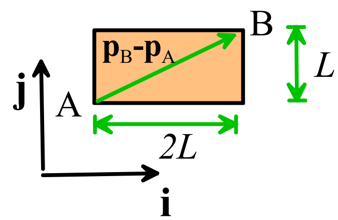
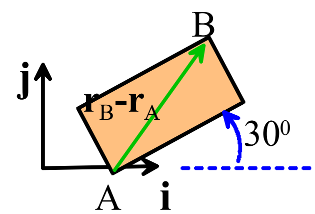
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 7: Rigid Body Kinematics, Inertial properties of rigid bodies
Due Friday April 21, 2017

1. Find the rotation tensor (matrix) that rotates the rectangle from its initial to its final position shown in the figure. Hence, express the rotated vector $\mathbf{r}_B - \mathbf{r}_A$ in (**i**,**j**) components.





Using the formula for 2D rotations

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

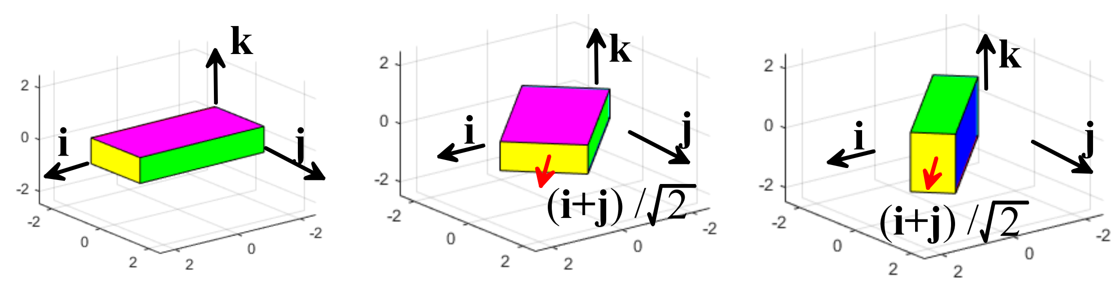
$\mathbf{r}_B - \mathbf{r}_A$ can be found using the mapping $\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 2L \\ L \end{bmatrix} = \begin{bmatrix} L(\sqrt{3} - 1/2) \\ L(1 + \sqrt{3}/2) \end{bmatrix}$$

Hence

$$\mathbf{r}_B - \mathbf{r}_A = L(\sqrt{3} - 1/2)\mathbf{i} + L(1 + \sqrt{3}/2)\mathbf{j}$$

[3 POINTS]



2. The rectangular prism shown in the figure is subjected to two sequential rotations:
(1) A 45 degree rotation about the **k** axis
(2) A 90 degree rotation about a line parallel to $\mathbf{n} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$

2.1 Write down the rotation tensor (matrix) for each rotation

We can use the general formula

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

For the first rotation $\theta = 45$, $n_x = n_y = 0$ $n_z = 1$

$$\mathbf{R}^{(1)} = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & \cos 45 + (1-\cos 45) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[2 POINTS]

For the second rotation $\theta = 90$ $n_x = n_y = 1/\sqrt{2}$ $n_z = 0$

$$\mathbf{R} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

[2 POINTS]

2.2 Find the rotation matrix $\mathbf{R} = \mathbf{R}^{(2)}\mathbf{R}^{(1)}$ that describes the combined effects of both rotations (you can use Mupad or matlab do do the matrix multiplication, but there is no need to submit the code or script).

Here's the solution in Mupad:

```
R1 := matrix([[1/sqrt(2),-1/sqrt(2),0],[1/sqrt(2),1/sqrt(2),0],[0,0,1]]):
R2 := matrix([[1/2,1/2,1/sqrt(2)],[1/2,1/2,-1/sqrt(2)],[-1/sqrt(2),1/sqrt(2),0]]):
R2*R1
```

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

[2 POINTS]

2.3 Find the axis **n** and rotation angle θ that will complete the rotation $\mathbf{R}$ directly.

We can use the fomula

$$1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz}$$

$$\mathbf{n} = \frac{1}{2\sin\theta}\left[\left(R_{zy} - R_{yz}\right)\mathbf{i} + \left(R_{xz} - R_{zx}\right)\mathbf{j} + \left(R_{yx} - R_{xy}\right)\mathbf{k}\right]$$

This gives $\cos\theta = \frac{1}{2}\left(\frac{\sqrt{2}}{2} - 1\right) \Rightarrow \theta = 1.71777\text{rad}$

$$\mathbf{n} = \frac{1}{2\sin(1.71777)}\left((1 + \frac{1}{\sqrt{2}})\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\right) = 0.8628\mathbf{i} + 0.3574\mathbf{j} + 0.3574\mathbf{k}$$

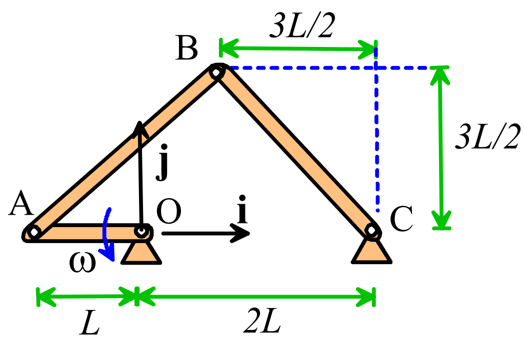
[3 POINTS]

Optional: You can check your answer by downloading a matlab script from the homework page of the course website that will animate a rigid rotation through an angle θ about an axis parallel to a unit vector **n**. You can use the code by navigating to the directory storing the file in the Matlab command window, and then typing

Animate_rotation(*angle*,[n_x, n_y, n_z])

Where *angle* is your solution for the rotation angle θ (in radians), and n_x, n_y, n_z are the components of your solution for the unit vector **n**.

3. The figure shows a four-bar chain mechanism. Member AO rotates counterclockwise at constant angular speed ω .



At the instant shown, calculate the angular velocity and acceleration of links AB and BC.

The rigid body formulas for the three members give

$$\mathbf{v}_A - \mathbf{v}_O = \omega\mathbf{k}\times(\mathbf{r}_A - \mathbf{r}_O) = \omega\mathbf{k}\times(-L\mathbf{i}) = -L\omega\mathbf{j}$$

$$\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{k}\times(\mathbf{r}_B - \mathbf{r}_A) = \omega_{AB}\mathbf{k}\times\frac{3L}{2}(\mathbf{i}+\mathbf{j}) = \frac{3L}{2}\omega_{AB}(\mathbf{j}-\mathbf{i})$$

$$\mathbf{v}_C - \mathbf{v}_B = \omega_{BC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_B) = \omega_{BC}\mathbf{k}\times\frac{3L}{2}(\mathbf{i}-\mathbf{j}) = \frac{3L}{2}\omega_{BC}(\mathbf{i}+\mathbf{j})$$

We know that $\mathbf{v}_O = \mathbf{v}_C = \mathbf{0}$ so if we add all the equations listed above we get

$$-L\omega\mathbf{j} + \frac{3L}{2}\omega_{AB}(\mathbf{j}-\mathbf{i}) + \frac{3L}{2}\omega_{BC}(\mathbf{i}+\mathbf{j}) = \mathbf{0}$$

The **i** component of this equation gives $-\omega_{AB} + \omega_{BC} = 0$

The **j** component gives $-\omega + \frac{3}{2}\omega_{AB} + \frac{3}{2}\omega_{BC} = 0$

Multiply the first equation by 3/2 and add: $-\omega + 3\omega_{BC} = 0 \Rightarrow \omega_{BC} = \omega/3 \qquad \omega_{AB} = \omega/3$

[3 POINTS]

For accelerations, use

$$\mathbf{a}_A - \mathbf{a}_O = -\omega^2(\mathbf{r}_A - \mathbf{r}_O) = L\omega^2\mathbf{i}$$

$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{AB}\mathbf{k}\times(\mathbf{r}_B - \mathbf{r}_A) - \omega_{AB}^2(\mathbf{r}_B - \mathbf{r}_A) = \frac{3L}{2}\alpha_{AB}(\mathbf{j}-\mathbf{i}) - \frac{\omega^2}{9}\frac{3L}{2}(\mathbf{i}+\mathbf{j})$$

$$\mathbf{a}_C - \mathbf{a}_B = \alpha_{BC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_B) - \omega_{BC}^2(\mathbf{r}_C - \mathbf{r}_B) = \omega_{BC}\mathbf{k}\times\frac{3L}{2}(\mathbf{i}-\mathbf{j}) = \frac{3L}{2}\alpha_{BC}(\mathbf{i}+\mathbf{j}) - \frac{\omega^2}{9}\frac{3L}{2}(\mathbf{i}-\mathbf{j})$$

At the stationary points $\mathbf{a}_O = \mathbf{a}_C = \mathbf{0}$, and add everything:

$$L\omega^2\mathbf{i} + \frac{3L}{2}\alpha_{AB}(\mathbf{j}-\mathbf{i}) - \frac{\omega^2}{9}\frac{3L}{2}(\mathbf{i}+\mathbf{j}) + \frac{3L}{2}\alpha_{BC}(\mathbf{i}+\mathbf{j}) - \frac{\omega^2}{9}\frac{3L}{2}(\mathbf{i}-\mathbf{j}) = \mathbf{0}$$

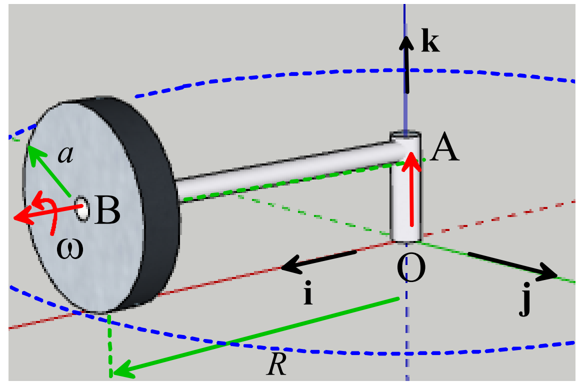
The **j** component gives $\alpha_{AB} + \alpha_{BC} = 0$. The **i** component gives

$$\omega^2 - \frac{3}{2}\alpha_{AB} + \frac{3}{2}\alpha_{BC} - \frac{\omega^2}{3} = 0 \Rightarrow \frac{2}{3}\omega^2 - \frac{3}{2}\alpha_{AB} + \frac{3}{2}\alpha_{BC} = 0$$

Multiply the first equation by 3/2 and add $3\alpha_{BC} + 2\omega^2/3 = 0 \Rightarrow \alpha_{BC} = -2\omega^2/9 \qquad \alpha_{AB} = 2\omega^2/9$

[3 POINTS]

4. The wheel shown in the figure rotates at angular speed ω about its axle, and the wheel rolls without slip. Calculate



4.1 The velocity vector of the center of the wheel B at the instant shown

We can use the rolling wheel formula

$\mathbf{v}_B = -a\omega\mathbf{j}$

[1 POINT]

4.2 Calculate the angular velocity of the axle AB (assume that there is a bearing at B that allows the wheel to rotate on the axle and a bearing at O that allows OA to rotate. The connection at A is rigid). Hence, determine the angular velocity vector of the wheel

We can use the rigid body rotation formula for the axle

$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega}_{AB} \times (\mathbf{r}_B - \mathbf{r}_A)$$

$$\Rightarrow -a\omega\mathbf{j} = \left(\omega_{xAB}\mathbf{i} + \omega_{yAB}\mathbf{j} + \omega_{zAB}\mathbf{k}\right) \times R\mathbf{i} = -\omega_{yAB}R\mathbf{k} + \omega_{zAB}R\mathbf{j}$$

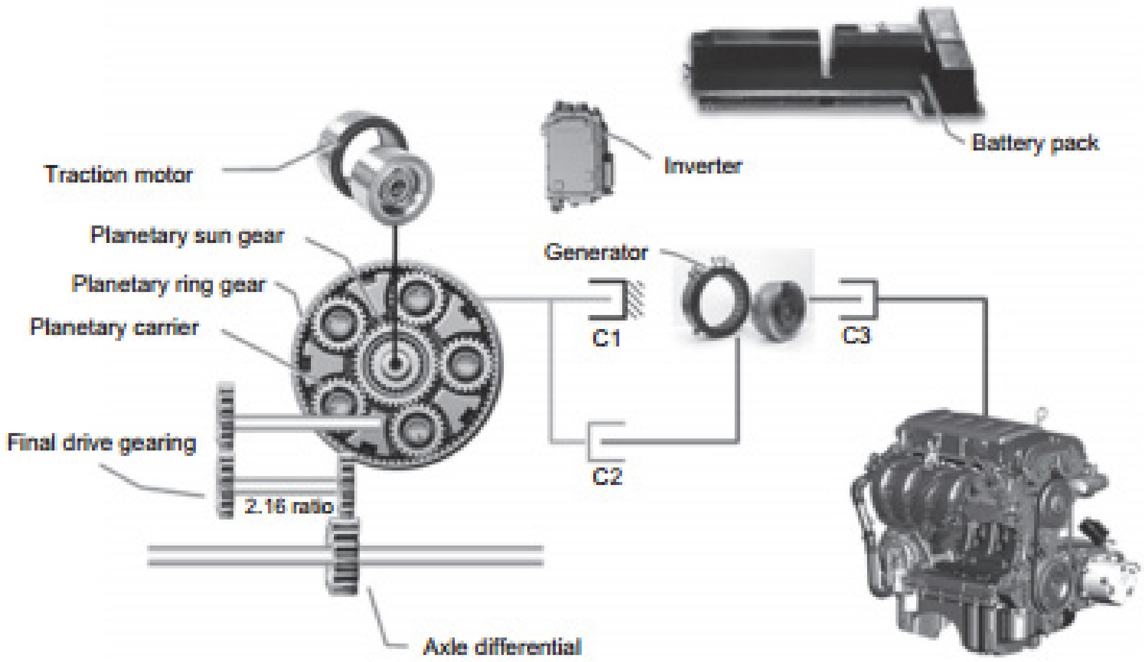
$$\Rightarrow \omega_{yAB} = 0 \qquad \omega_{zAB} = -\frac{a}{R}\omega$$

[2 POINTS]

The wheel's angular velocity is the sum of the axle and the angular velocity of the wheel about the axle

$$\boldsymbol{\omega} = \omega\mathbf{i} - \frac{a}{R}\omega\mathbf{k}$$

[1 POINT]



5. The figure (from this publication) shows a schematic of the split-power transmission system for the first generation Chevy Volt (the 2016 version is a lot more sophisticated). The wheels are always connected to the planet carrier through the 'final drive gearing'. The sun gear is driven by the electric 'traction motor', while the ring gear can either be locked (close clutch C1 and open C2), to make the car an electric vehicle, or connected to the internal combustion engine (open C1 and close C2,C3) to make the car a hybrid. We can find the following specifications for the vehicle:

- Tire diameter 16.5 inches
- The gear ratio between the planet carrier and wheels is $\omega_{PC} / \omega_W = 2.16$
- With the ring gear locked, the sun gear: planet carrier ratio is

5.1 Design the planetary gear system necessary to achieve the required gear ratio (i.e. specify the smallest integer numbers of teeth N_S, N_R, N_P on the sun, ring and planet gears that will achieve the required ratios). You could start by finding the number of teeth on the sun and ring gears, using the general epicyclic gear ratio formula from class and the given $\omega_{zS} / \omega_{zPC} = 3.24$. Once you have N_S, N_R you can find N_P from geometry (recall that the number of teeth on each gear is proportional to its radius)

The general formula is $\dfrac{\omega_{zR} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\dfrac{N_S}{N_R}$

With the data given:

$$\frac{-1}{3.24-1} = -\frac{N_S}{N_R}$$

The lowest integers N_R, N_S that achieve this ratio are $N_S = 25,\ N_R = 56$. The planet has $N_P = (N_R - N_S)/2 = 31/2$ teeth but this is not an integer, so we need to double everything: $N_S = 50,\ N_R = 112\ \ N_P = 31$

[3 POINTS]

5.2 If the car drives at 30 mph in EV mode (ring gear locked), calculate (i) the angular speed of the wheels; (ii) the angular speed of the planet carrier, and hence (iii) calculate the required speed of the electric motor.

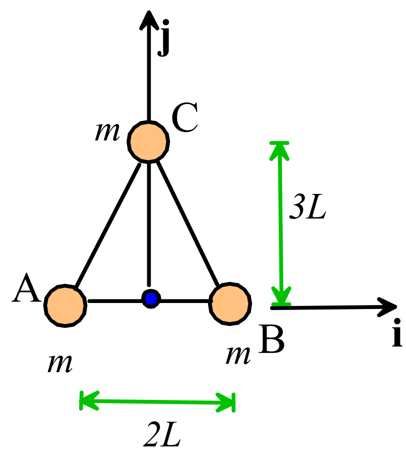
We can use the rolling wheel formula to calculate the angular speed of the wheel
$\omega = v / R = 13.4 / (0.4191 / 2) = 63.95 \; rad / s$

The angular speed of the planet carrier is 2.16 times bigger: 138.1 rad/s

The sun (and electric motor) run 3.24 times faster than this 447.5 rad/s (4274 rpm)

[3 POINTS]

6. The figure shows three particles with equal mass m connected by rigid massless links.



6.1 Calculate the position of the center of mass of the assembly

$$\mathbf{r}_G = \frac{1}{M}\sum_i m_i \mathbf{r}_i = \frac{1}{3m}\left(mL\mathbf{i} - mL\mathbf{i} + m3L\mathbf{j}\right) = L\mathbf{j}$$

[1 POINT]

6.2 Calculate the 2D mass moment of inertia of the system about the center of mass

$$I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2\right)$$

where $\mathbf{d}_i = d_{xi}\mathbf{i} + d_{yi}\mathbf{j} = \mathbf{r}_i - \mathbf{r}_G$ is the position vector of the ith particle with respect to the center of mass.

Use the formula $I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2\right) = 2m(L^2 + L^2) + m(2L)^2 = 8mL^2$

[1 POINT]

6.3 Suppose that the assembly rotates about its center of mass with angular velocity $\omega\mathbf{k}$ (the center of mass is stationary). What are the speeds of the particles A,B and C?

Use the circular motion formula. Particles A and B are distances $\sqrt{2}L$ from the COM, and particle C is a distance 2L from the COM. Therefore

$$V_A = V_B = \sqrt{2}L\omega \qquad V_C = 2L\omega$$

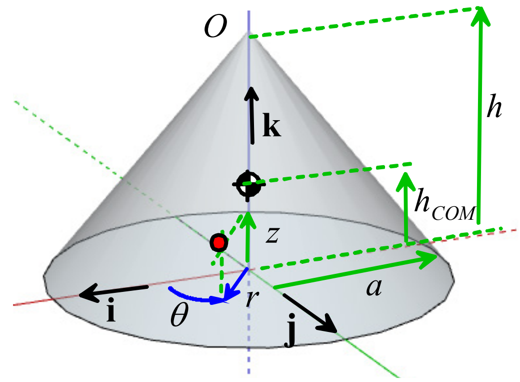
[1 POINT]

6.4 Calculate the total kinetic energy of the system (a) using your answer to 6.2; and (b) using your answer to 6.3

(a) The kinetic energy formula in terms of mass moment of inertia is $T = \frac{1}{2} I_{Gzz}\omega_z^2 = 4mL^2\omega^2$

(b) Summing the kinetic energies of the masses directly gives $T = \frac{1}{2}(2m\left(\sqrt{2}L\omega\right)^2 + m(2L)^2) = 4mL^2\omega^2$ **[2 POINTS]**

7. The figure shows a cone with height h and base radius a and uniform mass density ρ Using Mupad, calculate



7.1 The total mass M

$M = \frac{\pi}{3}\rho a^2 h$ (see below for Mupad)

[1 POINT]

7.2 The height of the center of mass h_{COM} above the base

$h_{COM} = h/4$ (see below for Mupad)

[2 POINTS]

7.3 The inertia tensor (matrix) about the center of mass, in the basis shown

The inertia matrix is (see below for Mupad)

$$\frac{3Ma^2}{20}\begin{bmatrix} 1+h^2/(4a^2) & 0 & 0 \\ 0 & 1+h^2/(4a^2) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Graders – equivalent solutions that include the density should get credit too [3 POINTS]

7.4 Using the parallel axis theorem, calculate the mass moment of inertia about the tip O.

The general formula is

$$\mathbf{I}_O = \mathbf{I}_G + M\begin{bmatrix} d_y^2+d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2+d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2+d_y^2 \end{bmatrix}$$

In the problem here $d_x = d_y = 0$ $d_z = 3h/4$ so

$$\mathbf{I}_O = \frac{3Ma^2}{20}\begin{bmatrix} 1+h^2/(4a^2) & 0 & 0 \\ 0 & 1+h^2/(4a^2) & 0 \\ 0 & 0 & 2 \end{bmatrix} + M\begin{bmatrix} 9h^2/16 & 0 & 0 \\ 0 & 9h^2/16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{3Ma^2}{20}\begin{bmatrix} 1+4h^2/a^2 & 0 & 0 \\ 0 & 1+4h^2/4a^2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

equivalent solutions that include the density should get credit [2 POINTS]

```
reset():
mass := int(int(int(`&rho;`*r,r=0..a*(1-z/h)),q=0..2*PI),z=0..h)
```

$$\frac{\pi \rho a^2 h}{3}$$

```
position := matrix([r*cos(q),r*sin(q),z]):
com := int(int(int(position*`&rho;`*r,r=0..a*(1-z/h)),q=0..2*PI),z=0..h)/mass
```

$$\begin{pmatrix} 0 \\ 0 \\ \frac{h}{4} \end{pmatrix}$$

```
dx := r*cos(q): dy := r*sin(q): dz := z-h/4:
InertiaIntegrand := matrix([[dy^2+dz^2,-dx*dy,-dx*dz],
                            [-dx*dy, dx^2+dz^2,-dy*dz],
                            [-dx*dz, -dy*dz, dx^2+dy^2]]):
Inertia := int(int(int(InertiaIntegrand*`&rho;`*r,r=0..a*(1-z/h))
                                                  ,q=0..2*PI),z=0..h)/mass
```

$$\begin{pmatrix} \frac{3a^2}{20} + \frac{3h^2}{80} & 0 & 0 \\ 0 & \frac{3a^2}{20} + \frac{3h^2}{80} & 0 \\ 0 & 0 & \frac{3a^2}{10} \end{pmatrix}$$

Please upload your Mupad script to Canvas as a solution this problem

8. Flywheel energy storage systems are an alternative to batteries. The Beacon BP400 flywheel is used in several experimental grid-scale energy storage facilities. It has the following specifications:

- Cylinder height: approx. 7 ft
- Cylinder diameter: 3ft
- Cylinder mass: 2500lb
- Angular speed: 15500 rpm

Calculate

8.1 The total energy stored in the flywheel (in Joules – please convert the info provided to SI)

The formula for kinetic energy is $T = \frac{1}{2} M \left| \mathbf{v}_G \right|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}$

Here, the COM is stationary and the cylinder spins about the **k** axis so

$I_{Gzz} = Ma^2 / 2$

$$T = \frac{1}{2}\frac{Ma^2}{2}\omega^2 = \frac{1}{4} \times 1134 \times \frac{1}{4} 0.9144^2 \times \left(\frac{15500}{60} \times 2\pi \right)^2 = 156 MJ$$

[1 POINT]

8.2 The energy density (energy per kg) For comparison he energy density of an Li ion battery is about 150 watt-hours/kg, which is 150x3600 J/kg =540kJ/kg

The energy density is 137.7 kJ/kg

[1 POINT]

8.3 The magnitude of the angular momentum vector of the cylinder, when spinning at full speed

The angular momentum is $I_{Gzz}\omega_z = \frac{1}{2} \times 1134 \times \frac{1}{4} \times 0.9144^2 \times \left(\frac{15500}{60} \times 2\pi \right) = 1.923 \times 10^5 kgm^2 / s^2$

[1 POINT]