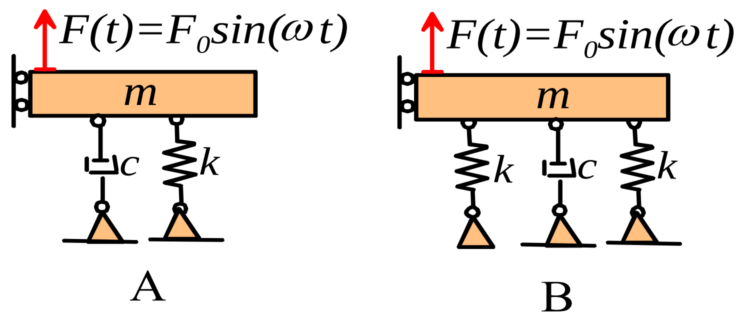
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations, Rigid Body Kinematics
Due Friday April 7, 2017



1. System A in the figure is **critically damped**. The amplitude of the force is 1 *kN*, and the frequency of the force is equal to the undamped natural frequency of the spring-mass system ($\omega = \omega_n$).

Its amplitude of vibration is measured to be 1mm.

1.1 What is the value of ζ for system A?

The system is critically damped, so $\zeta = 1$

[1 POINT]

1.2 What is the value of the magnification factor *M* for system A, (with force frequency $\omega = \omega_n$ and ζ in 1.1)?

The formula for *M* from the notes is

$$M = \frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \Rightarrow M = \frac{1}{2}$$

[1 POINT]

1.3 Use 1.2 and the given force amplitude and vibration amplitude to find a value for the spring stiffness in system A. (you can use the formulas for vibration amplitude from the notes, you don't have to re-derive them, but you are welcome to do so as a review if you like!)

The formula for vibration amplitude is $X_0 = KMF_0$. We know that *K=1/k* so

$$X_0 = \frac{1}{k}\frac{1}{2}F_0 \Rightarrow k = \frac{F_0}{2X_0} = \frac{1}{2}\times\frac{1000}{10^{-3}} = 500kN/m$$

[1 POINT]

1.4 System B is excited with the same force (i.e. the forces in acting on A and B have the same amplitude and frequency). What is the vibration amplitude of B?

For system B,

$$\omega_n = \sqrt{\frac{2k}{m}} \Rightarrow \omega / \omega_n = \frac{1}{\sqrt{2}} \qquad \zeta = \frac{c}{2\sqrt{2km}} = \frac{1}{\sqrt{2}}$$

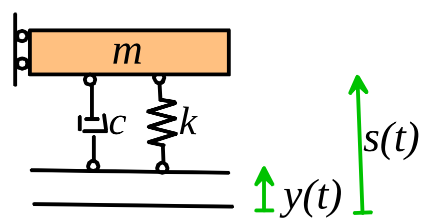
$$M = \frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{(1-1/2)^2 + \left(\frac{2}{2}\right)^2}} = \frac{2}{\sqrt{5}}$$

$$X_0 = \frac{1}{2k} M F_0 = \frac{2}{\sqrt{5}} \times \frac{1000}{2\times 500000} m = \frac{2}{\sqrt{5}} mm$$

[3 POINTS]

2. The goal of this problem is to design a vibration isolation system (i.e. recommend values for the spring stiffness, mass, and the dashpot coefficient in the idealization shown) with the following specifications:

(a) The system should return to equilibrium as quickly as possible following a disturbance
(b) The amplitude of vibration of the platform should be less than 10% of the amplitude of the base for all frequencies exceeding 100Hz
(c) The deflection of the platform when an object is placed on it should be minimized
(d) The mass of the system cannot exceed 20kg.



2.1 What value of damping factor ζ is needed to meet condition (a)?

We need critical damping $\zeta = 1$

[1 POINT]

2.2 We know that if the system is disturbed, its motion will have the form

$$x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n (x_0 - C)\right] t\right\} \exp(-\omega_n t)$$

What does this tell us about choosing ω_n needed to meet condition (a)? (It won't tell us the value of ω_n - only whether we should make ω_n big or small).

ω_n needs to be big

[1 POINT]

2.3 Write down the formula for the magnification *M* of the base-excited system with ζ from 2.1, in terms of ω / ω_n . Hence, use condition (b) to show that

$$\frac{\left\{1+\left(2\omega / \omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2 / \omega_n^2\right)^2+\left(2\omega / \omega_n\right)^2\right\}^{1/2}} < 0.1$$

for all frequencies $\omega > 200\pi$. Deduce that $\omega_n \le \dfrac{200\pi}{19.96}$ (you can use Mupad to solve the equation if you like), and hence use 2.2 to select the value for ω_n that will best meet the design specification.

The second condition says that

$$\frac{X_0}{Y_0} = \frac{\left\{1+\left(2\varsigma\omega / \omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2 / \omega_n^2\right)^2+\left(2\varsigma\omega / \omega_n\right)^2\right\}^{1/2}} < 0.1$$

for all frequencies $\omega > 200\pi$. We already know that $\zeta = 1$ so this condition can be solved for ω / ω_n : we find (see the mupad below) that $\omega / \omega_n \ge 19.96$

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sqrt(1+4*r^2)/sqrt( (1-r^2)^2 + 4*r^2 ) = 0.1
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$$\frac{\sqrt{4\,r^2+1}}{\sqrt{(r^2-1)^2+4\,r^2}} = 0.1$$

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solve(%,r,IgnoreSpecialCases)
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$$\{-19.95616668,\ 19.95616668,\ -0.4985864536\,i,\ 0.4985864536\,i\}$$

Since we want to maximize ω_n to meet the first condition, we have that $\omega_n = (200\pi)/19.96 = 31.485$ rad/s

[3 POINTS]

2.4 What does condition (c) tell us about the choosing the spring stiffness *k*? (It won't tell us the value, only whether *k* should be big or small).

The deflection of the platform when an object of mass *M* is placed on it is Mg / k . To minimize this, we want to use the stiffest possible spring.

[1 POINT]

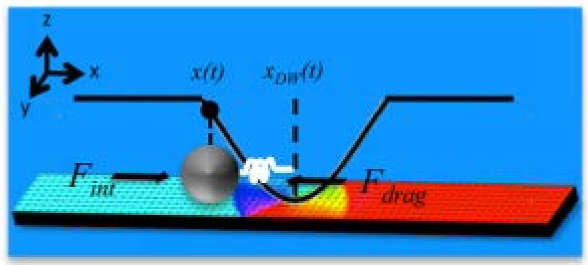
2.5 Use the solution to 2.3, 2.4 and condition (d) to select the best values for k and m, and use 2.1 to determine the necessary value for c.

Since $\omega_n = \sqrt{\dfrac{k}{m}}$, if we maximize k we also have to maximize m. So we pick

- m=20kg, which gives $k = m\omega_n^2 = 20 \times (31.485)^2 = 19.8kN/m$
- $\zeta = c/(2\sqrt{km}) \Rightarrow c = 2\zeta\sqrt{km} = 2\sqrt{20} = 1.259kNs/m$

[3 POINTS]

3. In this publication Rapaport and Beach show that it is possible to detect the diameter of a micro-bead by measuring its forced vibration response in a highly localized fluctuating magnetic field (this would provide a way to sort micron-scale particles or cells, for example).

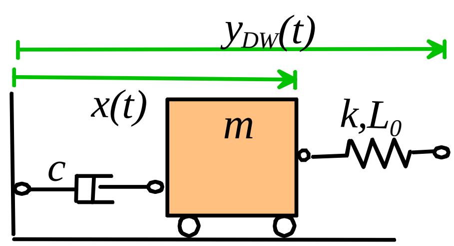


The figure (from their publication) illustrates the operating principle. A localized magnetic domain is created on a substrate by e-beam lithography. A paramagnetic particle with radius R and mass m is placed on the substrate. The domain attracts the particles to its center, and behaves like a spring with stiffness k. The position of the domain is varied harmonically $y_{DW} = Y_0 \sin\omega t$ by applying an external AC field to the substrate. The particles are in a fluid, and so are subjected to a viscous drag force $F_D = 6\pi\mu R v$, where μ is the fluid viscosity, and v is the speed of the particle.

3.1 The figure shows more clearly how the authors idealize their device. The dashpot represents viscous fluid forces acting on the bead: the dashpot coefficient is related to the bead radius and fluid viscosity by $c = 6\pi\mu R$. The motion of the end of the spring

$$y_{DW}(t) = Y_0 \sin\omega t$$

represents the fluctuation in the magnetic field.

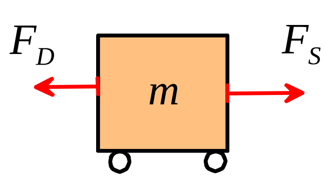


Use Newton's law to find the equation of motion for the system , and show that the equation of motion for the sphere is a 'Case IV' differential equation

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = C + Ky_{DW}(t)$$

with $\omega_n = \sqrt{k/m}$ $\qquad \zeta = \dfrac{c}{2\sqrt{km}}$ $\qquad K = 1$

A free body diagram for the mass is shown



Newton's law gives $m\frac{d^2x}{dt^2} = F_S - F_D$

The spring force law gives $F_S = k(y_{DW} - x)$ $\quad F_D = c\frac{dx}{dt}$, which gives

$$\frac{m}{k}\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x = Y_0 \sin\omega t$$

For this problem $\omega_n = \sqrt{k/m}$ $\quad \zeta = \frac{c}{2\sqrt{km}}$ $\quad K = 1$

[2 POINTS]

3.2 Estimate values for ζ, ω_n for the following values of parameters:

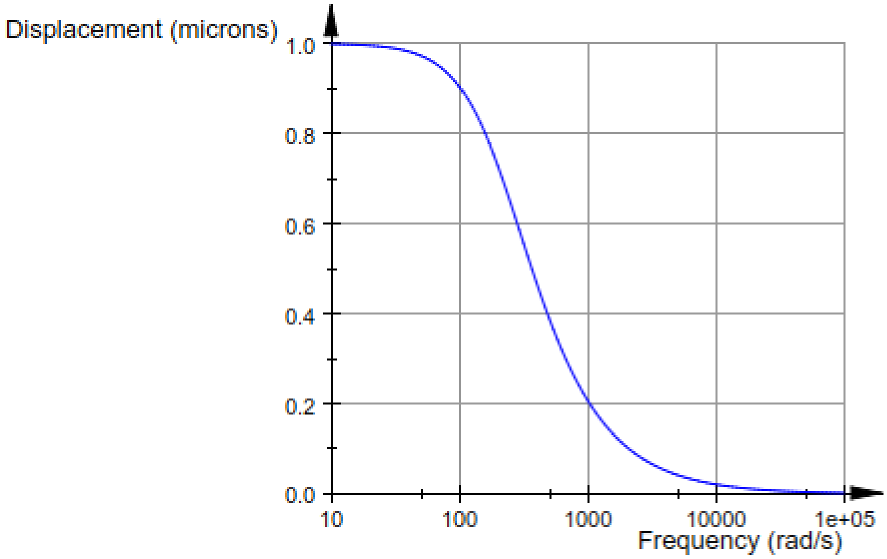
- Bead radius $1\mu m$
- Fluid viscosity (water) $10^{-3} Ns/m^2$
- Bead mass density 1800 kg/m^3
- Stiffness k $4\times10^{-6} N/m$

Substituting numbers gives $\zeta = 54.3,\ \omega_n = 23000 rad/s$

[1 POINT]

3.3 Use 3.2 and the solution for 'Case IV' vibrations to plot the expected amplitude of vibration of the bead as a function of frequency ω for an excitation amplitude $Y_0 = 1\ \mu m$, for frequencies in the range $10 < \omega < 100000$ Note that (because the system is very heavily damped) the resonant peak is not visible. Because the resonant peak is not visible, the usual trick of looking for resonance to learn something about a system does not work for this application. The authors needed to find some other way to detect the bead size from their measurements.

The figure is shown (see Mupad file for details).

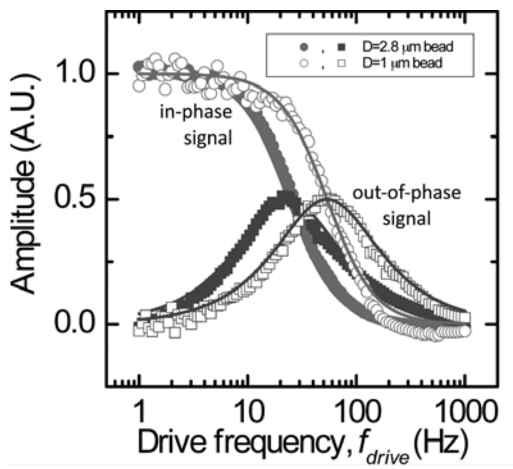


[2 POINTS]

3.4 Use the Case 4 steady-state solutions to show that the displacement x can be expressed as

$$x(t) = A(\omega)\sin\omega t - B(\omega)\cos\omega t$$

and find formulas for $A(\omega), B(\omega)$ for the parameters given in 3.2. (you will need to use the formula for the phase shift ϕ from the notes, and also use the double angle formula $\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$). Plot $A(\omega), B(\omega)$ for a displacement amplitude $X_0 = 1\ \mu m$ and $10 < \omega < 5000$. The peak in $B(\omega)$ depends on ζ , and so can be used to determine the bead radius (although the authors refer to the peak as 'resonance' in their paper they are not using the usual definition of resonance). You don't need to submit Mupad/Matlab code for this problem, just include a copy of your graphs



The solution for case 4 is

$$x(t) = Y_0 X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2}} \qquad \tan\phi = \frac{-2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

The double angle formula gives

$$x = Y_0 X_0 \left(\sin\omega t\cos\phi + \cos\omega t\sin\phi\right)$$

Note that

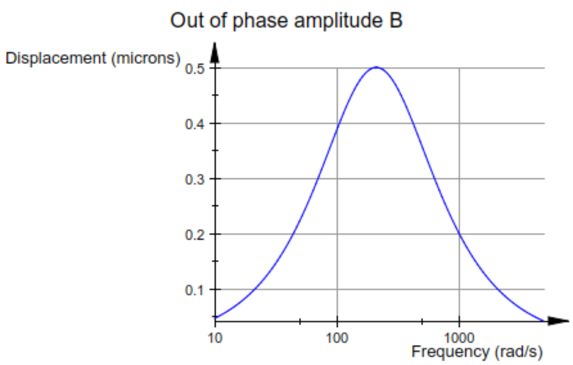
$$x(t) = Y_0 X_0 \sin(\omega t + \phi)$$

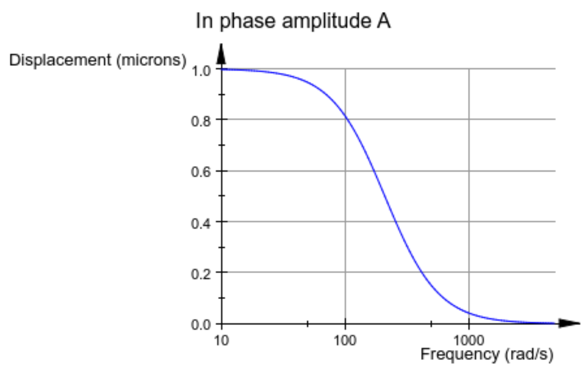
$$\sin\phi = \frac{-2\zeta\omega / \omega_n}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2}} \qquad \cos\phi = \frac{\left(1-\omega^2/\omega_n^2\right)}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2}}$$

Hence

$$x(t) = Y_0 X_0 \sin(\omega t + \phi)$$

$$B(\omega) = \frac{Y_0 2\zeta\omega/\omega_n}{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2} \qquad A(\omega) = \frac{\left(1-\omega^2/\omega_n^2\right)}{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2}$$





[4 POINTS]