School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Kinematics and Dynamics of Particles
Due Friday Feb 12, 2016

1. Last summer, a student group from ETH Zurich broke the record for the shortest time for an electric car to accelerate from rest to 100 km/hr. They report a time of 1.513 sec over a distance of 'less than 30m'.

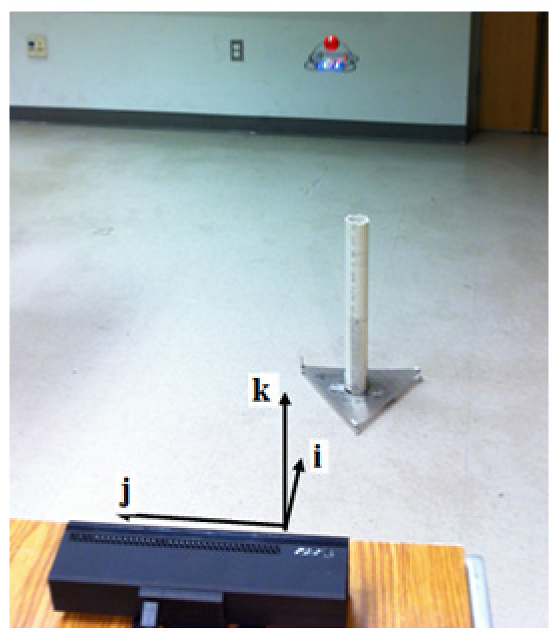
Using the constant acceleration formulas, estimate the acceleration of their vehicle (i) using the reported time; and (ii) using the reported 30m distance. How is it possible for the two estimates to differ?

- Straight line motion formula
  $v = v_0 + at \Rightarrow a = (v - v_0)/\Delta t = (100000/3600)1.513 = 18.36 m/s^2$
- Straight line motion formula $v^2 = v_0^2 + 2a\Delta s \Rightarrow a = (100000/3600)^2 / 60 = 12.8\,\text{m/s}^2$
- (You can also get another formula using $x = \frac{1}{2}at^2$ - if people did this it should get credit)

The two estimates might differ because the acceleration is not constant. This is investigated in more detail in problem 6.

[2 POINTS]

2. The quadcopter control system demonstrated in class uses a simple filter to estimate the velocity of the quadcopter. The goal of this problem is to check the accuracy of the filter.



The homework website provides a datafile that contains measurements of the position and velocity of the quadcopter taken during the test flight conducted in class.

The nine columns in the file specify the time, three components of position *(x,y,z)* (measured by the Kinect sensor), the velocities (v_x, v_y, v_z) estimated by the filter, and the accelerations (a_x, a_y, a_z) logged from the IMU (these were integrated in class). The coordinate system is such that x is the perpendicular distance of the quadcopter from the Kinect sensor, z is vertically upwards, and y is lateral. The data is in SI units (m and s).

Write a MATLAB code that will do the following:

(i) Read the datafile using the MATLAB data=csvread('*filename*') command. Note that the variable data will be a matrix: the first column of the matrix will be time values, the second column the x component of position, and so on.

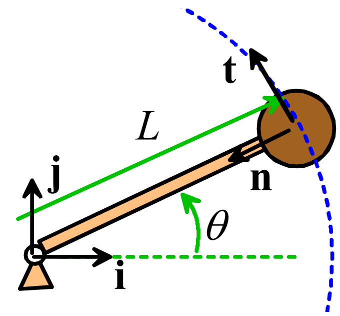
(ii) Estimate the position by integrating the velocity using the MATLAB ‘cumtrapz’ function,
(iii) Plot the position estimated using (ii) and the actual measured position on the same graph to compare the two.

Please upload your MATLAB code on canvas as a solution to this problem.

See separate MATLAB script for solution.

[5 POINTS]

3. The link AB shown in the figure is rigid and has length $L=2m$. The angle θ is given by $\theta = 4t^2$ (radians). Calculate the magnitude of the velocity of B and the magnitude of the acceleration of B at the instant when $\theta = \pi$ (there are many ways to solve the problem)



Using normal-tangential coordinates:

$$V = L\frac{d\theta}{dt} = 16t$$

$$t = \sqrt{\pi/4} \Rightarrow V = 8\sqrt{\pi}$$

[2 POINTS]

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} = L\frac{d^2\theta}{dt^2}\mathbf{t} + L\left(\frac{d\theta}{dt}\right)^2\mathbf{n} = 16\mathbf{t} + 2(8t)^2\mathbf{n} = 16\mathbf{t} + 32\pi\mathbf{n}$$

$$|\mathbf{a}| = \sqrt{16^2 + (32\pi)^2} = 16\sqrt{4\pi^2 + 1} = 101.8m/s^2$$

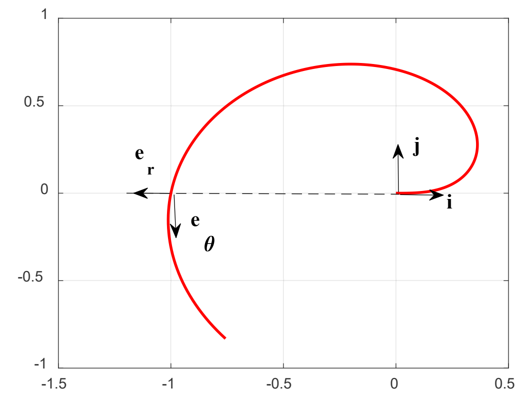
[2 POINTS]

Alternatively using polar coordinates

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = 16t\mathbf{e}_\theta = 8\sqrt{\pi}\mathbf{e}_\theta \Rightarrow |\mathbf{v}| = 8\sqrt{\pi}$$

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta = -2(8t)^2\mathbf{e}_r + 16\mathbf{e}_\theta \Rightarrow |\mathbf{a}| = \sqrt{16^2 + (32\pi)^2}$$

4. The position of a particle in polar coordinates is given by $\theta = t^2 \quad r = t/\sqrt{\pi}$ (meters). At the instant when $\theta = \pi$, calculate the following quantities:



4.1 The position vector in $\mathbf{i},\mathbf{j}$ components and in $\mathbf{e}_r,\mathbf{e}_\theta$ components

When $\theta = \pi, r = 1$ so

$$\mathbf{r} = -\mathbf{i} \ \ (meters)$$
$$\mathbf{r} = \mathbf{e}_r \ \ (meters)$$

[2 POINTS]

4.2 The velocity vector in $\mathbf{e}_r,\mathbf{e}_\theta$ and $\mathbf{i},\mathbf{j}$

The polar coordinate formula is $\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = \frac{1}{\sqrt{\pi}}\mathbf{e}_r + 2t\mathbf{e}_\theta = \frac{1}{\sqrt{\pi}}\mathbf{e}_r + 2\sqrt{\pi}\mathbf{e}_\theta$ m/s

By inspection we see that $\mathbf{i} = -\mathbf{e}_r, \mathbf{j} = -\mathbf{e}_\theta$ at the instant of interest, so $\mathbf{v} = -\frac{1}{\sqrt{\pi}}\mathbf{i} - 2\sqrt{\pi}\mathbf{j}$

[2 POINTS]

4.3 The acceleration vector in $\mathbf{e}_r,\mathbf{e}_\theta$ components

$$\mathbf{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta = -4t^2\mathbf{e}_r + (2 + 4\frac{1}{\sqrt{\pi}}t)\mathbf{e}_\theta = -4\pi\mathbf{e}_r + 6\mathbf{e}_\theta \ \ \text{m/s}^2$$

[1 POINT]

4.4 Unit vectors $\mathbf{t},\mathbf{n}$ tangent and normal to the path, in $\mathbf{e}_r,\mathbf{e}_\theta$. (Choose $\mathbf{n}$ to point towards the center of curvature)

We know $\mathbf{t}$ is parallel to $\mathbf{v}$ so

$$\mathbf{t} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{4\pi + 1/\pi}}\left(\frac{1}{\sqrt{\pi}}\mathbf{e}_r + 2\sqrt{\pi}\mathbf{e}_\theta\right) = \frac{1}{\sqrt{4\pi^2 + 1}}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right)$$

(graders please be careful to check for numerically equivalent answers – there are many possible forms for $\mathbf{t}$)

[1 POINT]

We can find **n** in three ways: first, we know **n** must be perpendicular to **t** and must lie in the **i,j** plane (and hence is perpendicular to **k**). Remember that you can create a vector perpendicular to two others using a cross product, so $\mathbf{n} = \pm\mathbf{k}\times\mathbf{t}$. The positive choice points towards the center of curvature (by inspection), and note $\mathbf{k}\times\mathbf{e}_r = \mathbf{e}_\theta \quad \mathbf{k}\times\mathbf{e}_\theta = -\mathbf{e}_r$ so $\mathbf{n} = \frac{1}{\sqrt{4\pi^2+1}}\left(-2\pi\mathbf{e}_r + \mathbf{e}_\theta\right)$

You can also use the condition $\mathbf{t}\cdot\mathbf{n} = 0$ - if we assume $\mathbf{n} = n_r\mathbf{e}_r + n_\theta\mathbf{e}_\theta$ then

$$\mathbf{t}\cdot\mathbf{n} = 0 \Rightarrow \frac{1}{\sqrt{4\pi^2+1}}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right)\cdot\left(n_r\mathbf{e}_r + n_\theta\mathbf{e}_\theta\right) = 0$$
$$\Rightarrow n_r + 2\pi n_\theta = 0$$

Any n_r, n_θ that satisfies this (eg $n_r = -2\pi, n_\theta = 1$) is perpendicular to **t**. But **n** must be a unit vector, and we know we want the vector to point towards the origin (because the center of curvature of the path is inside the turn). So we have to choose

$$\mathbf{n} = \frac{1}{\sqrt{4\pi^2+1}}\left(-2\pi\mathbf{e}_r + \mathbf{e}_\theta\right)$$

The last (cumbersome, but general) way to do the calculation is to note that $\mathbf{a} - (\mathbf{a}\cdot\mathbf{t})\mathbf{t}$ must be parallel to **n**.

$$-4\pi\mathbf{e}_r + 6\mathbf{e}_\theta - \left[\left(-4\pi\mathbf{e}_r + 6\mathbf{e}_\theta\right)\cdot\frac{1}{\sqrt{4\pi^2+1}}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right)\right]\frac{1}{\sqrt{4\pi^2+1}}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right)$$
$$= -4\pi\mathbf{e}_r + 6\mathbf{e}_\theta - \frac{8\pi}{4\pi^2+1}\left(\mathbf{e}_r + 2\pi\mathbf{e}_\theta\right) = -\frac{4\pi(3+4\pi^2)}{4\pi^2+1}\mathbf{e}_r + \frac{2(3+4\pi^2)}{4\pi^2+1}\mathbf{e}_\theta$$

Dividing by the magnitude of this vector (to create a unit vector) gives the same answer as before.

4.5 Tangential and normal components of acceleration a_t, a_n : we know (from ENGN30) that we can find the component of a vector in a basis by dotting it with the basis vectors, so

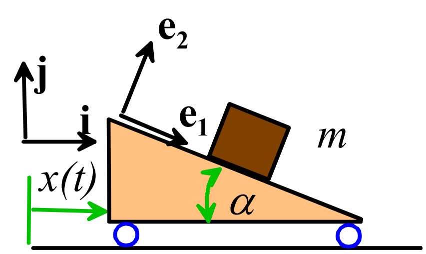
$$a_t = \mathbf{a}\cdot\mathbf{t} = \frac{8\pi}{\sqrt{1+4\pi^2}} \qquad a_n = \mathbf{a}\cdot\mathbf{n} = \frac{8\pi^2+6}{\sqrt{1+4\pi^2}} \text{ m/s}^2$$

You can also do this problem using the formula

$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$. This shows $a_t = \frac{dV}{dt}$, which is not too hard to calculate (but is a pain so I can't be bothered). To find $a_n = V^2/R$ we would either have to use our MA0200 super-powers to find the radius of curvature of the path, or else use $\mathbf{a} - \frac{dV}{dt}\mathbf{t} = +\frac{V^2}{R}\mathbf{n} = a_n\mathbf{n}$ and then take the magnitude of the vector on the left to get a_n . If anyone did the problem this way, they should of course get credit, and a great deal of sympathy for their suffering.

[2 POINTS]

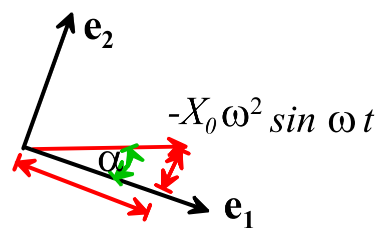
5. The figure shows an idealization of a vibrating conveyor The ramp moves back and forth with a horizontal displacement $x(t) = X_0 \sin \omega t$, where ω is a constant (the vibration frequency). The goal of this problem is to calculate a formula for the critical displacement X_0 that will ensure that the mass m begins to slip on the ramp.



5.1 Find a formula for the acceleration of the ramp. Give your answer in terms of components in both the $\mathbf{i},\mathbf{j}$ and $\mathbf{e}_1,\mathbf{e}_2$ bases shown in the figure.

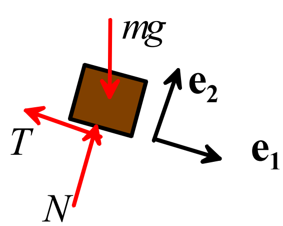
$$\mathbf{a} = \frac{d^2x}{dt^2} = -X_0\omega^2 \sin\omega t\mathbf{i} = -X_0\omega^2 \sin\omega t\left(\mathbf{e}_1\cos\alpha + \mathbf{e}_2\sin\alpha\right)$$

People were having trouble resolving the vector into $\mathbf{e}_1,\mathbf{e}_2$ - to do this just sketch the triangle and resolve the acceleration into the two components exactly as though it were a force on a FBD.



[2 POINTS]

5.2 Draw a free body diagram showing the forces acting on the mass m



[3 POINTS]

(OK to draw the friction force in the opposite direction since we assume no slip)

5.3 Assume that the mass does not slip on the ramp. Calculate expressions for the normal and tangential forces acting at the contact between the mass and ramp.

$$\mathbf{F} = m\mathbf{a} \Rightarrow (mg\sin\alpha - T)\mathbf{e}_1 + (N - mg\cos\alpha)\mathbf{e}_2 = -X_0\omega^2(\cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2)\sin\omega t$$

$$\Rightarrow T = mg\sin\alpha + mX_0\omega^2\cos\alpha\sin\omega t$$

$$N = mg\cos\alpha - mX_0\omega^2\sin\alpha\sin\omega t$$

[2 POINTS]

5.4 Find a formula for the critical value of $X_0\omega^2$ for the mass to begin to slip, in terms of μ, g, α

$|T| \le \mu N$ for no slip. Note that the maximum value of T and minimum value of N both occur when $\sin\omega t = 1$, so slip will first start to occur at these instants. This leads to

$$mg\sin\alpha + mX_0\omega^2\cos\alpha = \mu\left(mg\cos\alpha - mX_0\omega^2\sin\alpha\right)$$
$$\Rightarrow X_0\omega^2(\cos\alpha + \mu\sin\alpha) = g(\mu\cos\alpha - \sin\alpha)$$
$$\Rightarrow X_0\omega^2 = g\frac{(\mu\cos\alpha - \sin\alpha)}{(\cos\alpha + \mu\sin\alpha)}$$

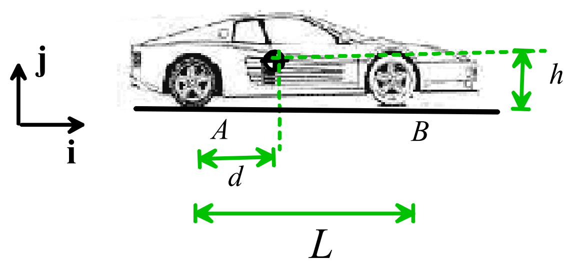
[2 POINTS]

5.5 Suppose that the ramp vibrates at a frequency of two cycles per second (this means $\omega = 4\pi$ rad/s), and $\mu = 0.5$. Calculate the necessary value of X_0 for a 5 degree ramp.

Substituting values gives $X_0 = 2.46cm$.

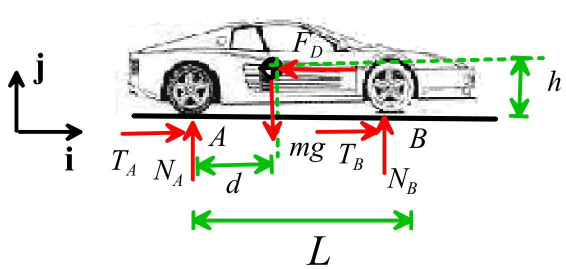
[1 POINT]

6. In this problem we reconsider the ETH Zurich racecar from problem 1. Assume that the vehicle has mass *m,* a wheel-base *L*, and that its center of mass is a distance *d* in front of the rear wheel and a height *h* above the ground.



6.1 Draw a free body diagram showing all the forces acting on the vehicle, as follows:

- Use the 2-D approximation in the figure[1]
- Note that the vehicle has four-wheel drive (why?) so friction forces act on both front and back wheels.
- Assume that the vehicle is also subjected to a drag force $F_D = c_D v^2$, which acts opposite to the direction of motion. Here c_D is a constant, and v is the vehicle's speed.
- To keep things simple, assume that the drag force acts on the center of mass.



[3 POINTS]

6.2 Using Newton's laws (consider both forces and moments) find formulas for normal forces N_A, N_B acting on the front and back wheels in terms of *h,d,m,g,*, and show that the tangential forces acting on the wheels satisfy $T_A + T_B = ma_x + c_D v^2$

F=m**a** gives $(T_A + T_B - F_D)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma_x\mathbf{i}$

[1] In the 2d model the forces on the wheels represent the total forces exerted by the two wheels at the front and back.

Moments about the COM gives $(T_A + T_B)h\mathbf{k} + N_B(L-d)\mathbf{k} - N_A d\mathbf{k} = 0$
This gives three equations

$$T_A + T_B - c_D v^2 = ma_x$$
$$N_A + N_B = mg$$
$$-N_A + N_B \frac{(L-d)}{d} = -(T_A + T_B)\frac{h}{d}$$

Eliminate T_A, T_B from the last equation using the first, add the last two equations, and simplify:

$$N_B = mg\frac{d}{L} - \left(ma_x + c_D v^2\right)\frac{h}{L}$$

Substitute back into the second equation and solve for N_A

$$N_A = mg - \left[mg\frac{d}{L} - \left(ma_x + c_D v^2\right)\frac{h}{L}\right] = mg\frac{(L-d)}{L} + \left(ma_x + c_D v^2\right)\frac{h}{L}$$

(some people might have substituted $T_A = \mu N_A \; T_B = \mu N_B$ instead of using the acceleration and drag – that's fine too. Don't worry too much about the algebra and the answer when grading this problem – if people are writing down **F**=m**a** and the moment equation correctly and have some reasonable approach to trying to solve the equations that should get 3 pts)

[4 POINTS]

6.3 Assume that the power distribution system ensures that all the wheels are just at the point of slip, so $T_A = \mu N_A$, $T_B = \mu N_B$, where μ is the friction coefficient. Show that the acceleration of the vehicle can be expressed in the form

$$a_x = \alpha - \beta v^2$$

and give formulas for the two constants α, β in terms of μ, c_D, m and the gravitational acceleration g.

From 6.2 $N_A + N_B = mg$, $T_A + T_B - c_D v^2 = ma_x$
Hence

$$T_A + T_B = ma_x + c_D v^2 = \mu mg$$
$$\Rightarrow a_x = \mu g - \frac{c_D}{m} v^2$$

Thus $\alpha = \mu g \quad \beta = \dfrac{c_D}{m}$

[2 POINTS]

6.4 Hence, show that the vehicle's speed is related to the distance traveled x by

$$v = \sqrt{\frac{\alpha}{\beta}}\sqrt{1 - \exp(-2\beta x)}$$

(Mupad can help with the integral if you like - although this one is easy to do by hand - use `assume(`α`>0): assume(`β`>0):` and don't forget to do a definite integral)

From the preceding problem

$$a_x = v\frac{dv}{dx} = \alpha - \beta v^2$$

$$\Rightarrow \int_0^v \frac{vdv}{\alpha - \beta v^2} = \int_0^x dx \Rightarrow -\frac{1}{2\beta}\left[\log(\alpha - \beta v^2) - \log(\alpha)\right] = x$$

$$\Rightarrow \log(1 - \frac{\beta}{\alpha}v^2) = -2\beta x \Rightarrow v = \sqrt{\frac{\alpha}{\beta}}\sqrt{1 - \exp(-2\beta x)}$$

[2 POINTS]

6.5 Use 6.4 to show that the distance traveled is related to time by

$$x = -\frac{1}{2\beta}\log\left[1 - \tanh^2(\sqrt{\alpha\beta}t)\right] = \frac{1}{2\beta}\log\left(\cosh^2\sqrt{\alpha\beta}t\right)$$

(You will need to separate variables and integrate both sides of the answer to problem 6.4. The integral on the left hand side is hard so use Mupad. This will give you an expression for t in terms of x - you rearrange this by hand to get the answer.). As a solution to this problem write down the value of the integral you get from Mupad, and the remaining derivations that you use to simplify the result. You don't need to submit your Mupad script).

6.4 gives

$$\frac{dx}{dt} = \sqrt{\frac{\alpha}{\beta}}\sqrt{1 - \exp(-2\beta x)}$$

$$\Rightarrow \int_0^x \frac{dx}{\sqrt{1 - \exp(-2\beta x)}} = \sqrt{\frac{\alpha}{\beta}}\int_0^t dt$$

The integral on the left can be done with mupad

```
reset():
assume(`&beta;`>0):
integrand := 1/sqrt(1-exp(-2*`&beta;`*x))
```

$$\frac{1}{\sqrt{1 - e^{-2\beta x}}}$$

```
int(integrand,x=0..xx)
```

$$\frac{\operatorname{arctanh}\left(\sqrt{1 - e^{-2\beta xx}}\right)}{\beta}$$

We can do the rest by hand

$$\sqrt{1 - \exp(-2\beta x)} = \tanh(\sqrt{\alpha\beta}t)$$

$$\Rightarrow \exp(-2\beta x) = 1 - \tanh^2(\sqrt{\alpha\beta}t) = \frac{1}{\cosh^2(\sqrt{\alpha\beta}t)}$$

$$\Rightarrow x = -\frac{1}{2\beta}\log\left(\frac{1}{\cosh^2(\sqrt{\alpha\beta}t)}\right) = \frac{1}{2\beta}\log\left(\cosh^2(\sqrt{\alpha\beta}t)\right)$$

[3 POINTS]

6.6 Use the fact that the vehicle reaches 100km/hr and travels 30m in 1.513s to calculate values for α, β . You will need to do this using Mupad. Note you can solve 6.4 for α in terms of β and substitute into 6.5. You can then solve 6.5 for β in Mupad using `numeric::solve(equation, `β`)`

We have two equations

$$30 = \frac{1}{2\beta}\log\left(\cosh^2(1.513\sqrt{\alpha\beta})\right)$$

$$\frac{100000}{3600} = \sqrt{\frac{\alpha}{\beta}}\sqrt{1-\exp(-60\beta)}$$

We can solve these in mupad

```
reset():
eq1 := 30=1/2/`&beta;`*log(cosh(1.513*sqrt(`&alpha;`*`&beta;`))^2)
```

$$30 = \frac{\ln\left(\cosh\left(1.513\ \sqrt{\alpha\ \beta}\right)^2\right)}{2\ \beta}$$

```
eq2 := 100000/3600=sqrt(`&alpha;`/`&beta;`)*sqrt(1-exp(-60*`&beta;`))
```

$$\frac{250}{9} = \sqrt{\frac{\alpha}{\beta}}\ \sqrt{1 - e^{-60\ \beta}}$$

```
alphasol := solve(eq2,`&alpha;`,IgnoreSpecialCases)[1]:
equation := subs(eq1,`&alpha;`=alphasol):
betaval := numeric::solve(equation,`&beta;`)[1]
```

0.05270996644

```
alphaval := float(subs(alphasol,`&beta;`=betaval))
```

42.46834663

Hence

$$\alpha = 42.468\ m/s^2$$

$$\beta = 0.0527\ m^{-1}$$

[2 POINTS]

6.7 What value does this calculation predict for the coefficient of friction μ ?

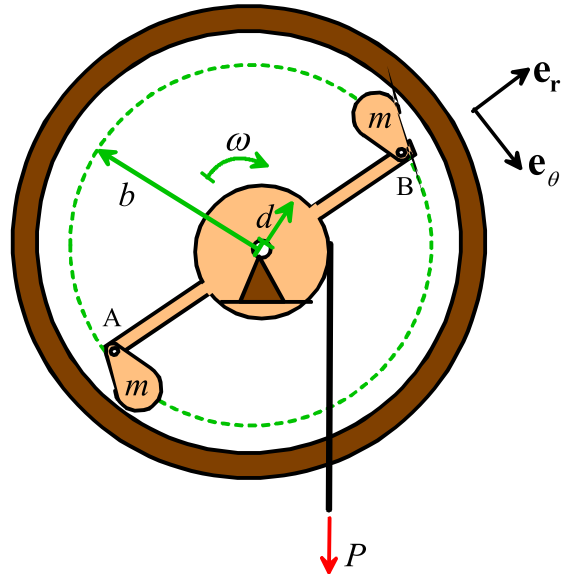
We know $\alpha = \mu g$ so this suggests that $\mu \approx 4$. Race car tires do have unusually high values of friction coefficient (and don't obey the Coulomb friction law very well) but this seems unreasonably high….

[1 POINT]

7. The figure shows a proposed design for an inertial fall arrest device. Note that:

- The external drum is stationary.
- During a fall, a constant force P acts on the rope, which causes the bar AB to rotate clockwise.
- We will consider steady-state motion, wherein AB rotates at constant angular speed ω .
- The two masses m pivot freely about A and B.
- Friction forces act at the contacts between the masses and the drum, which act as a brake.

The goal of this problem is to find a formula relating the force P on the rope to the angular speed ω.

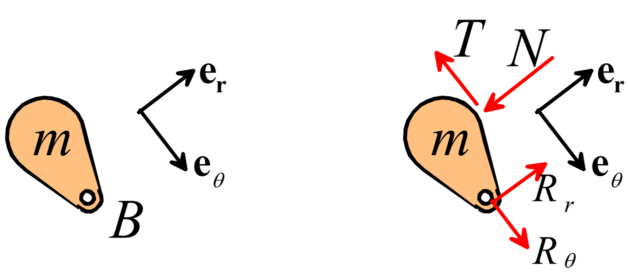


7.1 Write down a formula for the acceleration of the mass m attached to point B, in the polar coordinate system shown.

$$\mathbf{a} = -b\omega^2 \mathbf{e}_r$$

[1 POINT]

7.2 Draw a free body diagram showing the forces acting on the mass attached at point B (use a copy of the figure shown) **Neglect gravity.**



[OK to neglect the radial reaction at B – if we idealize the mass as a particle the net moment about COM must vanish, which means the radial must be small]

[3 POINTS]

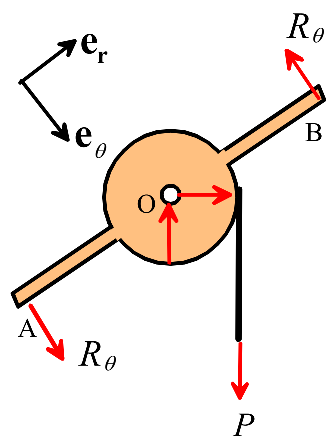
7.3 Use Newton's laws to find formulas for the forces acting on the mass, in terms of $b, \omega,$ and friction coefficient μ

Newton (neglecting the radial reaction at B): $-N\mathbf{e}_r + [R_\theta - T]\mathbf{e}_\theta = -mb\omega^2 \mathbf{e}_r$

Friction law $T = \mu N$. Hence $N = mb\omega^2$ $\quad T = \mu mb\omega^2$ $\quad R_\theta = \mu mb\omega^2$

[2 POINTS]

7.4 Draw a free body diagram showing the forces acting on the assembly shown. Assume that the bearings at O exert no moment.



[3 POINTS]

7.5 Since AB rotates at constant speed the net moment about O must be zero. Hence, find a formula for P in terms of b, d, ω, μ, m

Moments about O gives $[-Pd + 2bR_\theta]\mathbf{k} = \mathbf{0}$

Hence $P = 2\mu m \dfrac{b^2}{d}\omega^2$

[1 POINT]