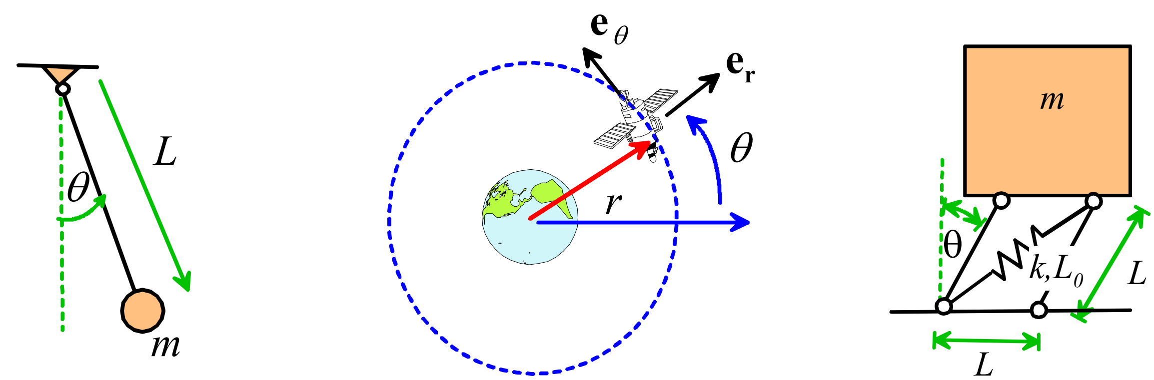
School of Engineering
Brown University

EN40: Dynamics and Vibrations
Homework 4: Work, Energy and Linear Momentum
Due Friday March 4th
[42 POINTS TOTAL]

1. For each of the systems shown in the figure, write down formulas for (i) the total potential energy and (ii) the total kinetic energy of the system, in terms of variables (and their time derivatives) provided in the figures (e.g. r , θ , etc).



(a) Pendulum (b) Orbiting satellite (do not assume r is constant) (c) Vibration isolator

Hint: the length of the spring for (c) is $2L\sin(\pi/4+\theta/2)$. Can you show this?

- For (a), the height of the mass (taking the pivot as datum) is $-L\cos\theta$ and its speed is $L\dfrac{d\theta}{dt}$.

  Therefore $V=-mgL\cos\theta \qquad T=\dfrac{1}{2}mL^2\left(\dfrac{d\theta}{dt}\right)^2$

[2 POINTS]

- For (b) the gravitational potential energy is $-\dfrac{GMm}{r}$. The velocity vector is $\dfrac{dr}{dt}\mathbf{e}_r+r\dfrac{d\theta}{dt}\mathbf{e}_\theta$.

  Therefore $V=-\dfrac{GMm}{r}\ T=\dfrac{1}{2}m|\mathbf{v}|^2=\dfrac{1}{2}m\left(\left(\dfrac{dr}{dt}\right)^2+r^2\left(\dfrac{d\theta}{dt}\right)^2\right)$

[2 POINTS]

- For (c) the gravitational potential energy is $mgh=mgL\cos\theta$; the potential energy of the spring is $\dfrac{1}{2}k\left\{2L\sin(\pi/4+\theta/2)-L_0\right\}^2$. The speed of the mass is $L\dfrac{d\theta}{dt}$ (this follows from the circular motion formula, but if you don't see this you can also write down the position vector of any point on the mass and differentiate it to find the velocity vector, and then compute the magnitude of the velocity for the KE formula). The KE is therefore $T=\dfrac{1}{2}mL^2\left(\dfrac{d\theta}{dt}\right)^2$ and the PE is the sum of gravity and the spring, i.e. $V=mgL\cos\theta+\dfrac{1}{2}k\left\{2L\sin(\pi/4+\theta/2)-L_0\right\}^2$

[2 POINTS]

2. The Airbus E-Fan 1.0 is an experimental electrically powered aircraft, with the following specifications

- Empty Weight: 500kg.
- Total engine power 60kW.
- Endurance 1 hr
- Maximum speed 220 km/hr
- Lift/drag ratio (optimal) 16
- Cruise speed 160 km/hr

Assume the pilot and her/his baggage adds 100kg.

2.1 Calculate the power consumption during level cruise (assume that the aircraft is flying at the optimal Lift/drag ratio)

- In level cruise the lift force must be equal to the weight; the drag force can then be calculated from the lift/drag ratio.
- The power-KE relation says total power (engine+lift+drag+gravity) is equal to the time derivative of kinetic energy – but KE is constant since speed is constant so its time derivative is zero.
- The drag power is $P_D = \mathbf{F}_D \cdot \mathbf{v} = -(F_L / 16)\mathrm{V}$ (the lift force and gravity do no work).
- Thus $P_{engine} = -P_D = 600 \times 9.81 \times \frac{1}{16} \times 160 \times \frac{1000}{3600} = 16350W$

[2 POINTS]

2.2 Calculate the total battery capacity required to achieve the stated endurance, plus a 30 min reserve (the total energy stored in the batteries). Give your answer in kilowatt-hours

- The energy stored is power*time, giving $E = Pt = 16350 \times 1.5hr = 24.5kWh$

(Airbus quotes a value of 29 kWhr – slightly larger, because the aircraft will not fly at the optimal lift-drag ratio throughout the flight)

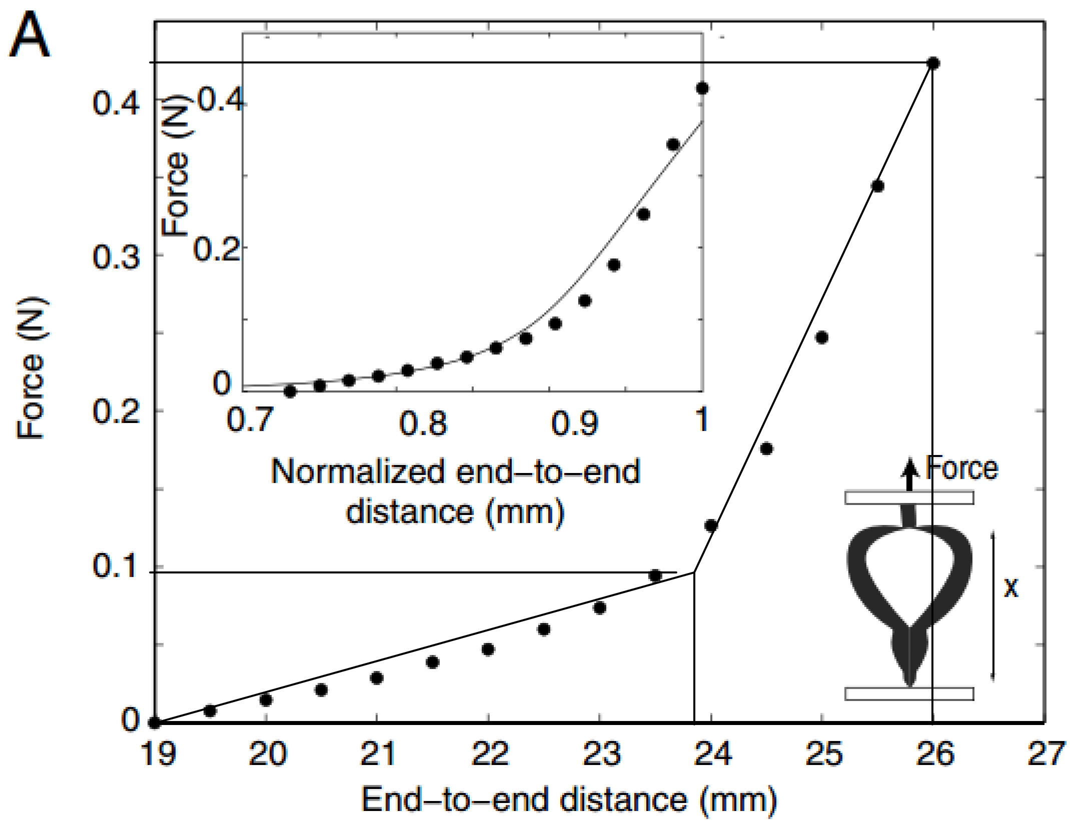
[1 POINT]

2.3 Calculate the maximum rate of climb of the aircraft at cruise speed (you can assume the horizontal speed is equal to the cruise speed. Give your answer in ft/min)

- For the maximum climb rate the plane must be flown with minimum drag, so the drag power will be the same as that for max endurance in part 2.1. Lift force is workless, and gravity power (from class) is $-mgv_y$ where v_y is the rate of climb. The power –KE relation gives
  $$P_e + P_D + P_{grav} + P_{Lift} = 0 \Rightarrow P_e = -P_D - mgv_y$$
- The max vertical speed is achieved by operating at max power, which gives
  $v_y = \frac{P_{\max}}{mg} - P_D = \frac{60000}{600 \times 9.81} - \frac{1}{16} \times \frac{160000}{3600} = 7.4m/s = 1457\,ft/\min$ (this is a reasonable number….).

(There are other ways to do the problem as well by doing a force balance on the aircraft flying at an angle…)

[2 POINTS]



3. ‘Jewelweed’ is an example of a plant that uses a biological spring mechanism to disperse its seeds (a process known as ‘dehiscence’) (you can see videos of the process here). The goal of this problem is to estimate the efficiency of the plants launch mechanism (the ratio of the kinetic energy of the seeds to the stored energy in the seed-pod). Use the following data.

- The measured force-v-extension curve for a seed-pod (from this reference) is shown in the figure
- The mass of a typical seed is 10.7 mg (Hayashi et al, 2009)
- A pod contains on average 3 seeds
- Seeds are launched with an average velocity of 1.25 m/s.

3.1 Estimate the energy stored in the seed-pod (i.e. the maximum amount of work that can be done by the force exerted by the seed pod on its seeds. You don’t need to do a very accurate calculation – approximating the force-displacement curve with two straight lines is sufficient).

- We can estimate the area under the force-displacement curve using the lines shown. This gives
$$W = \frac{1}{2}0.095 \times 4.9 \times 10^{-3} + \frac{1}{2}(0.095 + 0.42) \times 2.1 \times 10^{-3} = 0.7735 \times 10^{-3} J$$

[2 POINTS]

3.2 Calculate the total kinetic energy of the launched seeds

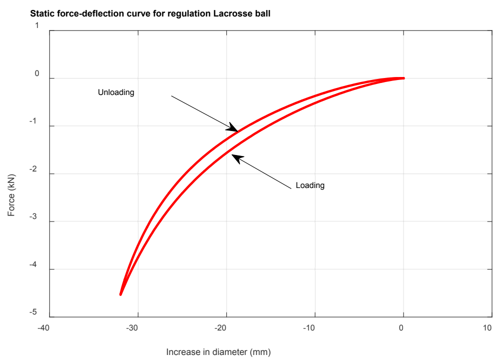
- The total kinetic energy is $T = \frac{1}{2}10.7 \times 10^{-6} \times 3 \times \left(1.25\right)^2 = 25 \times 10^{-6} J$

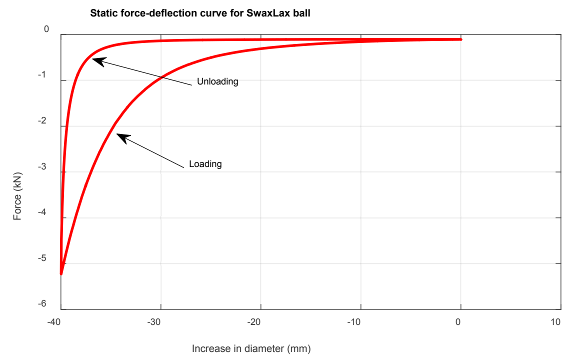
[1 POINT]

3.3 Hence, estimate the dynamic efficiency.

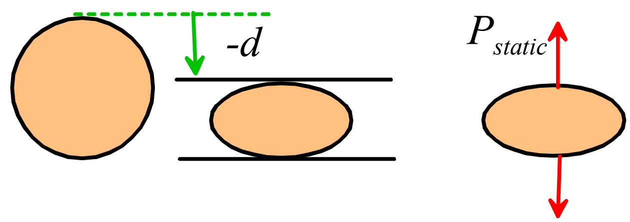
- The dynamic efficiency is $T / W = 3.23\%$

[1 POINT]

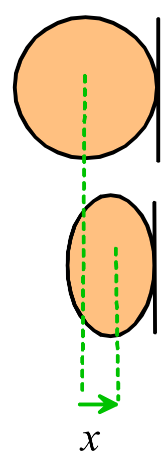




4. ‘Swaxlax’ manufactures lacrosse ‘training’ balls that are intended to reduce the risks of head injury. To obtain a rough estimate of the restitution coefficients for both regulation and Swaxlax lacrosse balls, their force-v-deflection curves were measured using a static compression test. The data from these tests is on the Homework page of the course website.



In the test, a ball is compressed between two rigid platens, as shown in the figure. The force $P_{static}(d)$ is measured. The test was conducted in an Instron in Prince Lab, which is set up to stretch, rather than compress, specimens by default, and so records both d and P_{static} as negative numbers. The figures show the force as a function of the change in diameter of the balls. Note that the $P_{static}(d)$ curve differs during loading and unloading, because the ball is permanently deformed. The goal of this problem is to estimate the coefficient of restitution for the two balls from this data.



For this purpose, we assume that the impact between a ball and a rigid surface can be idealized as a particle that is subjected to an external force, and estimate the impact force from the static force – deflection curve. Specifically, we assume that if the particle moves a distance x towards the wall after impact it is subjected to a force $P_{static}(x/2)$. The factor of 2 is needed because during the static test the ball is compressed between two rigid surfaces, whereas the collision occurs with only one surface. In the static test, both contacts experience the same force, and the change in diameter is the sum of the two deflections at the contacts.

4.1 Assume that just before contacting the wall the ball has a speed V_0. Using the work-energy relation, write down a formula relating the speed of the ball $v(x)$ to P_{static}.

- The work done by the contact force must equal the change in kinetic energy. Therefore
  $$\frac{m}{2}\left(v_0^2 - v(x)^2\right) = \int_0^x \left|P_{static}(-x/2)\right| dx$$
  (The absolute value and the sign in the argument aren't important as long as the idea is right…)

[2 POINTS]

4.2 Write a MATLAB script that will read the csv files of data provided, and plot the graphs of force-v-displacement shown in the figure. You can open the files with Microsoft excel to see the format of the data. To read them use

```
data = csvread('filename',2,0);
```

This will tell MATLAB to start reading data from the second row.

- The plots will be identical to those shown in the HW problem.
- A separate MATLAB code file has the solution.

[4 POINTS]

4.3 Use the data in the files provided to calculate the impact speed V_0 that would be expected to subject the balls to the maximum force in the file. The regulation Lax ball had a mass of 149 grams; the SwaxLax ball had a mass of 145 grams. You will need to find the row in the matrix that stores the csv data that has the largest force. You can do this using the matlab function

```
[value,index] = min(data(:,3))
```

Here the variable 'index' will return the row containing the minimum value of the 3rd column of the matrix 'data' – we use min because the data are negative. You can use the 'trapz' function evaluate the necessary integrals.

- Matlab (see code) gives the following results:
  - Regular ball impact velocity 17.3916 (m/s)
  - Swax ball impact velocity 15.0737 (m/s)

[4 POINTS]

4.4 Note that the formula in part 4.1 can also be used to calculate the rebound velocity, by using the force-displacement curve during the unloading portion of the graph. Calculate the expected rebound velocity for both regular and SwaxLax balls

- Matlab (see code) gives
- Regular ball rebound velocity 16.0993 (m/s)
- Swax ball rebound velocity 7.79084 (m/s)

[2 POINTS]

4.5 Hence, calculate a value for the restitution coefficients for both the regular and SwaxLax ball.

- Matlab (see code) gives
- Regular ball restitution coefficient 0.925695
- SwaxLax ball restitution coefficient 0.516851

[1 POINT]

4.6 Using the restitution coefficient calculated in 4.4, calculate the impulse exerted on a head by a collision with both a regulation and a SwaxLax ball. You will have to do this by (i) calculating the change in velocity of the ball/head during the collision, and then (ii) using the impulse/momentum formula for a single particle to calculate the impulse. Use the following data:

- Take the mass of a human head to be 4.5kg, treat the head as particle ('particle-head' is a useful insult),
- Assume that both balls travel at 40 m/s (about 90 mph)
- Assume the head is stationary before the collision
- Ignore the impulse exerted by the player's neck during the collision.

- This is a standard 1-D collision problem. Let particle A be the ball; B be the head; and let $v^{A0}, v^{A1}, v^{B0} = 0, v^{B1}$ denote the velocities before and after collision. Then momentum conservation and the restitution formula give

$$m_A v^{A0} = m_A v^{A1} + m_B v^{B1}$$
$$e v^{A0} = v^{B1} - v^{A1}$$

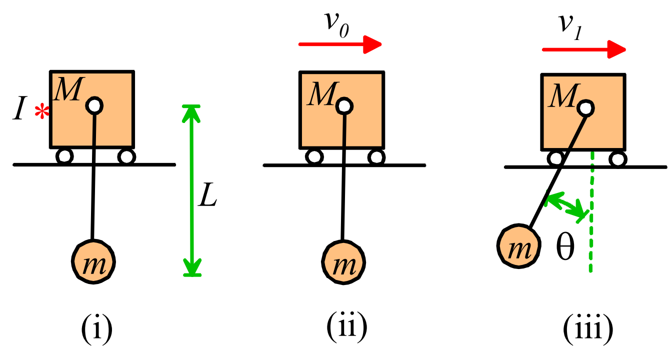
We can solve these for the velocity of the head (or ball, it doesn't matter) after impact:

$$(1+e)v^{A0} = \left(1 + \frac{m_B}{m_A}\right) v^{B1} \Rightarrow v^{B1} = \frac{m_A}{m_A + m_B}(1+e)v^{A0}$$

The impulse on the head follows as $I = m_B(v^{B1} - v^{B0}) = \dfrac{m_A m_B}{m_A + m_B}(1+e)v^{A0}$

[3 POINTS]

5. The figure shows a proposed design for a ballistic pendulum. At time $t=0$ (fig i) the mass M is subjected to an impulse I. This causes the mass to translate to the right (fig ii); which in turn will cause the pendulum to swing to the left of the mass (iii). The goal of this problem is to calculate a formula relating the maximum angle of swing of the pendulum to the impulse, and other relevant parameters.



5.1 Use the impulse-momentum formula to find a formula for the velocity of mass M just after it is struck.

- $I = Mv_0 \Rightarrow v_0 = I / M$

[1 POINT]

5.2 Now consider the system at the instant when the pendulum reaches the maximum angle of swing. Use momentum and energy conservation to show that

$$I = \sqrt{2M(M+m)gL(1-\cos\theta)}$$

- Note that the mass m at the end of the pendulum is stationary just after mass M is struck
- At the instant of maximum swing, the mass on the end of the pendulum moves with the same speed as mass M and has zero vertical velocity. Let v_1 denote the horizontal velocity of the two masses at this instant.
- Momentum is conserved after the impact. Therefore $(m+M)v_1 = Mv_0$
- Energy is conserved after the impact. Therefore $\frac{1}{2}Mv_0^2 - mgL = \frac{1}{2}(m+M)v_1^2 - mgL\cos\theta$
- Combining the last two equations gives

$$\frac{1}{2}Mv_0^2 - mgL = \frac{1}{2}(m+M)\left(\frac{Mv_0}{m+M}\right)^2 - mgL\cos\theta$$

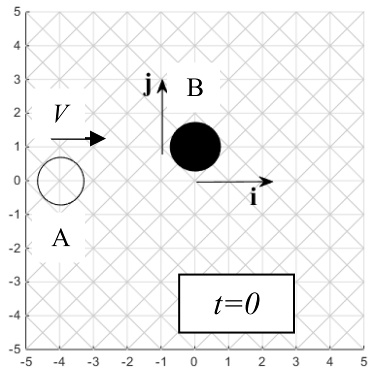
$$\Rightarrow \frac{1}{2}Mv_0^2\left(1-\frac{M}{m+M}\right) = mgL(1-\cos\theta)$$

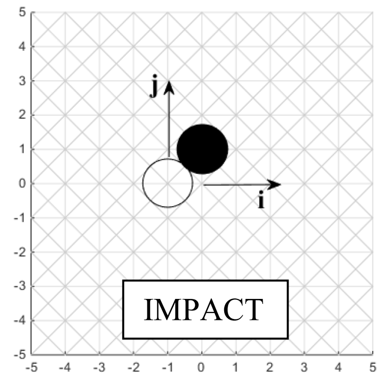
$$\Rightarrow \frac{M}{M+m}v_0^2 = 2gL(1-\cos\theta)$$

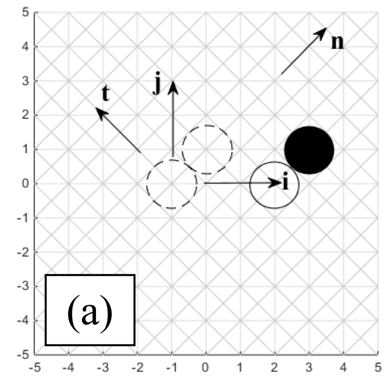
- Combining this with 5.1 gives the required answer.

[4 POINTS]

6. Two spheres with identical mass and restitution coefficient $e=0$ have initial positions shown in the figure below. Before impact sphere B is stationary and sphere A has velocity $V\mathbf{i}$. The collision is frictionless. By answering the true/false questions below, identify which of the figures (a-d) shows the correct position of the spheres after collision.







CIRCLE ONE RESPONSE TO EACH STATEMENT BELOW

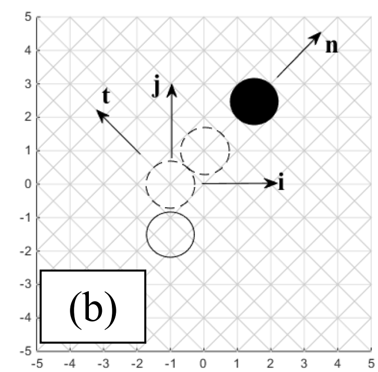
(a) Total Momentum is conserved in the **j** direction T F
Momentum of B is conserved in the **t** direction T F
The restitution formula is satisfied in the **n** direction T F

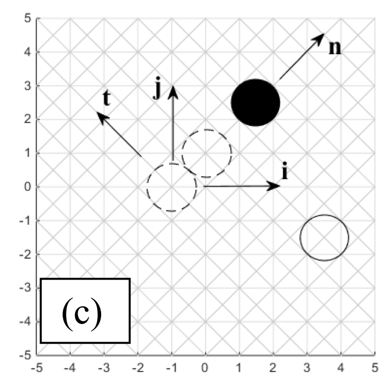
(b) Total Momentum is conserved in the **j** direction T F
Momentum of B is conserved in the **t** direction T F
The restitution formula is satisfied in the **n** direction T F

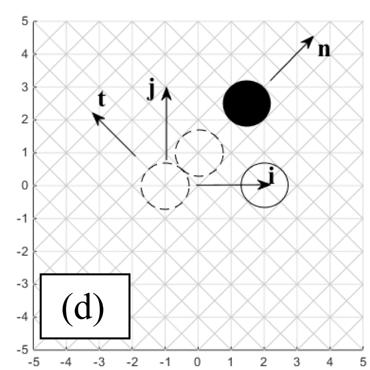
(c) Total Momentum is conserved in the **j** direction T F
Momentum of B is conserved in the **t** direction T F
The restitution formula is satisfied in the **n** direction T F

(d) Total Momentum is conserved in the **j** direction T F
Momentum of B is conserved in the **t** direction T F
The restitution formula is satisfied in the **n** direction T F

Correct figure: a b c d







Total momentum in the **j** direction before impact is zero, and must therefore be zero after impact. That means A and B must have equal and opposite vertical velocities (and must travel equal and opposite distances).
Momentum of B in the **t** direction is zero before impact and must be zero after impact. B must therefore travel parallel to **n**
Since the restitution coefficient is zero, the restitution formula in the **n** direction requires that both spheres have the same velocity in the **n** direction after impact. This means that both spheres must travel the same distance parallel to **n** after impact **[6 POINTS]**