



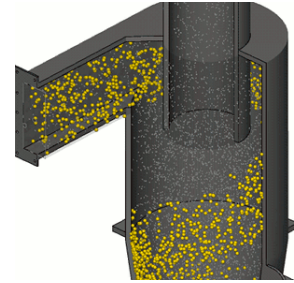
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 3: Kinematics and Dynamics of Particles Due Friday Feb 17, 2017

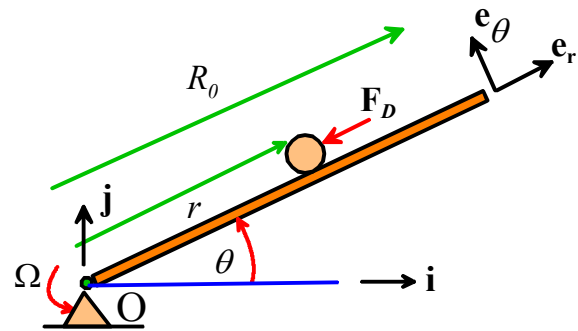
Please submit your solutions to the MATLAB coding problems by uploading a SINGLE file, with a .m extension, to Canvas.

1. A ‘[cyclonic separator](#)’ uses a spiraling airflow to remove particles from air. The rotating flow pushes particles towards the outer walls of the device, where they get trapped in a layer of still air (and eventually fall to the bottom of the air column). The goal of this problem is to develop a crude idealization of the device.



Suppose the air contains a distribution of spherical particles, with radius b and mass density ρ . As a rough approximation, we can represent the effect of the spiraling airflow as a rotating rigid bar that exerts a normal reaction force on particles that lie on its surface (see the figure). In addition to this normal force, particles are subjected to a radial drag force $\mathbf{F}_D = -6\pi\eta b \frac{dr}{dt} \mathbf{e}_r$, where η is the air viscosity.

1.1 Assume that the bar rotates at constant angular speed $\frac{d\theta}{dt} = \Omega$ about the pivot at O . Write down the acceleration vector of the particle shown in the figure, in terms of Ω, r , and time derivatives of r (i.e. the polar coordinate formula)



1.2 Draw a free body diagram showing the forces acting on the particle. Neglect gravity.

1.3 Use Newton’s law to show that the radial position of the particle satisfies the equation

$$\frac{d^2 r}{dt^2} + 2c \frac{dr}{dt} - r\Omega^2 = 0$$

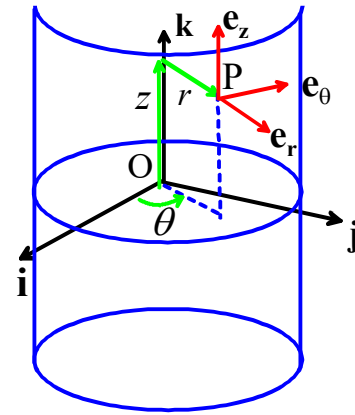
and find a formula for the constant c in terms of b, η, ρ (the factor of 2 in the equation is introduced to simplify the answer to 1.4).

1.4 Assume the particle starts from a position $r = r_0$ and has radial velocity $dr/dt = 0$ at time $t = 0$. Use Mupad to solve the differential equation (use `assume(c > 0)`). As a solution to this problem, just write down the answer from Mupad. You don’t need to submit your Mupad script. (**optional** – check your answer by showing that it can be rearranged into the form

$$r(t) = \frac{r_0 \exp(-ct)}{\sqrt{\Omega^2 + c^2}} \left[c \sinh\left(\sqrt{\Omega^2 + c^2} t\right) + \sqrt{\Omega^2 + c^2} \cosh\left(\sqrt{\Omega^2 + c^2} t\right) \right]$$

1.5 Consider a separator with the following parameters: radius $R_0 = 0.1m$, angular speed $\Omega = 125$ rad/s (about 1200 rpm), air viscosity $\eta = 1.8 \times 10^{-5}$ Ns/m², particle radius 10^{-6} m (1 micron), density $\rho = 1000kg / m^3$. If the air initially contains a spatially uniform distribution of N particles per unit volume, what is the average particle density (in terms of N) in the separator after 10 sec ? (Hint: particles that start with large values of r_0 will reach the outer wall of the separator and be removed. Particles close to the axis, with small r_0 will not have time to reach the outer wall and will remain in the air. Start by calculating the initial position r_0 of the particle that just reaches the outer wall in 10 sec)

2. In this problem we calculate the trajectory of a particle through a cyclonic separator in more detail. As in the previous problem, we assume that the air contains spherical particles, with mass density ρ and radius b . This time we will analyze the trajectory of a representative particle in three dimensions, using a more realistic description of the air flow and drag forces.



2.1 The position of a representative particle is specified by cylindrical polar coordinates (r, θ, z) . Write down the velocity and acceleration of the particle in terms of these variables and their time derivatives (i.e. the polar coordinate formulas – you can assume that the

component of acceleration in the e_z direction is $\frac{d^2z}{dt^2}$).

2.2 Assume that the particle is subjected to the following forces (i) a gravitational force, acting vertically downwards; and (ii) a drag force, which is given by

$$\mathbf{F}_D = 6\pi\eta b(\mathbf{v}^{air} - \mathbf{v}^{particle})$$

where η is the air viscosity , $\mathbf{v}^{particle}$ is the velocity of the particle, and \mathbf{v}^{air} is the air velocity.

[Simulations](#) suggest that the air velocity in a separator can be approximated by

$$\mathbf{v}^{air} = u(r)\mathbf{e}_\theta$$

$$u(r) = \begin{cases} U_0 \frac{(r/R_0)^2 - (r/R_0)^{20}}{0.04 + (r/R_0)^3} & r < R_0 \\ 0 & r \geq R_0 \end{cases}$$

where R_0 is the radius of the separator, and U_0 is the flow speed (both are constants with known values). Show that Newton's law of motion for the particle can be written as

$$m \left(\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + m \left(r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta + m \frac{d^2z}{dt^2} \mathbf{e}_z = 6\pi\eta b \left[u(r)\mathbf{e}_\theta - \left(\frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta + \frac{dz}{dt} \mathbf{e}_z \right) \right] - mg\mathbf{e}_z$$

where m is the particle mass (you can assume the component of velocity in the e_z direction is $\frac{dz}{dt}$)

Use this equation to obtain three second order differential equations for (r, θ, z) . Split each of these three equations into two first order equations, and hence show that the equations for (r, θ, z) and their time derivatives (v_r, ω, v_z) can be written in the MATLAB format

$$\frac{d}{dt} \begin{bmatrix} r \\ \theta \\ z \\ v_r \\ \omega \\ v_z \end{bmatrix} = \begin{bmatrix} v_r \\ \omega \\ v_z \\ -2cv_r + r\omega^2 \\ \{2c(u(r) - r\omega) - 2v_r\omega\} / r \\ -g - 2cv_z \end{bmatrix}$$

and give a formula for the constant c in terms of b, η, ρ

2.3 Write a MATLAB script that will solve the equations derived in problem 2.2 (use a conditional statement to handle the function $u(r)$ in the formula for air velocity). Plot the trajectory of the particle (in x, y, z coordinates – your script will need to convert the polar coordinates to x, y, z) Use the following parameters: $R_0 = 0.1m$, $U_0 = 10m/s$, $\eta = 1.8 \times 10^{-5} \text{ Ns/m}^2$, $\rho = 500kg/m^3$. Start the particle from $r = 0.01, \theta = 0, z = 0$, with velocity $\mathbf{v} = \mathbf{0}$ at $t = 0$ (i.e. $(v = 0, \omega = 0, v_z = 0)$). Plot trajectories (i.e. a 3D plot of x, y, z along the path) (on separate figures) for particles with 5, 10 and 20 micron radii (a micron is 10^{-6} m) (if you are curious you can find a table of sizes for common particles [here](#)). Run your simulations for 50secs. Submit your MATLAB code as a solution to this problem.

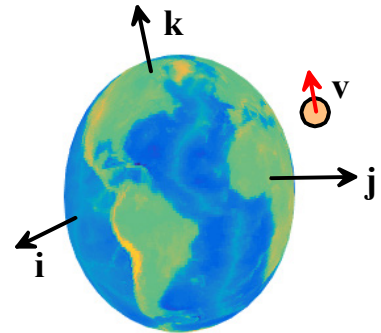
3. This problem analyzes ‘mirror trapping’ of charged particles by the earth’s magnetic field. Some background information is needed to set up the problem (don’t worry if these equations are unfamiliar... electricity & magnetism will be discussed in more detail in ENGN0510 next semester):

- A particle with charge q moving with velocity \mathbf{v} through a magnetic field with intensity \mathbf{B} experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$
- The earth’s magnetic field can be approximated by

$$\mathbf{B} = -B_0 \frac{R_e^3}{R^5} (3xz\mathbf{i} + 3yz\mathbf{j} + (2z^2 - x^2 - y^2)\mathbf{k}) \quad R = \sqrt{x^2 + y^2 + z^2}$$

where $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis with origin at the center of the earth, and \mathbf{k} pointing parallel to magnetic North, R_e is the earth’s radius, and B_0 is the magnetic field strength at the earth’s surface at the equator.



3.1 Let $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ denote the velocity of a charged particle. Use Newton’s law (neglect gravity) to show that

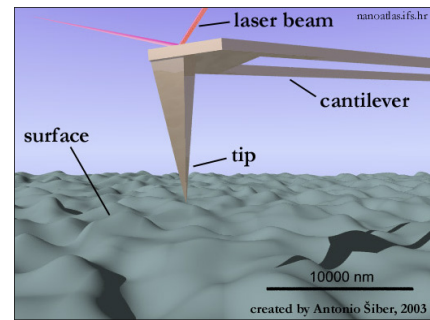
$$m \left\{ \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j} + \frac{d^2z}{dt^2} \mathbf{k} \right\} = -q (v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}) \times B_0 \frac{R_e^3}{R^5} (3xz\mathbf{i} + 3yz\mathbf{j} + (2z^2 - x^2 - y^2)\mathbf{k})$$

Expand out the cross product and hence obtain three second order differential equations for x, y, z . Split each equation into two first order equations, and hence show that the equations of motion for the particle can be expressed in the Matlab form

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ qB_0R_e^3(-v_y(2z^2 - x^2 - y^2) + 3v_z yz) / (mR^5) \\ qB_0R_e^3(v_x(2z^2 - x^2 - y^2) - 3v_z xz) / (mR^5) \\ -3qB_0R_e^3(v_x yz - v_y xz) / (mR^5) \end{bmatrix}$$

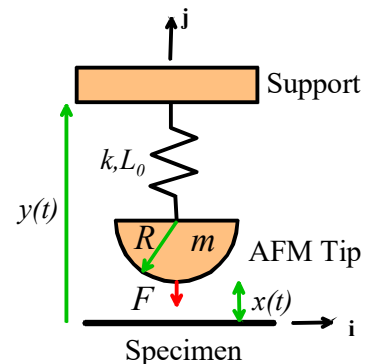
3.2 Write a MATLAB script that will solve the equations listed in 3.1. Use your code to predict (and plot) the trajectory of a proton with the following parameters: $m = 1.673 \times 10^{-27}$ kg, $q = 1.6022 \times 10^{-19}$ Coulombs, $B_0 = 3.07 \times 10^{-5}$ Tesla, $R_e = 6378$ km. Start the particle at $x = 4R_e$, $y = z = 0$, with initial velocity: $v_x = 0$, $v_y = 7.7 \times 10^6$ m/s, $v_z = 13.3 \times 10^6$ m/s. Run your simulation for 500sec. Use (eg) `options = odeset('RelTol', 0.00001);` to improve the accuracy of your simulation. Submit your MATLAB code on Canvas as a solution to this problem.

4. An atomic force microscope consists of a probe with a sharp tip that is mounted at the tip of a flexible cantilever. The device detects the forces that act between the probe tip and a specimen. By measuring changes in these forces as the tip scans over the specimen, it is possible to reconstruct high-resolution images of the surface topography.



Several different methods are used to detect the surface forces. In ‘tapping mode,’ the tip is vibrated just above the surface, and small changes in the vibration response are used to construct an image of the surface. The goal of this problem is to predict how surface forces influence the vibration of an AFM tip.

For design calculations, [the cantilever/tip assembly is usually idealized as a spring-mass system](#), as shown in the figure. The tip has mass m and radius R , and at time t is a height $x(t)$ above the surface. The spring has stiffness k and unstretched length L_0 . The top of the spring is moved vertically by a high-frequency actuator and has height $y(t) = \Delta + Y_0 \sin \omega t + R + L_0$ above the surface (here Δ is a constant, Y_0 is the amplitude of vibration, ω is the frequency of vibration).



The tip is subjected to: (i) a force from the spring; (ii) an attractive surface force F (acting downwards on the tip), which is related to x by $F = F_0 \frac{x}{d} \exp(-x/d)$; and (iii) a drag force resisting motion of the tip

$$\mathbf{F}_D = -c \frac{dx}{dt} \mathbf{j}$$

4.1 Draw a free body diagram showing the forces acting on the AFM tip.

4.2 Use Newton's law and the spring force law to show that the equation of motion for $x(t)$ can be arranged in the form

$$\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ \{k(\Delta + Y_0 \sin \omega t) - kx - cv - F\} / m \end{bmatrix} \quad F = F_0 \frac{x}{d} \exp(-x / d)$$

4.3 Write a MATLAB script to solve the equations in 4.2. Use the following values for the parameters:

- $k=40\text{N/m}$, $m= 8.27 \times 10^{-12}$ kg, $c = 2 \times 10^{-6}$ Ns/m, $F_0 = 10^{-7}$ N,
- $d = 5 \times 10^{-9}$ m, $Y_0 = 1 \times 10^{-9}$ m, $\omega = 2.4 \times 10^6$ rad/s.

Use your code to plot the cantilever tip deflection $\delta(t) = x(t) - \Delta$ as a function of time, for a time interval $0 < t < 10^{-4}$ s.

Show results for $\Delta = 10 \times 10^{-9}$ m and $\Delta = 40 \times 10^{-9}$ m, and use initial conditions $x(0) = \Delta, v(0) = 0$ for both cases. Upload your MATLAB code to Canvas as a solution to this problem.

Note that the vibration amplitude changes with Δ : if this can be detected, the gap between the tip and surface can be determined without touching it with the AFM tip. In practice, commercial AFMs use a feedback control loop to adjust Δ to maintain a constant vibration amplitude, and then use variations in the value of Δ to image the surface.