



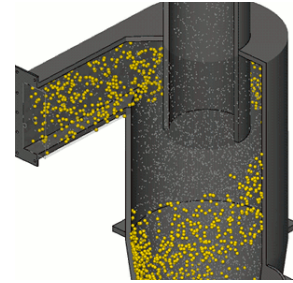
School of Engineering  
Brown University

## EN40: Dynamics and Vibrations

### Homework 3: Kinematics and Dynamics of Particles Due Friday Feb 17, 2016

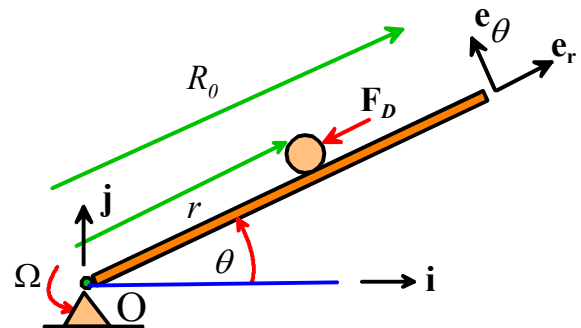
Please submit your solutions to the MATLAB coding problems by uploading a SINGLE file, with a .m extension, to Canvas.

1. A ‘[cyclonic separator](#)’ uses a spiraling airflow to remove particles from air. The rotating flow pushes particles towards the outer walls of the device, where they get trapped in a layer of still air (and eventually fall to the bottom of the air column). The goal of this problem is to develop a crude idealization of the device.



Suppose the air contains a distribution of spherical particles, with radius  $b$  and mass density  $\rho$ . As a rough approximation, we can represent the effect of the spiraling airflow as a rotating rigid bar that exerts a normal reaction force on particles that lie on its surface (see the figure). In addition to this normal force, particles are subjected to a radial drag force  $\mathbf{F}_D = -6\pi\eta b \frac{dr}{dt} \mathbf{e}_r$ , where  $\eta$  is the air viscosity.

1.1 Assume that the bar rotates at constant angular speed  $\frac{d\theta}{dt} = \Omega$  about the pivot at O. Write down the acceleration vector of the particle shown in the figure, in terms of  $\Omega, r$ , and time derivatives of  $r$  (i.e. the polar coordinate formula)

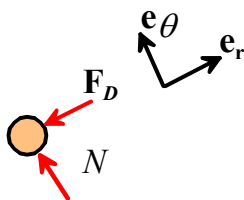


$$\mathbf{a} = \left( \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + \left( r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta$$

$$= \left( \frac{d^2 r}{dt^2} - r\Omega^2 \right) \mathbf{e}_r + \left( 2 \frac{dr}{dt} \Omega \right) \mathbf{e}_\theta$$

[1 POINT]

1.2 Draw a free body diagram showing the forces acting on the particle. Neglect gravity.



[1 POINT]

1.3 Use Newton's law to show that the radial position of the particle satisfies the equation

$$\frac{d^2r}{dt^2} + 2c \frac{dr}{dt} - r\Omega^2 = 0$$

and find a formula for the constant  $c$  in terms of  $b, \eta, \rho$  (the factor of 2 in the equation is introduced to simplify the answer to 1.4).

$\mathbf{F} = m\mathbf{a}$  gives

$$-6\pi\eta b \frac{dr}{dt} \mathbf{e}_r + N\mathbf{e}_\theta = m\mathbf{a} = \frac{4\pi}{3} b^3 \left( \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + \frac{4\pi}{3} b^3 \left( r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta$$

We know  $\frac{d\theta}{dt} = \Omega$ . Rearranging the radial part of this equation gives

$$\begin{aligned} \frac{4\pi}{3} b^3 \left( \frac{d^2r}{dt^2} - r\Omega^2 \right) &= -6\pi\eta b \frac{dr}{dt} \\ \Rightarrow \frac{d^2r}{dt^2} + \frac{6\pi\eta b}{4\pi b^3 / 3} \frac{dr}{dt} - r\Omega^2 &= 0 \end{aligned}$$

Comparing coefficients gives

$$2c = \frac{6\pi\eta b}{4\rho\pi b^3 / 3} \Rightarrow c = \frac{9}{4} \frac{\eta}{\rho b^2}$$

[2 POINTS]

1.4 Assume the particle starts from a position  $r = r_0$  and has radial velocity  $dr/dt = 0$  at time  $t=0$ . Use Mupad to solve the differential equation (use **assume** ( $c > 0$ )). As a solution to this problem, just write down the answer from Mupad. You don't need to submit your Mupad script. (**optional** – check your answer by showing that it can be rearranged into the form

$$r(t) = \frac{r_0 \exp(-tc)}{\sqrt{\Omega^2 + c^2}} \left[ c \sinh\left(\sqrt{\Omega^2 + c^2} t\right) + \sqrt{\Omega^2 + c^2} \cosh\left(\sqrt{\Omega^2 + c^2} t\right) \right]$$

The mupad is shown below

```
[ reset() :
[ assume(c>0) : assume(m>0) :
[ diffeq := diff(diff(r(t),t),t) + 2*c*diff(r(t),t) - r(t)*`&Omega;`^2=0:
[ IC := r(0)=r0, r'(0)=0:
[ r(0) = r0, r'(0) = 0
[ rsol := simplify(solve(ode({diffeq,IC}, r(t)), IgnoreSpecialCases) [1])
[ 
$$\frac{r_0 e^{-t(c-\sigma_1)} (c+\sigma_1)}{2\sigma_1} - \frac{r_0 e^{-t(c+\sigma_1)} (c-\sigma_1)}{2\sigma_1}$$

[ where
[ 
$$\sigma_1 = \sqrt{\Omega^2 + c^2}$$

```

To simplify to the form given, note that

$$\cosh(x) = (\exp(x) + \exp(-x)) / 2 \quad \sinh(x) = (\exp(x) - \exp(-x)) / 2$$

[3 POINTS]

**1.5** Consider a separator with the following parameters: radius  $R_0 = 0.1m$ , angular speed  $\Omega = 125$  rad/s (about 1200 rpm), air viscosity  $\eta = 1.8 \times 10^{-5}$  Ns/m<sup>2</sup>, particle radius  $10^{-6}$  m (1 micron), density  $\rho = 1000 \text{ kg/m}^3$ . If the air initially contains a spatially uniform distribution of  $N$  particles per unit volume, what is the average particle density (in terms of  $N$ ) in the separator after 10 sec? (Hint: particles that start with large values of  $r_0$  will reach the outer wall of the separator and be removed. Particles close to the axis, with small  $r_0$  will not have time to reach the outer wall and will remain in the air. Start by calculating the initial position  $r_0$  of the particle that just reaches the outer wall in 10 sec)

In any given time, particles with initial radius greater than a critical value

$$r_0 \geq \frac{R_0 \sqrt{\Omega^2 + c^2}}{\exp(-tc) \left[ c \sinh(\sqrt{\Omega^2 + c^2} t) + \sqrt{\Omega^2 + c^2} \cosh(\sqrt{\Omega^2 + c^2} t) \right]}$$

will be removed. Here is the Mupad substitution

```

[ subs(rsol, {c=9*1.8*10^(-5)/4/1000/(1*10^(-6))^2, t=10, `&Omega;`=125})
  0.000002381479709 r0 e-810001.929 + 0.9999976185 r0 e1.929007752
[ solve(%=0.1, r0)
  {0.0145292639}

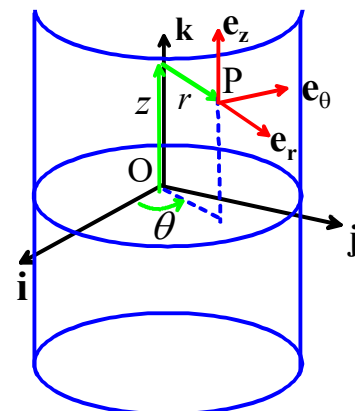
```

The number of particles per unit volume that remain in the air volume is  $Nr_0^2 / R_0^2$ , which gives  $0.021N$ .

[3 POINTS]

**2.** In this problem we calculate the trajectory of a particle through a cyclonic separator in more detail. As in the previous problem, we assume that the air contains spherical particles, with mass density  $\rho$  and radius  $b$ . This time we will analyze the trajectory of a representative particle in three dimensions, using a more realistic description of the air flow and drag forces.

**2.1** The position of a representative particle is specified by cylindrical polar coordinates  $(r, \theta, z)$ . Write down the velocity and acceleration of the particle in terms of these variables and their time derivatives (i.e. the polar coordinate formulas).



$$\mathbf{v} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta + \frac{dz}{dt} \mathbf{e}_z$$

$$\mathbf{a} = \left( \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + \left( r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta + \frac{d^2z}{dt^2} \mathbf{e}_z$$

[2 POINTS]

2.2 Assume that the particle is subjected to the following forces (i) a gravitational force, acting vertically downwards; and (ii) a drag force, which is given by

$$\mathbf{F}_D = 6\pi\eta b(\mathbf{v}^{air} - \mathbf{v}^{particle})$$

where  $\eta$  is the air viscosity,  $\mathbf{v}^{particle}$  is the velocity of the particle, and  $\mathbf{v}^{air}$  is the air velocity.

[Simulations](#) suggest that the air velocity in a separator can be approximated by

$$\mathbf{v}^{air} = u(r)\mathbf{e}_\theta$$

$$u(r) = \begin{cases} U_0 \frac{(r/R_0)^2 - (r/R_0)^{20}}{0.04 + (r/R_0)^3} & r < R_0 \\ 0 & r \geq R_0 \end{cases}$$

where  $R_0$  is the radius of the separator, and  $U_0$  is the flow speed. Show that the equations for  $(r, \theta, z)$  and their time derivatives  $(v_r, \omega, v_z)$  can be written in the MATLAB format

$$\frac{d}{dt} \begin{bmatrix} r \\ \theta \\ z \\ v_r \\ \omega \\ v_z \end{bmatrix} = \begin{bmatrix} v_r \\ \omega \\ v_z \\ -2cv_r + r\omega^2 \\ \{2c(u(r) - r\omega) - 2v_r\omega\} / r \\ -g - 2cv_z \end{bmatrix}$$

and give a formula for the constant  $c$

$\mathbf{F} = m\mathbf{a}$  gives

$$m \left( \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + m \left( r \frac{d^2\theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta + m \frac{d^2z}{dt^2} \mathbf{e}_z$$

$$= 6\pi\eta b \left\{ u(r)\mathbf{e}_\theta - \frac{dr}{dt} \mathbf{e}_r - r \frac{d\theta}{dt} \mathbf{e}_\theta - \frac{dz}{dt} \mathbf{e}_z \right\} - mg\mathbf{e}_z$$

Introduce  $v_r = \frac{dr}{dt}$      $\omega = \frac{d\theta}{dt}$      $v_z = \frac{dz}{dt}$ , and collect the components of the basis vectors:

$$\frac{dv_r}{dt} = r\omega^2 - \frac{6\pi\eta b}{m}v_r$$

$$\frac{d\omega}{dt} = \frac{1}{r} \left\{ -2v_r\omega + \frac{6\pi\eta b}{m}(u(r) - r\omega) \right\}$$

$$\frac{dv_z}{dt} = -g - \frac{6\pi\eta b}{m}v_z$$

Setting  $m = 4\pi\rho b^3 / 3$  gives

$$2c = \frac{6\pi\eta b}{4\pi\rho b^3 / 3} \Rightarrow c = \frac{9}{4} \frac{\eta}{\rho b^2}$$

[3 POINTS]

2.3 Write a MATLAB script that will solve the equations derived in problem 2.2 (use a conditional statement to handle the function  $u(r)$  in the formula for air velocity). Plot the trajectory of the particle (in  $x, y, z$  coordinates – your script will need to convert the polar coordinates to  $x, y, z$ ) Use the following parameters:  $R_0 = 0.1m$ ,  $U_0 = 10m/s$ ,  $\eta = 1.8 \times 10^{-5} \text{ Ns/m}^2$ ,  $\rho = 500kg/m^3$ . Start the particle from  $r = 0.01, \theta = 0, z = 0$ , with velocity  $\mathbf{v} = \mathbf{0}$  at  $t = 0$ . Plot trajectories (separately) for particles with 5, 10 and 20 micron radii (a micron is  $10^{-6} \text{ m}$ ) (if you are curious you can find a table of sizes for common particles [here](#)). Run your simulations for 50secs. Submit your MATLAB code as a solution to this problem.

See MATLAB code for solution

[5 POINTS]

3. This problem analyzes ‘mirror trapping’ of charged particles by the earth’s magnetic field. Some background information is needed to set up the problem (don’t worry if these equations are unfamiliar... electricity & magnetism will be discussed in more detail in ENGN0510 next semester):

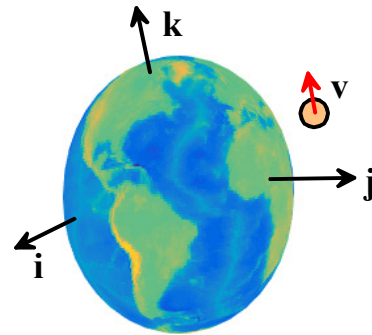
- A particle with charge  $q$  moving with velocity  $\mathbf{v}$  through a magnetic field with intensity  $\mathbf{B}$  experiences a force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

- The earth’s magnetic field can be approximated by

$$\mathbf{B} = -B_0 \frac{R_e^3}{R^5} (3xz\mathbf{i} + 3yz\mathbf{j} + (2z^2 - x^2 - y^2)\mathbf{k}) \quad R = \sqrt{x^2 + y^2 + z^2}$$

where  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  is a basis with origin at the center of the earth, and  $\mathbf{k}$  pointing parallel to magnetic North,  $R_e$  is the earth’s radius, and  $B_0$  is the magnetic field strength at the earth’s surface at the equator.



3.1 Let  $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$  denote the velocity of a charged particle. Use Newton’s law (neglect gravity) to show that the equations of motion for the particle can be expressed in the Matlab form

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ qB_0 R_e^3 (-v_y(2z^2 - x^2 - y^2) + 3v_z yz) / (mR^5) \\ qB_0 R_e^3 (v_x(2z^2 - x^2 - y^2) - 3v_z xz) / (mR^5) \\ -3qB_0 R_e^3 (v_x yz - v_y xz) / (mR^5) \end{bmatrix}$$

$\mathbf{F}=\mathbf{ma}$  gives

$$-(v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}) \times qB_0 \frac{R_e^3}{R^5} (3xz \mathbf{i} + 3yz \mathbf{j} + (2z^2 - x^2 - y^2) \mathbf{k}) = m \left( \frac{dv_x}{dt} \mathbf{i} + \frac{dv_y}{dt} \mathbf{j} + \frac{dv_z}{dt} \mathbf{k} \right)$$

$$\Rightarrow \frac{dv_x}{dt} = \frac{qB_0 R_e^3}{m R^5} (-v_y(2z^2 - x^2 - y^2) + v_z 3yz)$$

$$\frac{dv_y}{dt} = \frac{qB_0 R_e^3}{m R^5} (v_x(2z^2 - x^2 - y^2) - v_z 3xz)$$

$$\frac{dv_z}{dt} = -3 \frac{qB_0 R_e^3}{m R^5} (v_x yz - v_y xz)$$

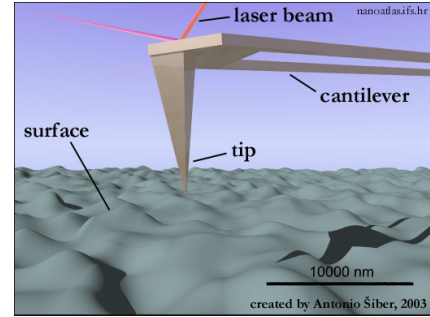
This, together with the position-velocity relation gives the answer stated.

**[3 POINTS]**

**3.2** Write a MATLAB script that will solve the equations listed in 3.1. Use your code to predict (and plot) the trajectory of a proton with the following parameters:  $m = 1.673 \times 10^{-27}$  kg,  $q = 1.6022 \times 10^{-19}$  Coulombs,  $B_0 = 3.07 \times 10^{-5}$  Tesla,  $R_e = 6378$  km. Start the particle at  $x = 4R_e$ ,  $y = z = 0$ , with initial velocity:  $v_x = 0$ ,  $v_y = 7.7 \times 10^6$  m/s,  $v_z = 13.3 \times 10^6$  m/s. Run your simulation for 500sec. Use (eg) `options = odeset('RelTol', 0.00001);` to improve the accuracy of your simulation. Submit your MATLAB code on Canvas as a solution to this problem.

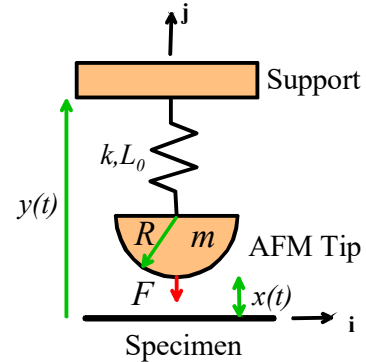
**[5 POINTS]**

4. An atomic force microscope consists of a probe with a sharp tip that is mounted at the tip of a flexible cantilever. The device detects the forces that act between the probe tip and a specimen. By measuring changes in these forces as the tip scans over the surface, it is possible to reconstruct high-resolution images of the surface topography.



Several different methods are used to detect the surface forces. In ‘tapping mode,’ the tip is vibrated just above the surface, and small changes in the vibration response are used to construct an image of the surface. The goal of this problem is to predict how surface forces influence the vibration of an AFM tip.

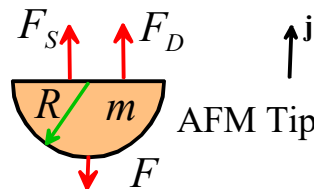
For design calculations, the cantilever/tip assembly is usually idealized as a spring-mass system, as shown in the figure. The tip has mass  $m$  and radius  $R$ , and at time  $t$  is a height  $x(t)$  above the surface. The spring has stiffness  $k$  and unstretched length  $L_0$ . The top of the spring is moved vertically by a high-frequency actuator and has height  $y(t) = \Delta + Y_0 \sin \omega t + R + L_0$  above the surface.



The tip is subjected to: (i) a force from the spring; (ii) an attractive surface force  $F$ , which is related to  $x$  by  $F = F_0 \frac{x}{d} \exp(-x/d)$ ; and

(iii) a drag force resisting motion of the tip  $\mathbf{F}_D = -c \frac{dx}{dt} \mathbf{j}$

4.1 Draw a free body diagram showing the forces acting on the AFM tip.



[2 POINTS]

4.2 Use Newton’s law to show that the equation of motion for  $x(t)$  can be arranged in the form

$$\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ \{k(\Delta + Y_0 \sin \omega t) - kx - cv - F\} / m \end{bmatrix} \quad F = F_0 \frac{x}{d} \exp(-x/d)$$

Newton’s law and the spring force law gives

$$m \frac{d^2 x}{dt^2} = k(y - x - R - L_0) - c \frac{dx}{dt} - F$$

Introduce  $v = dx / dt$ , and substitute for  $y$  to get the result needed.

[3 POINTS]

4.3 Write a MATLAB script to solve the equations in 4.2. Use the following values for the parameters:

- $k=40\text{N/m}$ ,
- $m= 8.27 \times 10^{-12} \text{ kg}$ ,
- $c = 2 \times 10^{-6} \text{ Ns/m}$ ,
- $F_0 = 10^{-7} \text{ N}$ ,
- $d = 5 \times 10^{-9} \text{ m}$ ,
- $Y_0 = 1 \times 10^{-9} \text{ m}$ ,
- $\omega = 2.4 \times 10^6 \text{ rad/s}$ .

Use your code to plot the cantilever tip deflection  $\delta(t) = x(t) - \Delta$  as a function of time, for a time interval  $0 < t < 10^{-4} \text{ s}$ .

Show results for  $\Delta = 10 \times 10^{-9} \text{ m}$  and  $\Delta = 40 \times 10^{-9} \text{ m}$ , and use initial conditions  $x(0) = \Delta, v(0) = 0$  for both cases.

Upload your MATLAB code to Canvas as a solution to this problem.

Note that the vibration amplitude changes with  $\Delta$ : if this can be detected, the gap between the tip and surface can be determined without touching it with the AFM tip. In practice, commercial AFMs use a feedback control loop to adjust  $\Delta$  to maintain a constant vibration amplitude.

**[5 POINTS]**