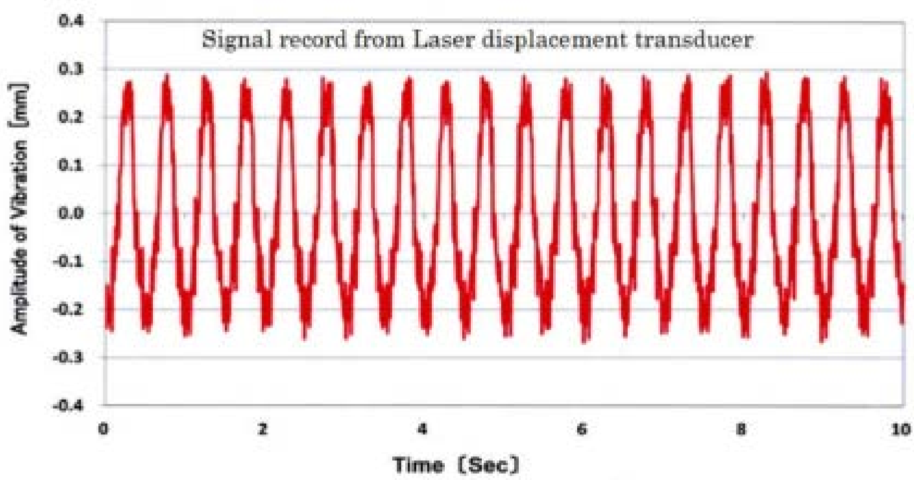
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday March 24, 2017



1. The figure (from this publication) shows a vibration measurement from a displacement transducer attached to a bridge. Calculate:
1.1 The amplitude of vibration

The amplitude is about 0.55mm peak-peak, which gives $A=0.28$mm amplitude

[1 POINT]

1.2 The period of the vibration

There are 20 cycles in 10sec so period is 0.5sec

[1 POINT]

1.3 The frequency (in Hertz) and angular frequency (in rad/s)
Frequency is 2 Hertz (cycles per sec), or 4π rad/s

[1 POINT]

1.4 The amplitude of the velocity

The amplitude of the velocity follows as $\Delta v = \Delta X\, \omega = 3.5 mm/s$

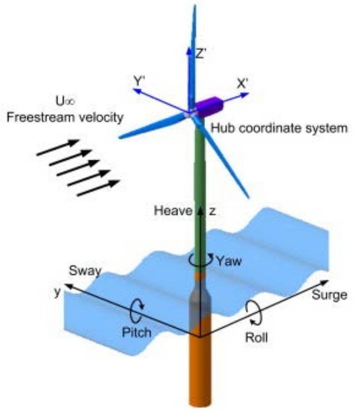
[1 POINT]

1.5 The amplitude of the acceleration.

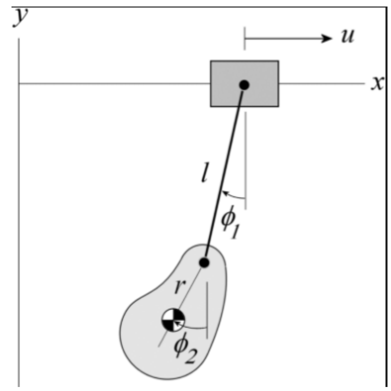
The amplitude of the acceleration is $\Delta a = \Delta X\, \omega^2 = 44 mm/s^2$

[1 POINT]

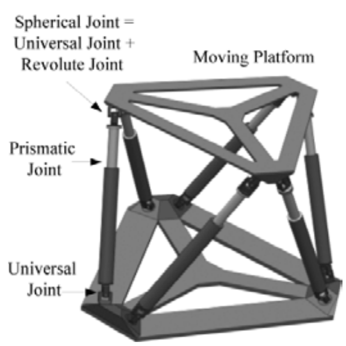
2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures



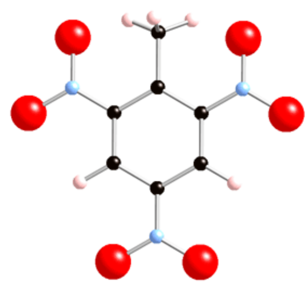
Floating wind turbine



2D Idealization of an overhead crane



Stewart platform (the base is fixed)



TNT Molecule

For (a), the assembly has 3 translational and 3 rotational DOF, and the prop can spin about its axis, so 7 total. Alternatively 2 rigid bodies (12 DOF) and 5 constraints at the axle gives 7 DOF. Counting vibration modes for this system is a bit tricky - presumably it is anchored, and the water keeps it upright, and something keeps it facing the wind, which means there can't be any rigid body modes. This would give 7 vibration modes (if the prop is not balanced). You would get 1 rigid body mode (the prop spinning about its axis) if it is balanced – giving 6. (also OK to assume the generator rotates on the mast – this would give 1 more DOF and 1 more vibratin mode.

For (b) 3 DOF since the coordinates are drawn on the figure! Or 3 rigid bodies, 6 constraints, 2D problem gives # DOF = 9-6=3 There is one rigid body mode (horizontal translation) so 2 vibration modes.

For (c) 13 rigid bodies (two for each of the members, plus the platform on top, 3 constraints at each revolute joint, 4 constraints at each universal joint, 5 constraints at each of the prismatic joints gives 3*6+4*6+30=72 constraints. Hence DOF = 6*13-72=6. The point of the mechanism is to be able to position the platform with any orientation or position, which is 6 DOF. This system has no rigid body modes so 6 vibration modes

For (d) we have 21 atoms, so 63DOF. 6 rigid body modes, so 57 vibration modes.

[4 POINTS]

3. Solve the following differential equations

3.1 $\dfrac{d^2y}{dt^2}+16y=4 \qquad y=\dfrac{1}{4} \qquad \dfrac{dy}{dt}=-8 \qquad t=0$

We can re-write this as a case-I equation

$$\frac{1}{4^2}\frac{d^2y}{dt^2}+y=\frac{1}{4}$$
$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+y=C$$

Comparing the equations shows that $C=1/4 \qquad \omega_n=4$. The solution is

$$x(t)=C+(x_0-C)\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t$$

We are given $x_0=1/4 \qquad v_0=-8$ so

$$y(t)=\frac{1}{4}-2\sin 4t$$

[3 POINTS]

3.2 $\dfrac{d^2y}{dt^2}+6\dfrac{dy}{dt}+9y=18\sin 3t+36\cos 3t \qquad y=0 \qquad \dfrac{dy}{dt}=0 \qquad t=0$

We can rearrange this as a Case 5 equation

$$\frac{1}{3^2}\frac{d^2y}{dt^2}+\frac{2}{3}\frac{dy}{dt}+y=2\left(\sin 3t+\frac{2}{3}3\cos 3t\right)$$
$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=K\left(\sin 3t+\frac{2\zeta}{\omega_n}\frac{d}{dt}\sin 3t\right)$$

It appears that $\omega=3,\ K=2,\ \omega_n=3,\ \zeta=1$.

The steady-state solution follows as

$$x_p(t)=X_0\sin(3t+\phi)$$
$$X_0=\frac{2\left\{1+2^2\right\}^{1/2}}{\left\{2^2\right\}^{1/2}}=\sqrt{5} \qquad \phi=\tan^{-1}\left(\frac{-2}{4}\right)=-0.4636$$

The homogeneous solution is

$$x_h(t)=\left\{x_0^h+\left[v_0^h+\omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

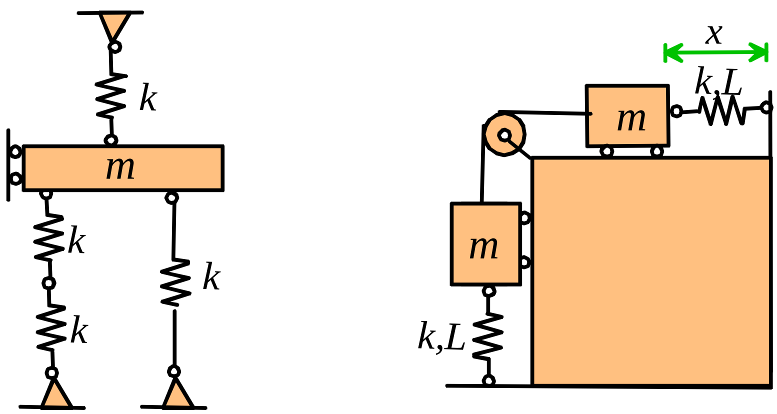
$$x_0^h = x_0 - C - X_0 \sin\phi = \sqrt{5}\sin\left(\tan^{-1}\frac{1}{2}\right) = 1$$

$$v_0^h = v_0 - X_0\omega\cos\phi = -3\sqrt{5}\cos\left(\tan^{-1}\frac{1}{2}\right) = -6$$

The total solution is therefore $\sqrt{5}\sin(3t - 0.4636) + \{1 - 3t\}\exp(-3t)$

[3 POINTS]

4. Find formulas for the natural frequency of vibration for the systems shown in the figure



For the first system we can replace the springs with an equivalent single spring and use the standard result for a spring-mass system. The two springs in series have combined stiffness $k/2$. This spring is in parallel with two springs with stiffness k so the effective stiffness is $5k/2$. The natural frequency is therefore

$$\omega_n = \sqrt{\frac{5k}{2m}}$$

[1 POINT]

For the second system we can use energy conservation to get the EOM. Note we can't directly write down the length of the vertical spring, or the height of the mass, but this doesn't matter (it will not influence the natural frequency).

$$T + V = 2 \times \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2}k(x - L)^2 + \frac{1}{2}k\left(D - x - L\right)^2 + mg(D - x)$$

where D is a constant (the length of the spring when $x=0$). We can take the time derivative:

$$\frac{d}{dt}(T + V) = 2m\left(\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + k(x - L)\frac{dx}{dt} - k\left(D - x - L\right)\frac{dx}{dt} - mg\frac{dx}{dt} = 0$$

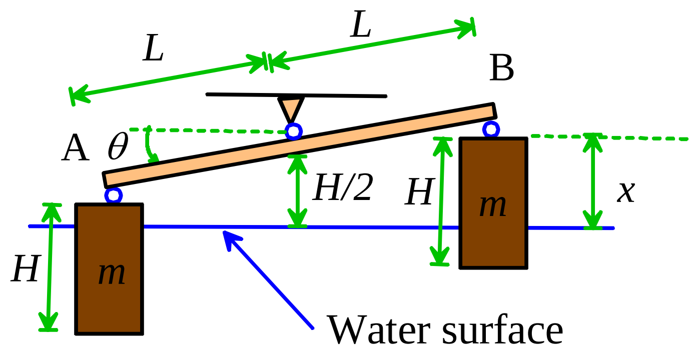
$$\Rightarrow 2m\frac{d^2x}{dt^2} + 2kx = -kD + mg$$

$$\Rightarrow \frac{m}{k}\frac{d^2x}{dt^2} + x = -\frac{D}{2} + \frac{mg}{2k}$$

Thus $\omega_n = \sqrt{k/m}$

[3 POINTS]

5. The figure shows a schematic diagram of a wave energy harvesting device. It consists of two cylindrical buoys with cross-sectional area A, height H and mass m, attached to a bar with total length $2L$ and negligible mass. The buoys are immersed in water with mass density ρ.



During operation waves cause the bar AB to rock through a time-varying angle θ as shown. When $\theta = 0$ both buoys are half submerged $(x = H/2)$.

The goal of this problem is to analyze oscillation of the device in still water (no waves). No torque or friction acts at the pivot. Neglect the mass of AB and viscous drag of the water.

5.1 The buoyancy force acting (upwards) on one cylinder can be expressed in terms of its height x above the water surface as

$$F_B = \rho g A(H - x)$$

Show that (for $0 < x < H$) the potential energy of the buoyancy force is $V = -\rho g A\left(Hx - \frac{x^2}{2}\right) + C$ where C is a constant

From the definition of potential energy $V = -\int_0^x F_B dx + C = -\int_0^x \rho g A(H - x)dx + C = -\rho g A(Hx - \frac{x^2}{2}) + C$

[1 POINT]

5.2 Write down a formula for the speed of the masses (which is equal to the speed of ends A and B of the bar) in terms of $L, d\theta/dt$

A and B describe circular motion about the pivot, therefore $v = L\frac{d\theta}{dt}$

[1 POINT]

5.3 Hence, write down the total potential and kinetic energy of the system, in terms of ρ, A, L, m, g and θ and its time derivatives (assume that $0 < x < H$ at all times).

The height of the mass attached to B above the water is $x_B = H/2 + L\sin\theta$; the height of the mass attached to A is $x_A = H/2 - L\sin\theta$. The total potential energy is

$$
\begin{aligned}
V &= -\rho g A\left(Hx_A - \frac{x_A^2}{2}\right) - \rho g A\left(Hx_B - \frac{x_B^2}{2}\right) + 2C + mgx_A + mgx_B \\
&= 2C + (mg - \rho g A H)(x_A + x_B) + \frac{1}{2}\rho g A(x_A^2 + x_B^2) \\
&= 2C + (mg - \rho g A H)H + \frac{1}{2}\rho g A\left(H/2 + L\sin^2\theta\right)^2 + \frac{1}{2}\rho g A\left(H/2 - L\sin^2\theta\right)^2 \\
&= 2C + (mg - \rho g A H)H + \rho g A H^2/4 + \rho g A L^2 \sin^2\theta
\end{aligned}
$$

(not necessary to fully simplify solution to receive credit, and also OK to drop all the constant terms since they are arbitrary. Also OK to use some other datum for gravity)

The total kinetic energy is $T = \frac{1}{2}2m\left(L\frac{d\theta}{dt}\right)^2$

[2 POINTS]

5.4 Hence, show that θ satisfies the equation of motion

$$
m\frac{d^2\theta}{dt^2} + \rho g A\sin\theta\cos\theta = 0
$$

We can use energy conservation to derive the EOM:

$$
\begin{aligned}
&\frac{d}{dt}\left(T+V\right) = 0 \Rightarrow 2mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \rho g A L^2 2\sin\theta\cos\theta\frac{d\theta}{dt} = 0 \\
&\Rightarrow m\frac{d^2\theta}{dt^2} + \rho g A\sin\theta\cos\theta = 0
\end{aligned}
$$

[2 POINTS]

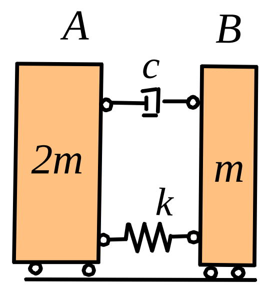
5.5 Linearize the equation of motion in 21.4 for small θ and hence find a formula for the natural frequency of vibration of the energy harvester, in terms of ρ, g, A, m.

We can use $\sin\theta \approx \theta \quad \cos\theta \approx 1$ to see that

$$
\begin{aligned}
&m\frac{d^2\theta}{dt^2} + \rho g A\theta = 0 \\
&\Rightarrow \frac{m}{\rho g A}\frac{d^2\theta}{dt^2} + \theta = 0 \\
&\Rightarrow \omega_n = \sqrt{\frac{\rho g A}{m}}
\end{aligned}
$$

[2 POINTS]

6. When mass A is held fixed, and mass B vibrates, the system shown in the figure has a natural frequency ω_n and damping factor ζ. Find formulas for the new natural frequency and damping factor (in terms of ω_n ζ) when mass B is held fixed, and A vibrates.

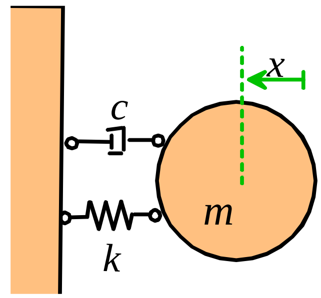


If (A) is fixed then $\omega_n = \sqrt{\dfrac{k}{m}}$ $\qquad \zeta = \dfrac{c}{2\sqrt{km}}$

If (B) is fixed, then $\sqrt{\dfrac{k}{2m}} = \dfrac{1}{\sqrt{2}}\omega_n \qquad \dfrac{c}{2\sqrt{k2m}} = \dfrac{\zeta}{\sqrt{2}}$

[2 POINTS]

7. In this problem, we will estimate the contact stiffness and damping of the baseball that you analyzed in Homework 4. During impact, the baseball can be idealized as a spring-mass-damper system: the mass represents the interior core of the baseball; and the spring and damper represents the deformed part of the ball that is in contact with the steel plate. We can characterize the system using the natural frequency ω_n and damping factor ζ in the usual way.



7.1 Assume that the center of the ball (i.e. the mass) is at position $x=0$ and has velocity $dx/dt = -v_0$ just before impact. Write down an equation for the displacement *x(t)* and velocity $v(t)$ of the center of the ball (represented by the mass in the figure) during the impact, in terms of the natural frequency ω_n and damping factor ζ for the system.

From the solutions to vibration EOM (using the underdamped case – the other two could be given two, but of course they would not rebound from the wall)

$$x(t) = \frac{v_0}{\omega_d}\exp(-\zeta\omega_n t)\sin\omega_d t$$

$$v(t) = \frac{v_0}{\omega_d}\exp(-\zeta\omega_n t)\left(\omega_d\cos\omega_d t - \omega_n\zeta\sin\omega_d t\right)$$

[2 POINTS]

7.2 If ζ is small, we can assume that the rebounding ball loses contact with the plate when $x(t) \approx 0$ after the first cycle of vibration. With this approximation, show that the restitution coefficient for the collision is $e \approx \exp(-\pi\zeta)$.

The formula for *x(t)* from 7.1 suggests that

$$\omega_d T_{contact} \approx \pi \Rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{\pi}{T_{contact}} \Rightarrow \omega_n = \frac{\pi}{\sqrt{1-\zeta^2}\,T_{contact}}$$

The velocity at the point where the ball loses contact follows as

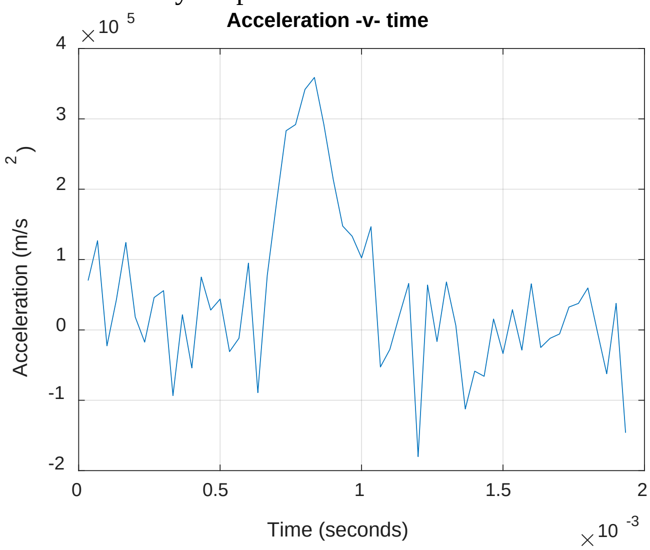
$$\frac{v_0}{\omega_d}\exp(-\zeta\omega_n T_{contact})\left(\omega_d\cos\pi-\omega_n\zeta\sin\pi\right)=-v_0\exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

The restitution coefficient is $e=-v_1/v_0 = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)\approx\exp(-\zeta\pi)$

[2 POINTS]

7.3. Use the data from Homework 4 problem 4 to estimate values of ω_n,ζ (you can use your plot of acceleration –v- time to estimate the time that the ball is in contact with the plate) . Use the known mass of the ball to calculate values for the effective spring stiffness k and dashpot coefficient c .

There is no need to submit Matlab code for this problem, just explain how you did the calculation in enough detail that the TAs can follow your procedure.



The acceleration –v- time plot from HW4 suggests that the ball is in contact with the plate for about 0.0004 seconds. We also estimated $e\approx 0.6$ in HW4. Thus

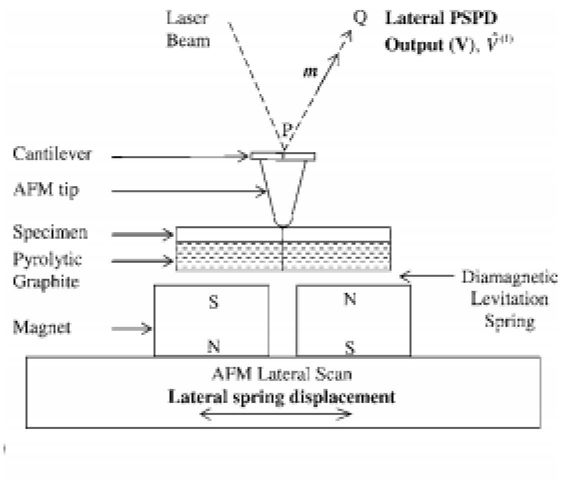
$$\zeta\approx-\frac{1}{\pi}\log(0.6)=\zeta\approx 0.16$$

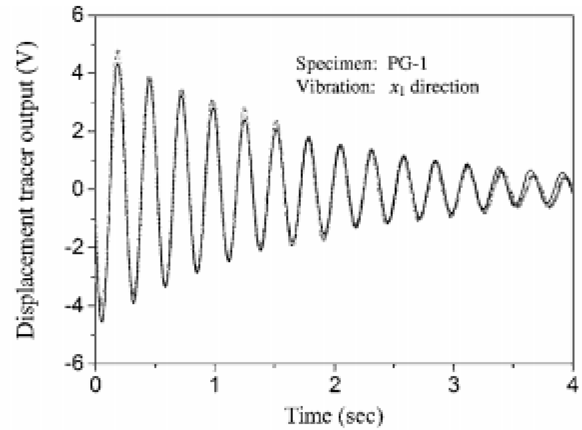
$$\omega_n\approx\frac{\pi}{T_{contact}}=2500\pi=7850 rad/s$$

The baseball mass is 0.145kg, so $\omega_n=\sqrt{\frac{k}{m}}\Rightarrow k=m\omega_n^2=1.14kN/m$

$$\zeta=\frac{c}{2\sqrt{km}}\Rightarrow c=2\zeta\sqrt{km}=4.2Ns/m$$

[2 POINTS]





8. In this publication, professor K-S Kim's group at Brown describe a new approach to measure the stiffness of an atomic force microscope tip. Their approach is to push on a very soft spring with the AFM tip. If the spring stiffness is known, the force on the AFM tip can be calculated. Their 'spring' actually consists of a thin sheet of graphite suspended over four magnets (see the figure): the magnets exert restoring forces that keep the sheet centered over the magnets, and so act like a spring. The spring stiffness can't easily be calculated, so instead, they measured it by doing a vibration experiment. The figure shows the free vibration response of the graphite sheet (measured using a laser interferometer).

8.1 Estimate the period and the log decrement from the vibration response.

The first peak has amplitude 4.3V; the 6th peak has amplitude 2V. The formula for log decrement is

$$\delta = \frac{1}{5}\log\frac{4.3}{2} = 0.15309$$

There are 15 cycles in 4 secs, so the period is 4/15=0.2666sec

[2 POINTS]

8.2 Hence, calculate the natural frequency ω_n and damping coefficient ζ for the system

The formulas give

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.024$$

$$\omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T} = 23.5\text{rad/s}$$

[2 POINTS]

8.3 The graphite sheet has mass m=15.052 milligrams. Calculate the effective spring stiffness of the magnetic levitation system.

We know that $\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2 = 8.33\times10^{-3}\,N/m$

[1 POINT]