School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Conservation Laws for Particles
Due Friday March 3, 2017

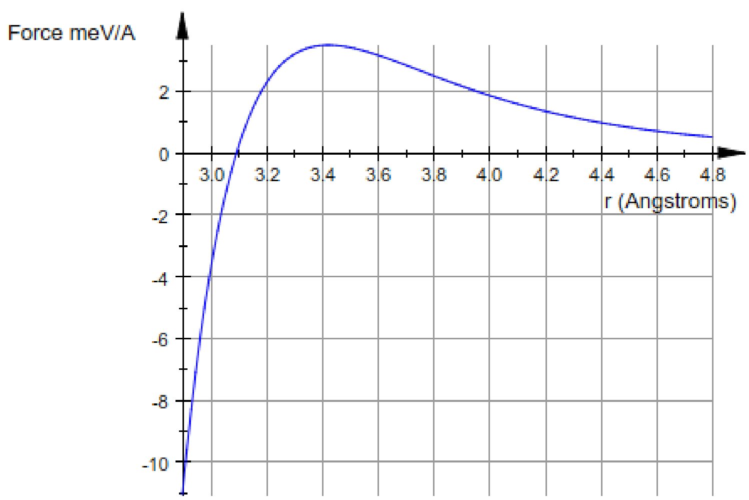
1. This publication proposes an interatomic potential (called the 'Improved Lennard Jones' potential) to characterize forces acting between inert gas molecules. As discussed in class, the potential specifies the potential energy V of the attractive force between two atoms as a function of the distance r between them. The potential has the form

$$V = \varepsilon\left(\frac{m}{n(r)-m}\left(\frac{r_m}{r}\right)^{n(r)} - \frac{n(r)}{n(r)-m}\left(\frac{r_m}{r}\right)^{m}\right) \qquad n(r) = \beta + 4\left(\frac{r}{r_m}\right)^2$$

Where $\varepsilon, m, \beta, r_m$ are constants that can be chosen to describe forces acting between different molecules.

For the Ne-Ne bond the authors propose $m=6$, $\beta = 9$, $\varepsilon = 3.66$ meV [1], $r_m = 3.094\overset{o}{\mathrm{A}}$ (Angstroms) (1 Angstrom is 10^{-10}m)

1.1 Use Mupad to find a formula for the force, and hence plot a graph of the force-v-separation curve for two Neon atoms for separations r in the range $2.9 < r < 4.8\overset{o}{\mathrm{A}}$. Submit a copy of your graph (there is no need to upload the Mupad code).



[2 POINTS]

1.2 Calculate the stiffness of the Ne-Ne bond (i.e. $k = dF/dr$) at the equilibrium separation $(r = r_m)$.

Give your answer in both units of $meV/(\overset{o}{\mathrm{A}})^2$ and N/m.

$$k = 29.8 \;\; meV/(\overset{o}{\mathrm{A}})^2 = 0.478 N/m \text{ (see the Mupad code for details)}$$

[2 POINTS]

[1] 1 eV=1.60218e-19 Joules. meV is milli-electron volts – there are 1000 meV in 1 eV.

1.3 Find the force required to separate two Ne atoms. To do this you will need to find the maximum value of the force – you can't find a formula for this, but can use the Mupad `numeric::solve()` function to find the value. Give your answer in $meV/(\overset{o}{\mathrm{A}})$ and Newtons.

$$F_{\max} = 3.49 meV/(\overset{o}{\mathrm{A}}) = 5.59 \times 10^{-12} N$$ (see the Mupad code for details)

[2 POINTS]

2. The Revelstoke mountain resort has a ski run that is served by a Leitner-Pomona high-speed chairlift with the following specifications:

- Max capacity: 3000 passengers/hr
- Max speed 2.8 m/s
- Vertical lift: 632m
- Ride time: 7 mins.
- Chair capacity: 4 persons

Using a passenger mass of 80kg, estimate:

2.1 The number of passengers on the lift at max capacity

1 hour consists of 60/7 seven minute trips. Since 3000 passengers are carried in this time the number of passengers per trip is 350. (This is presumably passengers riding the chairs going up – if they all ride back down again the number would be 700. Both answers should get credit)

[2 POINTS]

2.2 The spacing between chairs on the lift

The length of the run is 2.8 m/s x 7 x 60 = 1176m. 350 passengers are on the ride, with four per chair. If the chairs are equally spaced, the spacing between chairs is therefore (1176/350) x 4=13m (about 43ft) which is reasonable.

[1 POINT]

2.3 The total kinetic energy of lift passengers at max capacity

The KE per passenger is $\frac{1}{2}mv^2 = 40(2.8)^2 = 313.6J$.

The total KE is 350 times this, i.e. 109.76kJ (double this value is OK as well)

[1 POINT]

2.4 The power required to operate the chair at maximum capacity

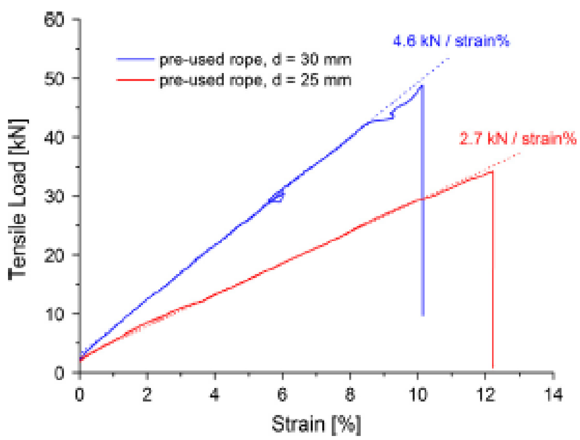
The PE cost per passenger is *mgh*=80*10*632=505.6 kJ.

The power is (PE cost + KE cost)*passengers/time, which gives 505.91*3000/3600 = 421.6 kW

GRADERS – ESTIMATES FOR ALL PARTS OF THIS PROBLEM WILL VARY PLEASE AWARD POINTS FOR METHOD, NOT THE NUMERICAL ANSWER

[2 POINTS]

This is a significant power consumption – enough that ski resorts find it economical to supplement grid power with renewable energy conversion to try to reduce costs and emissions.



3. The figure (from an accident investigation in this publication) shows the force –v- strain (% change in length) curves for two marine ropes.

3.1 Estimate the total energy stored in a 4m length of each rope at the point of failure.

The energy can be calculated in several different ways - eg find the area under the curves, or use the slopes given in the figure.

The slopes give the force per %strain: the stiffness (force/change in length) is therefore k = (force per %strain)x100/length

The energy stored can then be found using the spring force law

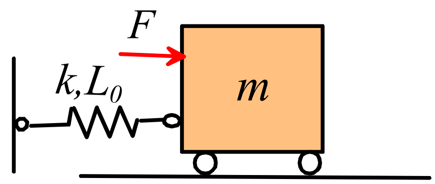
$$W = \frac{1}{2}kx^2 = \begin{cases} \frac{1}{2}4600 \times \frac{100}{4} \times (0.1 \times 4)^2 = 9.2kJ & 30mm\ diameter\ rope \\ \frac{1}{2}2700 \times \frac{100}{4} \times (0.121 \times 4)^2 = 7.9kJ & 25mm\ diameter\ rope \end{cases}$$

You can also do the problem graphically – the work is related to the area under the curve by $W = AL_0$ (since the area gives the work done per unit unstretched length).

GRADERS – ESTIMATES WILL VARY PLEASE AWARD POINTS FOR METHOD, NOT THE NUMERICAL ANSWER

[2 POINTS]

3.2 Assume that the rope can be idealized as a spring, and the boat as a mass (see the figure). The system is at rest at time $t=0$ with no force in the spring. The wind then exerts a constant force F on the ship. Using energy methods, find an expression for the maximum force in the spring (rope) during the subsequent motion, in terms of F.



The system is conservative – we can regard the wind as an external force. The energy conservation equation gives

$$W_{ext} = (T_1 + V_1) - (T_0 + V_0)$$

At $t=0$ both KE and PE are zero; at the instant of maximum force the mass is stationary, so KE is zero. The PE of the spring is $(kx^2/2)$ If the boat moves a distance d the work done by F is $W_{ext} = Fd$
Therefore:

$$Fd = \frac{1}{2}kd^2$$

The force in the spring is kd so

$$Fd = \frac{1}{2}F_s d \Rightarrow F_s = 2F$$

[2 POINTS]

3.3 The investigation estimated the wind load at 40kN. Assuming the ship is moored by two parallel ropes, determine whether a 40kN wind load would be expected to break the ropes.

One rope will be subjected to twice the wind load, two will share the load and so will experience just the wind load (40kN).

From the graph, the thick and thin rope fail at forces of 46 and 32kN, respectively. The thicker rope should survive (although the safety factor is small), the thinner one will break.

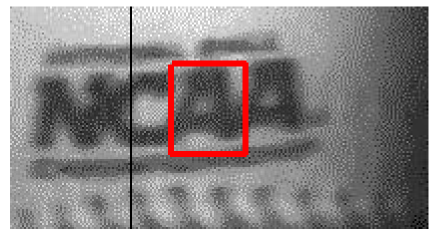
[1 POINT]

4. The Baseball Research Institute has posted a nice high-speed movie of the impact of a baseball against a rigid surface. In this problem you will use their data to estimate the forces acting on the ball during the impact, the restitution coefficient, and the impulse exerted on the baseball during the impact.

You will need to
1. Download the movie file baseball.avi from the EN40 Homework webpage
2. Download the Matlab script track_baseball.m from the EN40 Homework webpage.

Save both files in the same directory.

Then run the script to create a graph and a csv file of position –v- time for the baseball. The script will ask you to



1. Click on two diametrically opposed points on the baseball. The script counts the pixels between these points and uses the known baseball diameter to determine the number of pixels per cm.
2. Select a rectangular region near the center of the baseball with a well-defined pattern that can be detected in subsequent images. The NCAA letters are a good choice (if you have time, you can experiment with different reference images until you get good results)
3. Click on a point inside the rectangle that you would like to track in subsequent images.

The script will plot a graph of the horizontal position of your reference point (in cm, measured from the left of the image) as a function of time. The data will be saved in a csv file that you can read in your own code for further analysis (you can read a csv file using `data = csvread('filename.csv');`)

4.1 Write a MATLAB script to calculate and plot graphs of the velocity and acceleration of the ball during the test. You will need to differentiate the position –v- time data: you can do this by calculating the change in position between two successive readings, and dividing by the time difference between them, e.g. if $x(i)$ denotes the ith value of x, then

$$v_x(i) = \left(x(i) - x(i-1)\right) / \left(t(i) - t(i-1)\right)$$

The data will be noisy: you can use a simple first-order filter to smooth it, as follows: Let **y** be a vector (a list of numbers, eg velocity) that needs to be filtered. A vector **z** containing the filtered signal (a second list of numbers) can be constructed as follows:

$$z_1 = y_1$$
$$z_i = \alpha y_i + (1-\alpha) z_{i-1} \qquad i = 2,3,4....n$$

where $0 < \alpha < 1$ is a parameter that controls the cutoff frequency of the filter ($\alpha = 0.5$ works for the data in this problem, but you can try other values), and y_i, z_i denotes the *ith* value of **y** and **z**, respectively, and n is the length of the vector **y**.

Please upload your Matlab script to Canvas as a solution to this problem.

See the MATLAB code for solution

[4 POINTS]

4.2 Use the data to estimate the restitution coefficient for the ball

The velocity after imact is about 31m/s. Before impact it is about 51 m/s. The restitution coefficient is therefore 31/51=0.6.

Values will vary depending on points used for camera calibration, how the velocity is estimated.

[2 POINTS]

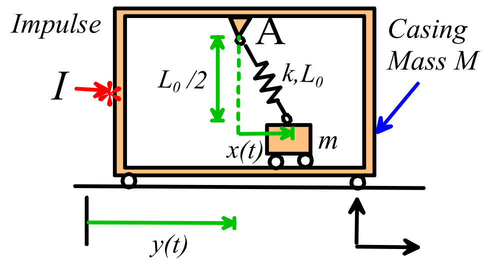
4.3 Calculate the impulse exerted on the ball during the impact, by (i) using its change in momentum; and (ii) using your MATLAB code from 4.1 to calculate and integrate the force acting on the ball during the collision (use the matlab 'trapz' function to integrate the force). Assume the baseball mass is 0.145kg. Why are the two values not exactly equal? Which is likely to be more accurate?

See MATLAB code

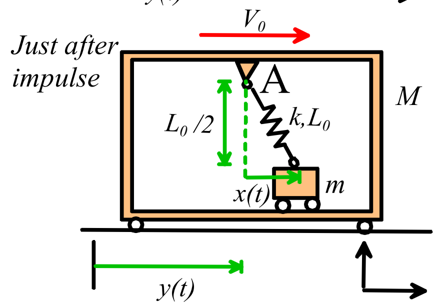
[1 POINT]

GRADERS – ESTIMATES FOR ALL PARTS OF THIS PROBLEM WILL VARY PLEASE AWARD POINTS FOR METHOD, NOT THE NUMERICAL ANSWER

5. The figure shows a device that is intended to detect an impulse. If the casing is subjected to an impulse that exceeds a critical magnitude, the mass will flip from its initial position to the right of the pivot to a new stable position to the left of the pivot at A. The goal of this problem is to calculate the critical value of impulse for which this will occur.



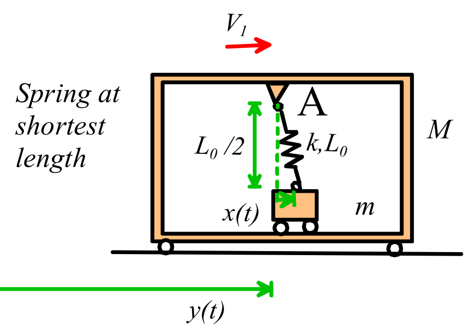
5.1 At time $t=0$ the system is at rest and the spring is un-stretched. The casing is then subjected to a horizontal impulse I. Write down a formula for the speed of the casing just after the impulse. Note that the spring exerts no force on either the casing or the mass m during the impulse.



$$MV_0 = I \Rightarrow V_0 = I/M$$

[1 POINT]

5.2 Find expressions for the total linear momentum and total kinetic energy of the system just after the impulse, in terms of I and the mass M of the casing.



$$T = \frac{1}{2}MV_0^2 = \frac{1}{2M}I^2$$
$$\mathbf{p} = MV_0\mathbf{i} = I\mathbf{i}$$

[2 POINTS]

5.3 Consider the system at the instant when the spring reaches its shortest length (assume $x>0$). Using energy and/or momentum conservation show that at this instant

$$x = \sqrt{\left(L_0 - I\sqrt{\frac{m}{kM(M+m)}}\right)^2 - \frac{L_0^2}{4}}$$

(You can find the spring length using Pythagoras' theorem)

- At this instant both the casing and the proof mass move with the same (unknown) speed v_1 .
- Momentum conservation requires that $(m+M)v_1 = I$
- Energy conservation requires that $\frac{1}{2M}I^2 = \frac{1}{2}(M+m)v_1^2 + \frac{1}{2}k\left\{L_0 - \sqrt{x^2 + L_0^2/4}\right\}^2$

  Here the first term on the right is the KE, the second is the PE of the spring. We used $L_0 - \sqrt{x^2 + L_0^2/4}$ instead of $\sqrt{x^2 + L_0^2/4} - L_0$ for convenience (it makes the quantity inside the $\{\ \}$ positive)

- Eliminate v_1 :

$$\frac{1}{2}\left(\frac{1}{M}-\frac{1}{M+m}\right)I^2=\frac{1}{2}k\left\{L_0-\sqrt{x^2+L_0^2/4}\right\}^2$$

$$\Rightarrow x=\sqrt{\left(L_0-I\sqrt{\frac{m}{kM(M+m)}}\right)^2-\frac{L_0^2}{4}}$$

[5 POINTS]

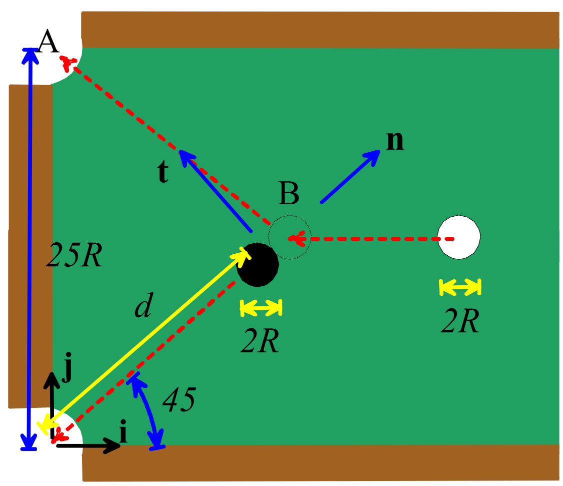
5.4 Hence, find a formula for the critical value of *I* that will flip the mass past *x=0*.

- Set *x=0* in the preceding problem and solve for *I:* $I=\frac{L_0}{2}\sqrt{kM\left(\frac{M}{m}+1\right)}$

[2 POINTS]

6. The goal of this problem is to set up a pool shot that sinks both the black and white balls shown in the figure. It is given that:

- Both balls have mass m.
- The collision has restitution coefficient *e*=0.75
- Friction can be neglected
- The black ball will lie on a 45 degree line from the bottom corner pocket
- The white ball will have initial velocity $\mathbf{v}^{W0}=-V_0\mathbf{i}$



The goal is to find a formula for the distance *d* (in terms of the ball radius *R*) that will ensure that the white ball will go in the top corner pocket.

6.1 Write down the initial velocity of the white ball in the $\mathbf{n},\mathbf{t}$ basis (i.e. find v_t^{W0}, v_n^{W0} such that $\mathbf{v}^{W0}=v_t^{W0}\mathbf{t}+v_n^{W0}\mathbf{n}$)

Note that we can write the **i** vector as $(\mathbf{n}-\mathbf{t})/\sqrt{2}$, so $\mathbf{v}^{W0}=-V_0(\mathbf{n}-\mathbf{t})/\sqrt{2}$

[1 POINT]

6.2 Write down the total initial linear momentum of the system, in both the $\mathbf{i}, \mathbf{j}$ and $\mathbf{n}, \mathbf{t}$ bases shown in the figure.

The linear momentum is just m**v**, so
$\mathbf{p}^0 = mV_0\mathbf{i} = mV_0(\mathbf{t}-\mathbf{n})/\sqrt{2}$

[1 POINT]

6.3 Which of the following statements are true?

(a) Momentum of the entire system in the **n** direction is conserved during the collision
(b) Momentum of the entire system in the **t** direction is conserved during the collision
(c) Momentum of the black ball in the **n** direction is conserved during the collision
(d) Momentum of the white ball in the **n** direction is conserved during the collision
(e) Momentum of the black ball in the **t** direction is conserved during the collision
(f) Momentum of the white ball in the **t** direction is conserved during the collision

We answer this with the impulse momentum formula $\mathbf{I} = \mathbf{p}_1 - \mathbf{p}_0$ (either for a system or for a single particle) Momentum is conserved (i.e. **p**=constant) whenever **I** is zero (i.e. if no impulse acts on a particle, or no *external* impulse acts on a system.

(a), (b), (e) and (f) are all true. For (a) (b) - Total momentum is conserved in all directions because no external impulse acts on the system during collision (the force exerted by one ball by the other is an *internal* force and is not counted). For (c) (d), (e) (f) we note that no forces act on either ball during the collision in the **t** direction (no friction), so momentum of each ball is conserved in the **t** direction. Momentum is not conserved for either ball in the **n** direction because each experiences an impulse in this direction during the collision. The total momentum of the system is conserved in the **n** direction, however.

[2 POINTS]

6.4 Let $\mathbf{v}^{B1} = v_n^{B1}\mathbf{n} + v_t^{B1}\mathbf{t}$ $\quad \mathbf{v}^{W1} = v_n^{W1}\mathbf{n} + v_t^{W1}\mathbf{t}$ denote the velocities of the black and white balls after collision. Use the answers to 6.1 and 6.2 to write down v_t^{W1}, v_t^{B1} , in terms of V_0

Since the momentum of each ball is conserved in the **t** direction, their **t** components of velocity do not change. Hence

$$v_t^{W1} = \frac{1}{\sqrt{2}}V_0, \qquad v_t^{B1} = 0$$

[1 POINT]

6.5 Use momentum in the **n** direction to write down an equation relating v_n^{W1}, v_n^{B1} and V_0

Total momentum conservation in the **n** direction gives
$mv_n^{W1} + mv_n^{B1} = -m\frac{1}{\sqrt{2}}V_0$

[1 POINT]

6.6 Use the restitution formula in the **n** direction to write down a second equation relating v_n^{W1}, v_n^{B1} and V_0

The restitution formula in the normal direction is the same as the 1-D version:

$$v_n^{W1} - v_n^{B1} = -e(v_n^{W0} - v_n^{B0}) \Rightarrow v_n^{W1} - v_n^{B1} = e\frac{1}{\sqrt{2}}V_0$$

[1 POINT]

6.7 Use 6.4 – 6.6 to solve for $\mathbf{v}^{B1} = v_n^{B1}\mathbf{n} + v_t^{B1}\mathbf{t} \qquad \mathbf{v}^{W1} = v_n^{W1}\mathbf{n} + v_t^{W1}\mathbf{t}$

Divide 6.5 by m and add to 6.6: $2v_n^{W1} = (e-1)\frac{1}{\sqrt{2}}V_0$

Divide 6.5 by m and subtract 6.6: $2v_n^{B1} = -(e+1)\frac{1}{\sqrt{2}}V_0$

Hence $\mathbf{v}^{B1} = -\frac{1}{2\sqrt{2}}(1+e)V_0\mathbf{n} \qquad \mathbf{v}^{W1} = -\frac{1}{2\sqrt{2}}(1-e)V_0\mathbf{n} + \frac{1}{\sqrt{2}}V_0\mathbf{t}$

[2 POINTS]

6.8 Check your answer for $\mathbf{v}^{W1}$ in 6.7 using the general solution derived in class

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Substituting values gives

$$\mathbf{v}^{W1} = \frac{1}{\sqrt{2}}V_0(\mathbf{t}-\mathbf{n}) + \frac{1}{2}(1+e)\left[\left(\mathbf{0} - \frac{1}{\sqrt{2}}V_0(\mathbf{t}-\mathbf{n})\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{W1} = \frac{1}{\sqrt{2}}V_0\mathbf{t} + V_0\left(\frac{1}{2\sqrt{2}}(1+e) - \frac{1}{\sqrt{2}}\right)\mathbf{n} = \frac{1}{\sqrt{2}}V_0\mathbf{t} + V_0\frac{1}{2\sqrt{2}}(e-1)\mathbf{n}$$

[2 POINTS]

6.9 Suppose that the center of the black ball is a distance *d* (to be determined) from the bottom left pocket. Using geometry (eg use vector subtraction), find a formula for the vector AB along the required direction of travel of the white ball after collision. Express your answer in $\mathbf{n}, \mathbf{t}$ basis.

The required direction is parallel to the vector BA – we can construct this as

$$\mathbf{r}_A - \mathbf{r}_B = \frac{25}{\sqrt{2}} R(\mathbf{t} + \mathbf{n}) - (d + 2R)\mathbf{n} = \frac{25}{\sqrt{2}} R\mathbf{t} + (\left(\frac{25}{\sqrt{2}} - 2\right) R - d)\mathbf{n}$$

[1 POINT]

6.10 Hence, find the distance *d* in terms of *R*.

$\mathbf{v}^{W1}$ must be parallel to $\mathbf{r}_A - \mathbf{r}_B$ so the cross product of these two vectors must be zero. This gives

$$\left(\mathbf{t} + \frac{1}{2}(e-1)\mathbf{n}\right) \times \left\{ \frac{25}{\sqrt{2}} R\mathbf{t} + (\left(\frac{25}{\sqrt{2}} - 2\right) R - d)\mathbf{n} \right\} = \left(\frac{25}{\sqrt{2}} - 2\right) R - d - \frac{25R}{2\sqrt{2}}(e-1) = 0$$

$$\Rightarrow d = \left(\frac{75 - 25e}{2\sqrt{2}} - 2\right) R = 17.8874R$$

[3 POINTS]

6.11 **Optional (no credit)** Check your answer by downloading the MATLAB p-code from the HW website. Run the code by

- opening Matlab,
- navigating to the directory with the downloaded code, and then
- typing check_hw4_p6(*number*), in the Matlab command window, w here *number* is your value of *d/R* from problem 6.9. You will see an animation of your shot (the white ball will drop into the pocket for a narrow range of values close to the exact correct answer. The black one will always be sunk, of course).