 POINTS]

(Designing earthquake resistant storage racks is a non-trivial process as this detailed report shows