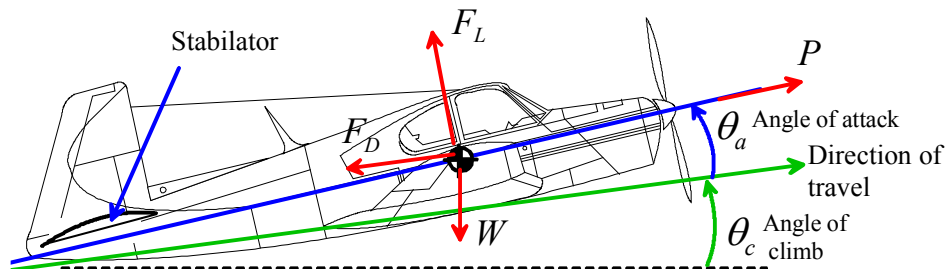## EN40: Dynamics and Vibrations

## Homework 3: Solving equations of motion for particles
### Due Friday Feb 19th

**School of Engineering**
  **Brown University**

**1.** In this problem you will use MATLAB to create a simple dynamical model to predict the flight of an aircraft (and will get a free flight lesson at the same time).



The aircraft has wing area $A_W$ and flies at (variable) speed $V$. It is subjected to four forces:

- Engine thrust $P$, acting parallel to the longitudinal axis of the aircraft
- Gravity $W$, acting vertically
- A lift force, magnitude $F_L = \frac{1}{2}\rho V^2 C_L A_W$, where $C_L$ is the lift coefficient and $\rho$ is air density acting perpendicular to the direction of motion of the aircraft (relative to the air, but we will assume no wind…)
- A drag force, magnitude $\frac{1}{2}\rho V^2 C_D A_W$, where $C_D$ is the drag coefficient, acting opposite to the direction of motion.

The lift and drag coefficients are related to the angle of attack $\theta_a$ (the angle between the direction of travel through the air and the wing chord) by

$$C_L = k_L \theta_a$$
$$C_D = k_{Dp} + k_{DI}\theta_a^2$$

Here $k_L, k_{DI}, k_{Dp}$ are three constants ( $k_{Dp}$ quantifies 'parasitic drag' and $k_{DI}$ quantifies 'induced drag')[1]

The pilot controls the climb rate and the aircraft speed by adjusting the angle of attack $\theta_a$ (which is controlled by the angle of the stabilator) and the engine thurst $P$. The goal of this problem is to write a MATLAB script that will predict the path of the aircraft.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$ denote the position and velocity vectors of the aircraft.

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[1] The formulas for lift don't account for a stall – they predict that the lift force increases indefinitely with AOA, but in reality, if the angle of reaches a critical value, the wing stalls. Lift decreases rapidly as angle of attack is increased beyond this critical angle.

1.1 Write down (a) a formula for the speed $V$ of the aircraft in terms of $v_x, v_y$ (b) formulas for unit vectors parallel and perpendicular to the direction of motion of the aircraft, in terms of $v_x, v_y, V$ , and (c) a formula for the angle of climb $\theta_c$ in terms of $v_x$ and $V$.

1.2 Using Newton's law to derive equations of motion, and re-arranging the results, show that the differential equations governing $(x, y, v_x, v_y)$ can be expressed as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ P\cos(\theta_a + \theta_c)/m - Bv_x - Dv_y \\ P\sin(\theta_a + \theta_c)/m - Bv_y + Dv_x) - g \end{bmatrix} \qquad \begin{aligned} B &= \frac{1}{2m}\rho A_W V(k_{Dp} + k_{DI}\theta_a^2) \\[2mm] D &= \frac{1}{2m}\rho A_W V k_L \theta_a \\[2mm] \theta_c &= \cos^{-1}(v_x/V) \end{aligned}$$

1.3 Write a MATLAB script that will calculate values of $(x, y, v_x, v_y)$ at discrete time intervals, using the 'ode45' differential equation solver. Be careful to use radians for the angles inside the trig functions (but note that, since $k_L, k_{DI}, k_{Dp}$ are specified using degrees below, the angle of attack must be in degrees in the formulas for B and D). Use the following parameters (representative of a small light aircraft):
- Wing area $A_W$ : 24 m$^2$
- Lift coefficient $k_L = 0.1$ (deg$^{-1}$)
- Parasitic drag coefficient $k_{Dp} = 0.03$
- Induced drag coefficient $k_{DI} = 0.001$ (deg$^{-2}$)
- Air density $\rho = 1.225 kg/m^3$
- Aircraft mass $m$=1134kg

Run the code with initial position (x=y=0), initial velocity $v_x = 61.34 m/s$ , an engine thrust of 1881.2N. and an angle of attack of $\theta_a = 2\deg$ . Plot a graph showing the vertical and horizontal velocity for a time interval of 120sec (2 minutes of flight). Note that this configuration corresponds to level cruise.

1.4 Next, use the code to investigate the effects of changing the control settings (these experiments will teach you to fly an airplane – but only in a straight line….). For each case plot a graph showing the variation of vertical and horizontal velocity for a time interval of 240sec. Assume that the aircraft is in steady cruise at time $t=0$, with position (x=y=0) and initial velocity $v_x = 61.34 m/s$

- At time $t=0$ the pilot increases the angle of attack to 5 degrees (raising the nose of the aircraft slightly), with power kept constant at 1881.2N. Notice that the change in pitch causes a small climb rate, but its effect is mostly to slow down the aircraft. Aircraft speed is controlled by the angle of attack – if you want to slow down, pull back on the control column; if you want to speed up, push forward. Once you have the angle of attack you need, you may need to make a small power adjustment to stop the climb or descent.
- At time $t=0$ the pilot increases the thrust to $P=3400$ with angle of attack kept constant at 2 degrees. Notice that the power change produces almost no change in speed, but causes the aircraft to climb. Use engine power to control climb or descent rate.
- At time $t=0$ the pilot increases the thrust to $P=3400$ and increases the angle of attack to 8 degrees. This setting causes oscillations in speed and altitude that never die out – they are called 'phugoid' oscillations. Federal aviation regulations have a lot to say about these and other aircraft flight modes…

**2.** In this problem we re-visit a more complicated version of the race-car problem from the preceding HW. A vehicle starts from rest at A and travels around a circular track. The car will skid if the magnitude of its acceleration exceeds $\mu g$. The goal of this problem is to find the shortest possible time for the vehicle to complete the first lap of the course.
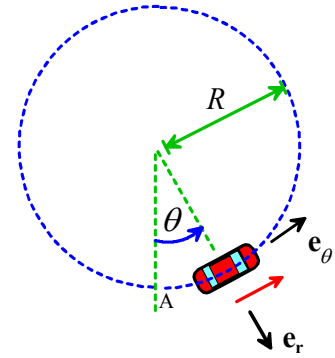
2.1 Use polar coordinates to describe the motion of the vehicle. By writing down a formula for the magnitude of its acceleration in polar coordinates, show that if the car travels with the maximum acceleration, the angle $\theta$ satisfies

$$\frac{d^2\theta}{dt^2} = \begin{cases} \sqrt{\left(\frac{\mu g}{R}\right)^2 - \left(\frac{d\theta}{dt}\right)^4} & \frac{d\theta}{dt} < \sqrt{\frac{\mu g}{R}} \\ 0 & \frac{d\theta}{dt} = \sqrt{\frac{\mu g}{R}} \end{cases}$$

2.2 Show that this equation can be re-arranged into the following pair of equations

$$\begin{bmatrix} \dfrac{d\theta}{dt} \\ \dfrac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} \omega \\ f(\omega) \end{bmatrix} \qquad f(\omega) = \begin{cases} \sqrt{\left(\dfrac{\mu g}{R}\right)^2 - \omega^4} & \dfrac{\mu g}{R} > \omega^2 \\ 0 & \dfrac{\mu g}{R} \le \omega^2 \end{cases}$$

2.3 Write a MATLAB code that uses the ODE solvers to calculate values of $\theta$ at a series of increasing values of time $t$. You will need to use a conditional statement (if… else…. end) to calculate $f(\omega)$, otherwise MATLAB ends up computing complex values for $d\omega/dt$ and gets confused. Add an 'event' function that will stop the calculation when $\theta = 2\pi$ (we will discuss 'event' functions in Tuesday's lecture, but if you are in a hurry they are discussed in Sect 13.8 of the matlab tutorial).

Use the code to plot
(i) The angular speed $\omega = d\theta/dt$ as a function of $\theta$, for $\mu g/R = 0.001, 0.01, 0.1, 1$ s$^{-2}$.

(ii) A graph of the time to complete the first lap as a function of $\sqrt{R/\mu g}$. Of course, MATLAB won't plot this automatically; you will have to find the values of the time when $\theta = 2\pi$ from the simulations in part (i), and store these values along with the values of $\sqrt{R/(\mu g)}$ in separate MATLAB vectors (you can do this by hand, or if you prefer you can have MATLAB do it for you with a loop), and then plot the graph.

2.4 **Optional (but extra credit):** The tests in the preceding problem show that (i) Regardless of the friction coefficient or curve radius, the vehicle always increases its speed between $0 < \theta < \pi/4$, and then travels at constant speed (the speed is proportional to $\sqrt{\mu g R}$, as you know from HW2); (ii) The time to complete the first lap is proportional to $\sqrt{R/(\mu g)}$. Prove these observations by calculating formulas for the speed of the car as a function of $\theta$, and for the time to complete the lap. You will need to use Mupad to do the relevant integrals.

**3.** The figure (from this publication) shows a proposed design for an 'energy harvester.' The mass vibrates vertically as the case is shaken, and drives an electromagnetic generator. The design differs from conventional passive energy harvesters by using actuators to apply a time-dependent stretch to the two springs. In this problem you will write a MATLAB code that shows . Make the following assumptions:

- The box vibrates vertically with a displacement $y(t) = A\sin\omega t$
- The springs have stiffness $k$ and un-stretched length $L_0$
- The actuators vary the horizontal distance $d$ according to the formula $d(t) = d_0(1 + \beta\sin\Omega t)$
- The electromagnetic generator exerts a force $-c\, dx/dt\,\mathbf{j}$ on the mass, where $c$ is a constant
- The power generated by the electromagnetic generator is $P = c(dx/dt)^2$
- **Gravity may be neglected**
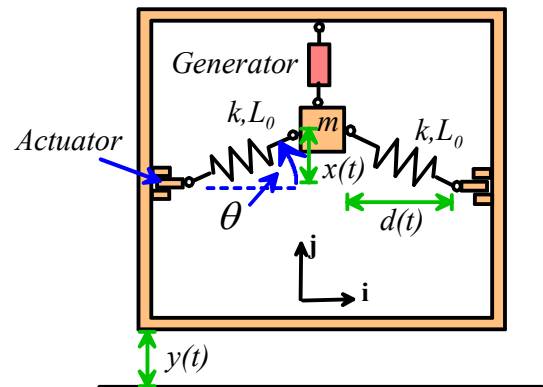
3.1 Write down an equation for $\sin\theta$ shown in the figure in terms of the deflection $x$ and the distance $d$ (this is just simple geometry)

3.2 Write down a formula for the force in the springs as a function of $x$, $d$, and other relevant variables

3.3 Write down a formula for the acceleration of the mass in terms of (time derivatives of) $x, y$

3.4 Hence show that the equations of motion for $x$ can be written in the following Matlab-friendly form

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -2(k/m)x - (c/m)v + \dfrac{2L_0 kx}{m\sqrt{x^2 + d^2}} + A\omega^2\sin\omega t \end{bmatrix} \qquad d = d_0(1 + \beta\sin\Omega t)$$

3.5 Write a MATLAB script that will calculate values of $x$ and $t$ at discrete time intervals using the ode45 function. Use the following values for parameters:

- Spring stiffness $k = 75N/m$
- Spring unstretched length $L_0 = 0.11m$
- Mean spring length $d_0 = 0.1m$
- Generator coefficient $c = 0.5Ns/m$
- Mass $m=0.1$kg
- Actuator frequency $\Omega = 24rad/s$
- Excitation frequency $\omega = 12rad/s$ (about 2 cycles per sec; a typical walking speed)
- Excitation amplitude $A=0.01m$.

Initial conditions $x = v = 0$

Use your code to plot (i) a graph of the displacement of the mass $x(t)$ as a function of time, with $(a)\beta = 0$ (this is a passive harvester, without the actuators); and (b) $(a)\beta = 0.1$ (show the results on the same plot,

using a time interval 0<*t*<10s); and (ii) the power generated by the device $P(t) = cv^2$, for the same two values of $\beta$ and the same time interval.

3.6 The average power generated by the device over a time interval $T$ can be calculated using the formula

$$\overline{P} = \frac{1}{T}\int_0^T P(t)dt$$

Use the 'trapz' function to calculate the average power harvested from the device over a time interval of 80sec, for the same two values of $\beta$ used in part 3.5

3.7 Of course, the solution in 3.6 over-estimates the total power that can be extracted, because it costs energy to operate the actuator. The power necessary to drive the actuator can be shown to be $P_a(t) = 2d_0\beta\Omega F_s \cos\theta\cos\Omega t$. Use this function to calculate the total useful power $\overline{P} - \overline{P}_a$ developed over 80 seconds of operation, for $\beta = 0$ and $\beta = 0.1$ (You will need to use a loop to calculate a vector of values of $P_a$ using the time and solution vectors returned by ode45, then use the trapz function to compute the time integral).