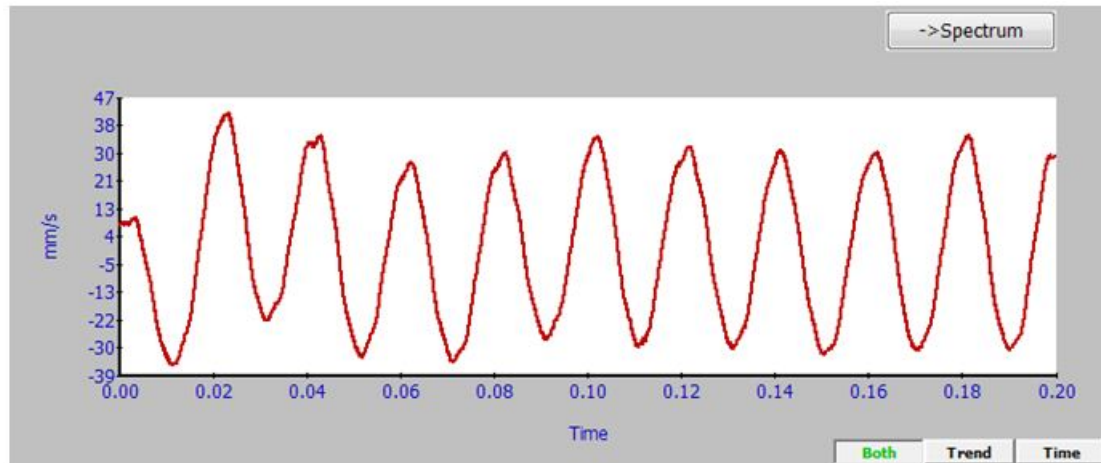EN40: Dynamics and Vibrations

Homework 5: Free Vibrations - Solutions
Due Friday March 25th 2015
55 POINTS TOTAL

School of Engineering
Brown University

1. The figure (from this reference) shows the measured velocity of lateral vibration of an out-of-balance rotor.   Calculate

1.1 The amplitude, the period, and frequency of the vibration (give the frequency both in Hertz and in radians per second)

-   There are 11 peaks in 0.2 secs. The period is therefore 0.2/11=0.018s
-   The frequency (in Hertz) is 55 ($f = 1/T$)
-   The frequency in rad/s is $\omega = 2\pi f = 346 rad/s$
-   The amplitude is about 30 mm/s

[3 POINTS]

1.2 The amplitude of the acceleration.

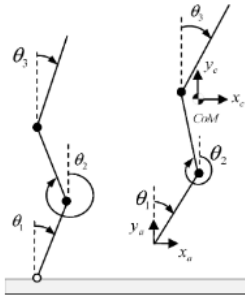-   The amplitude of the acceleration is $A = \omega V = 10.37 ms^{-2}$
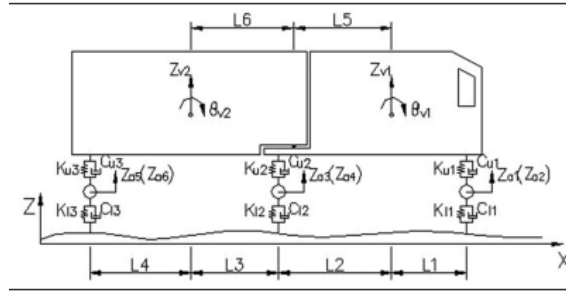
[1 POINT]

1.3 The amplitude of the displacement.

-   The amplitude of the displacement is $\Delta X = \Delta V / \omega = 86.8 \mu m$

[1 POINT]

2. State the number of degrees of freedo m for each of the systems shown below. For systems (a), (b) and (d) state the number of vibration modes

(a) Model of a hopping robot (motion is confined to the x,y plane. Consider the robot in the air only)

(b) 2D Model of an articulated truck (the model idealizes the wheels as particles, which can only move perpendicular to the truck body. Assume that the connection between the two parts of the body is a pin joint.
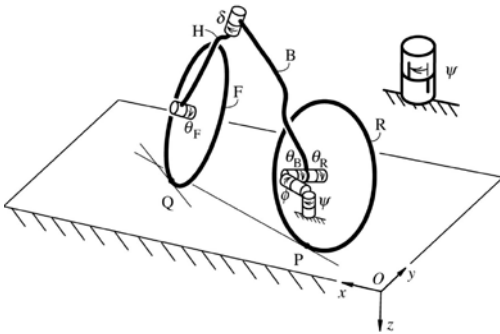
(c) Model of a bicycle The wheels and both links B and H are rigid bodies. (A riderless bicycle is usually unstable and so has no vibration modes. It is stable for a range of speeds, in which case there is one vibration mode. A simple explanation has not been found for this behavior).

(d) Benzene molecule (the spheres are particles, the rods are springs)

(a): There are 3 angles, and the entire system can move horizontally and vertically, so 5DOF. Alternatively, 3 rigid bodies give 9 DOF, and two pin joints with two constraints at each give 9-4=5DOF. The system has 3 rigid body modes (translation horizontally and vertically, and rotation in the plane) so 2 natural frequecies.

[2 POINTS]

(b): There are 3 wheels with 1 DOF (motion perpendicular to the truck body); each piece of the truck has 3 DOF. The connection between the two parts of the truck imposes 2 constraints (the x,y, components of velocity are equal at the connection). This 7 DOF. The system has 1 rigid body mode (translation horizontally) so there are 6 natural frequencies.

[2 POINTS]

(c) The bicycle can translate in one direction, can tilt, and the handlebar can be turned. This gives 3 DOF. Alternatively the two wheels plus the two parts of the body have 6 DOF each giving 24 DOF. Each axle has 5 constraints (all relative motion and relative rotation about 2 axes are prevented). The bearing that permits steering (just below the handlebar) also has 5 constraints. The two contact points with the ground have 3 constraints each (they allow relative rotation of the wheel and ground, but prevent relative motion of the contact point on the wheel and the ground). This gives 24DOF and 21 constraints, again 3DOF. (The large number of coordinates used in the figure are misleading, because they are not all independent. They are unavoidable, because the system is 'non-holonomic' – the constraints on velocity at the contact between the wheels and the ground cannot be integrated to write them as an algebraic equation relating positions and angles).

[1 POINT]

(d) 12 particles, 3 DOF each. 36 DOF in total. The molecule would have 36-6 =30 vibration modes.

[2 POINTS]

3. Solve the following differential equations (use the [Solutions to Differential Equations](#))

3.1 $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 0$ $\qquad y = 0$ $\quad \dfrac{dy}{dt} = 2$ $\quad t = 0$

- The formula sheet gives the solution to $\dfrac{1}{\omega_n^2}\dfrac{d^2x}{dt^2} + \dfrac{2\varsigma}{\omega_n}\dfrac{dx}{dt} + x = C$ with

  $x = x_0$ $\quad \dfrac{dx}{dt} = v_0$ $\quad t = 0$ The solution depends on the value of ς .

- Rearrange the equation in the problem into the appropriate form $\dfrac{1}{2^2}\dfrac{d^2y}{dt^2} + \dfrac{2}{2}\dfrac{dy}{dt} + y = 0$, so

  here $C = 0$ $\omega_n = 2, \varsigma = 1, \ x_0 = 0, \ v_0 = 2$

- For this case the solution is $x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n(x_0 - C)\right]t\right\}\exp(-\omega_n t)$

- so $x(t) = 2t\exp(-2t)$

[3 POINTS]

3.2 $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 16\sin(2t)$ $\qquad y = -2$ $\quad \dfrac{dy}{dt} = 1$ $\quad t = 0$

- The equation in standard form is $\dfrac{1}{2^2}\dfrac{d^2y}{dt^2} + \dfrac{2}{2}\dfrac{dy}{dt} + y = 4\sin(2t)$ $\qquad y = -2$ $\quad \dfrac{dy}{dt} = 1$ $\quad t = 0$ so

  $\varsigma = 1$ $\omega_n = 2$ $\omega = 2$ $KF_0 = 4$

- The formula gives $x(t) = C + x_h(t) + x_p(t)$ with steady state solution

$$x_p(t) = X_0\sin\left(\omega t + \phi\right)$$

$$X_0 = \dfrac{KF_0}{\left\{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} = 2 \qquad \phi = \tan^{-1}\dfrac{-2\varsigma\omega/\omega_n}{1 - \omega^2/\omega_n^2} = -\dfrac{\pi}{2}$$

- The system is critically damped, so the transient solution is

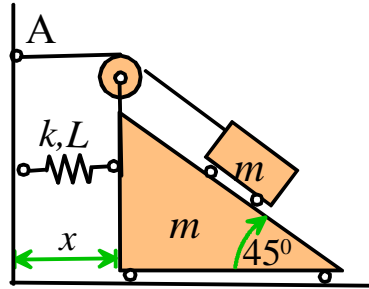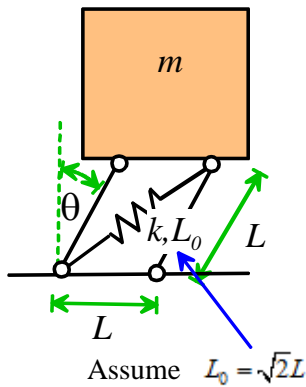$$x_h(t) = \left\{x_0^h + \left[v_0^h + \omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

  where

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0\sin\phi = 0$$

$$v_0^h = v_0 - \left.\dfrac{dx_p}{dt}\right|_{t=0} = v_0 - X_0\omega\cos\phi = 1$$

- Thus $x(t) = 2\sin\left(2t - \dfrac{\pi}{2}\right) + t\exp(-2t)$ (of course $2\sin(2t - \pi/2) = -2\cos(2t)$)

[3 POINTS]

m

θ

k, L_0

L

L

Assume $L_0 = \sqrt{2}L$

A

k, L

m

m

x

45^0

4. For the two conservative single-degree of freedom systems shown in the figure (note that in (a) the unstretched spring length is $\sqrt{2}L$):

4.1 Derive the equation of motion (use energy methods, and include gravity. The pulley and cable are massless). State whether the equation of motion is linear or nonlinear.

- The potential and kinetic energy of the first system were calculated in the preceding HW: we have that

$$V = mgL\cos\theta + \frac{1}{2}k\left\{2L\sin(\pi/4+\theta/2)-L_0\right\}^2, \quad T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$$

- Taking the time derivative of $T+V$=constant gives

$$mL^2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} - mgL\sin\theta\frac{d\theta}{dt} + k\left\{2L\sin(\pi/4+\theta/2)-L_0\right\}L\cos(\pi/4+\theta/2)\frac{d\theta}{dt} = 0$$

- The equation of motion is therefore (any of the results below are acceptable – they are simplified using various trig formulas). The equation is nonlinear (because of the trig terms)

$$\frac{d^2\theta}{dt^2} - \frac{g}{L}\sin\theta + \frac{k}{m}\left\{2\sin(\pi/4+\theta/2)-\sqrt{2}\right\}\cos(\pi/4+\theta/2) = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} - \frac{g}{L}\sin\theta + \frac{k}{m}\left\{\sin(\pi/2+\theta)-\sqrt{2}\cos(\pi/4+\theta/2)\right\} = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} - \frac{g}{L}\sin\theta + \frac{k}{m}\left\{\cos\theta-\sqrt{2}\cos(\pi/4+\theta/2)\right\} = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} - \frac{g}{L}\sin\theta + \frac{k}{m}\left\{\sin(\theta/2)+\cos\theta-\cos(\theta/2)\right\} = 0$$

[3 POINTS]

- For the second system, it helps to define a variable d quantifying the length of the cable. This variable disappears from the final results, but it simplifies the derivation.

- The position vector of the mass on the end of the cable with respect to point A can be expressed as

$$\mathbf{r} = x\mathbf{i} + (d-x)\cos(45)\mathbf{i} - (d-x)\sin(45)\mathbf{j}$$
$$= x\mathbf{i} + (d-x)(\mathbf{i}-\mathbf{j})/\sqrt{2}$$

- The velocity of this mass follows as
$$\frac{dx}{dt}\mathbf{i} - \frac{dx}{dt}(\mathbf{i}-\mathbf{j})/\sqrt{2}$$

- Recall that $T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\left(v_x^2 + v_y^2\right)$. The total KE of the two masses is therefore

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2\left(1-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{dx}{dt}\right)^2\left(-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}m\left(3-\sqrt{2}\right)\left(\frac{dx}{dt}\right)^2$$

- We can ignore the gravitational potential energy of the wedge since it does not move vertically. The total potential energy of the system (taking A as datum for the mass on the end of the cable) is

$$\frac{1}{2}k(x-L)^2 - mg(d-x)\sin(45) = \frac{1}{2}k(x-L)^2 - mg(d-x)/\sqrt{2}$$

- Take the time derivative of $T+V=C$ to get the following linear equation

$$m\left(3-\sqrt{2}\right)\left(\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + k(x-L)\left(\frac{dx}{dt}\right) + mg\left(\frac{dx}{dt}\right)/\sqrt{2} = 0$$

$$\Rightarrow m\left(3-\sqrt{2}\right)\frac{d^2x}{dt^2} + kx = kL - \frac{mg}{\sqrt{2}}$$

[3 POINTS]

4.2 If appropriate, linearize the equation of motion for small amplitude vibrations (that means doing that Taylor series stuff discussed in class. "Linearizing" means replacing the nonlinear function of the variable with an approximate linear function)

- We only need to linearize the first system. We can use the approximation $\sin\theta \approx \theta, \cos\theta \approx 1$ (or do the Taylor series with Mupad) to see that

$$\frac{d^2\theta}{dt^2} + \left(\frac{k}{2m} - \frac{g}{L}\right)\theta = 0$$

[2 POINTS]

4.3 Arrange the (linearized) equation of motion into standard form, and find an expression for the natural frequency of vibration.

- The equations in standard form reduce to

$$\frac{1}{\left(\dfrac{k}{2m}-\dfrac{g}{L}\right)}\frac{d^2\theta}{dt^2}+\theta=0 \qquad\qquad \frac{m}{k}\left(3-\sqrt{2}\right)\frac{d^2x}{dt^2}+x=L-\frac{mg}{k\sqrt{2}}$$

- The natural frequencies are therefore

$$\omega_n=\sqrt{\left(\frac{k}{2m}-\frac{g}{L}\right)} \qquad\qquad \omega_n=\sqrt{\frac{k}{(3-\sqrt{2})m}}$$

[2 POINTS]

5. Replace the system shown in the figure with an equivalent spring-mass system consisting of a mass with only one spring and dashpot. Hence, determine a formula for the undamped natural frequency and the damping factor for the system.
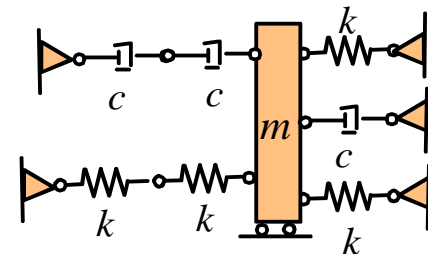
- The two end-to-end dampers are in series; and (together) are in parallel with the third damper. The effective dashpot coefficient is therefore

$$c^{eff}=\left(\frac{1}{c}+\frac{1}{c}\right)^{-1}+c=\frac{3c}{2}$$

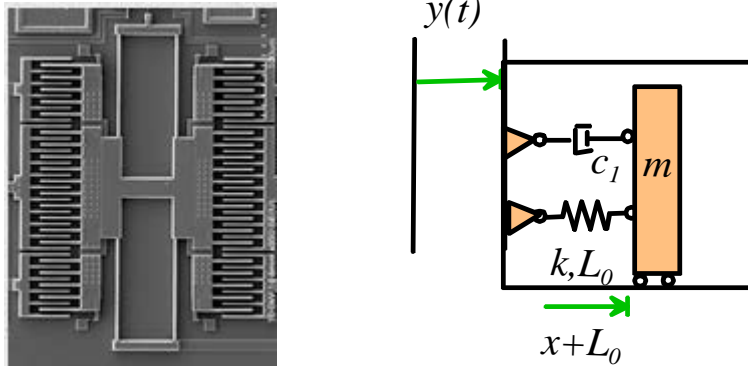- Similarly, the two end-to-end springs are in series; these are in parallel with both the other two springs. Therefore

$$k^{eff}=\left(\frac{1}{k}+\frac{1}{k}\right)^{-1}+2k=\frac{5k}{2}$$

- The natural frequency and damping factor therefore follow as

$$\omega_n=\sqrt{\frac{k_{eff}}{m}}=\sqrt{\frac{5k}{2m}} \qquad\qquad \zeta=\frac{c^{eff}}{2\sqrt{k^{eff}m}}=\frac{3c}{4\sqrt{5km/2}}$$

[3 POINTS]

$y(t)$

c_1 m

k, L_0

$x + L_0$

6. The figure shows a MEMS accelerometer (the figure on the left is from <u>this company</u>). It consists of of a proof mass m inside a sealed casing. The mass is suspended on springs and its motion is damped electrostatically. If the accelerometer accelerates to the right, the spring is compressed. The capacitative combs provide an electrical signal that senses the position x and hence provides a signal proportional to the acceleration of the device.

6.1 Show that the equation of motion for the distance x shown in the figure has the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = -\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

Give formulas for ω_n, ζ, K in terms of the m, k, c.

- The figure shows a FBD for the mass. The force in the damper is $F_D = c\frac{dx}{dt}$, and the force in the spring is $F_S = k(x + L_0 - L_0) = kx$.
- Note that the position of the mass with respect to a fixed origin is $x+y$.
- Newton's law gives

$$m\frac{d^2(x+y)}{dt^2} = -c\frac{dx}{dt} - kx$$

- Rearrange this in the standard form

$$\frac{m}{k}\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x = -\frac{m}{k}\frac{d^2y}{dt^2}$$

- If we define $\omega_n = \sqrt{k/m}$ $\quad \zeta = c/(2\sqrt{km})$ $\quad K = 1$ this reduces to the form stated.

F_D

m

F_S
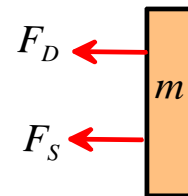
[2 POINTS]

6.3 Show that $x(t) \to -a/\omega_n^2$ as $t \to \infty$, so (once the transient has died out) x is proportional to a. This means that, once the transient motion has stopped, the signal will correctly measure acceleration.

- If the acceleration is constant, the governing equation reduces to

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = -\frac{1}{\omega_n^2}a$$

- The term on the right hand side is just a constant, so we can use Solution 2 from the handout. If the system starts from rest at time $t=0$ then $x_0 = v_0 = 0, C = -a/\omega_n^2$. Hence (for the underdamped case)

$$x(t) = -\frac{a}{\omega_n^2} + \frac{a}{\omega_n^2}\exp(-\zeta\omega_n t)\left\{\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right\}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

- The second term in the solution goes to zero as $t \to \infty$. This gives the correct solution.
- The over-damped and critically damped solutions will have the same behavior….

[3 POINTS]

6.4 The accelerometer designed in this publication has a resonant frequency of 41 kHz (don't forget the dreaded 2π factor between frequency and ω_n) and a damping factor of order $\zeta \approx 0.05$. Plot $x(t)$ with $a=1$g for this accelerometer (you can do the plot from $t=0$ to $t=0.4$ milliseconds). Use the graph to estimate how long it takes for the accelerometer reading to settle to within 5% of the correct value.

- The correct reading is achieved when x reaches its steady state value, which is

$$x(t) \to -a/\omega_n^2 = -9.81/(41000\times 2\times\pi) = -1.478\times10^{-10}$$

- The plot is shown below, along with the values of x 5% below and above the steady-state value.
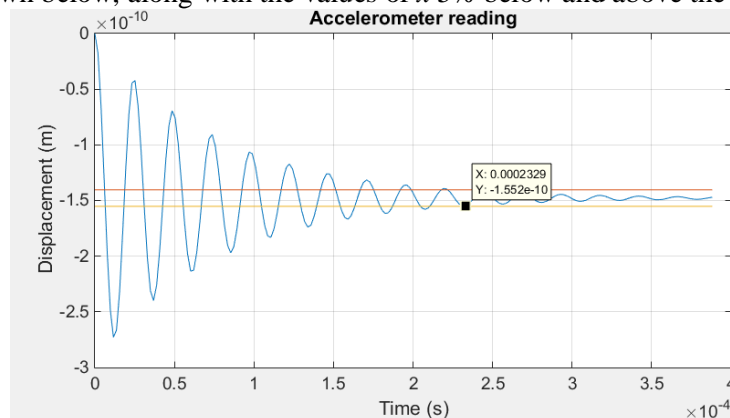
The reading reaches a value within 5% of the steady-state after about 0.23 milliseconds.

[4 POINTS]

7. Determine the steady-state amplitude of vibration for the base excited spring-mass systems shown in the figure (you don't need to derive the equations of motion – this is a standard textbook systems and you can just use the standard formulas). The mass $m=20$kg, the stiffness $k=2000N/m$ $c=20Ns/m$. The base motion is $y(t)=0.01\sin 20t$ N.
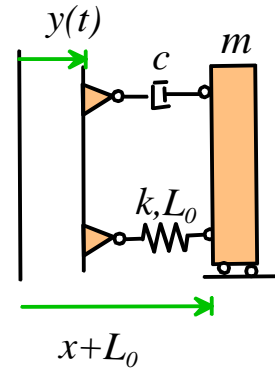
- The natural frequency and damping coefficient are

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \qquad \zeta = \frac{c}{2\sqrt{km}} = 0.05$$

- The amplitude of vibration is

$$X_0 = \frac{KY_0\left\{1+\left(2\zeta\omega/\omega_n\right)^2\right\}^{1/2}}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} = \frac{0.01\left\{1+\left(2\times0.05\times20/10\right)^2\right\}^{1/2}}{\sqrt{(1-20^2/100)^2 + (0.2)^2}} = 3.4mm$$

[3 POINTS]

8 The figure shows a simple idealization of a force sensor. Its purpose is to measure the force F, by providing an electrical signal that is proportional to the length s of the spring.

At time $t=0$ the system is at rest, and $F=0$. At time $t=1s$ a constant force of $F=100N$ is applied to the mass. The figure below shows the variation of s with time for $0<t<5s$.
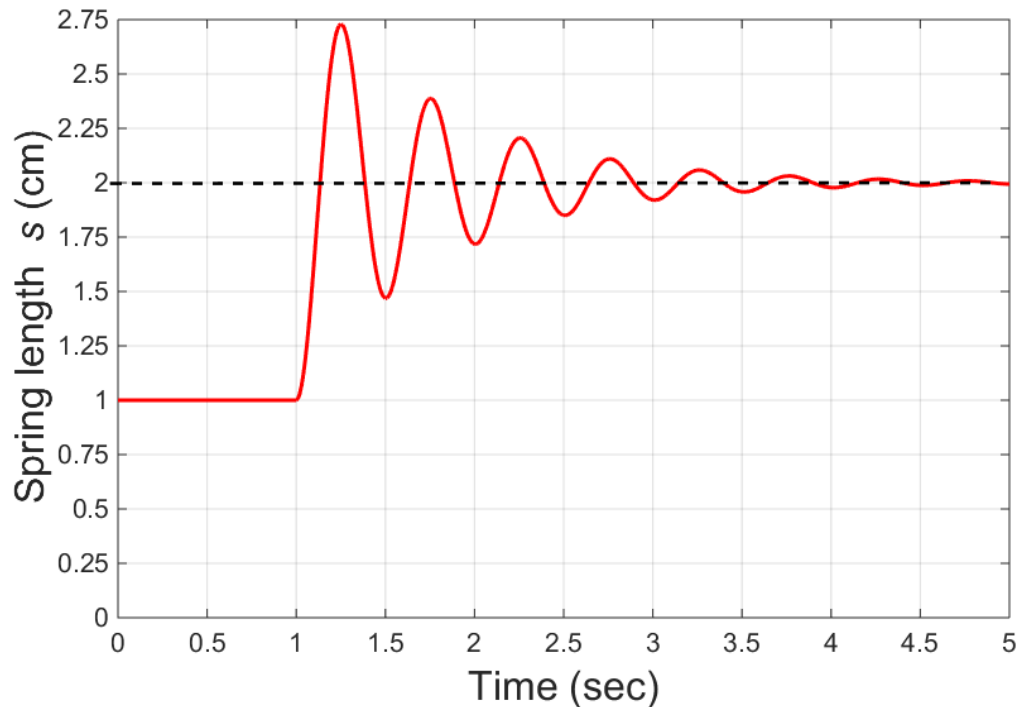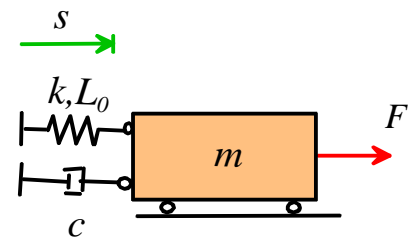
8.1 Using the graph provided, calculate values for the following quantities.

(a) The period of vibration (**1 POINT**)

2 cycles takes 1 sec so $T=0.5$s.

(b) The damped natural frequency ω_d (**1 POINT**)

$$\omega_d = 2\pi / T = 4\pi \, rad/s$$

(c) The log decrement of the vibration δ (be careful to use the correct origin) (**1 POINT**)

The first peak has amplitude 0.7; the third has amplitude 0.2 so the formula for log decrement gives $\delta = \dfrac{1}{2} \log(0.7/0.2) = 0.626$

(d) The damping factor of the system ζ (**1 POINT**)

From the formula $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.1$

(e) The undamped natural frequency of the system ω_n (**1 POINT**)

From the formula $\omega_n = \dfrac{\sqrt{4\pi^2 + \delta^2}}{T} = 12.62 \, rad/s$

(f) The un-stretched length of the spring L_0 (**1 POINT**)

The length of the spring must be equal to its unstretched length before the force is applied, so $L_0 = 1cm$

(g) The spring stiffness k (**1 POINT**)

After the oscillations die out, the spring has stretched by 1cm after the 100N force is applied. Therefore $k = 100/0.01 = 10000 N/m$

(h) The mass m (**1 POINT**)

We know that $\omega_n = \sqrt{k/m} \Rightarrow m = k/\omega_n^2 = 10000/(12.62)^2 = 62.8kg$ (this must be the world's heaviest force sensor – don't design one like this!)

(i) The dashpot coefficient c. (**1 POINT**)

We have $\zeta = c/2\sqrt{km} \Rightarrow c = 2\zeta\sqrt{km} = 158 Ns/m$

8.2 The sensor is now used to measure a force that vibrates harmonically $F(t) = F_0 \sin \omega t$. The figure below shows the steady-state variation of the spring length s with time. Calculate the amplitude of the force F_0.
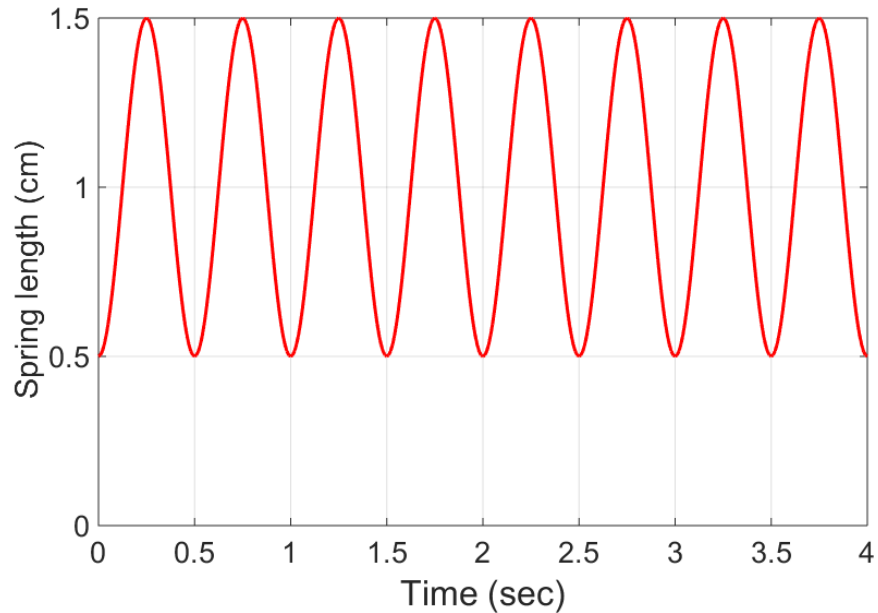
Note that the frequency of the force is equal to the natural frequency (the period of vibration is equal to the period in the first figure). This means the system is at resonance, and we can use the formula for the amplitude at resonance

$$X_0 = KF_0 M_{max} \approx \frac{1}{k} F_0 \frac{1}{2\zeta} \Rightarrow F_0 = 2\zeta k X_0 = 2 \times 0.1 \times 10000 \times 0.005 = 10N$$

(3 POINTS)