



School of Engineering  
Brown University

## EN40: Dynamics and Vibrations

### Homework 2: Kinematics and Dynamics of Particles Due Friday Feb 9, 2018

1. 'Bloodhound' is an experimental vehicle designed to break the land speed record (with a goal of eventually achieving 1000mph). The vehicle has a mass of 7500kg. It is accelerated to its maximum speed by a combined jet and rocket engine; and then slowed by aerodynamic braking. The average acceleration is expected to be about 1g; the average deceleration is about 0.8g.



1.1 How long will the vehicle take to reach 1000mph?

The constant acceleration formula gives  $v = at \Rightarrow t = \frac{447.04}{1.0 \times 9.81} = 45.57s$

[1 POINT]

1.2 How much distance is required for the test (including both acceleration and deceleration)?

There are several ways to do this calculation – we can use the time from 1.1 to calculate the distance traveled during acceleration  $x = \frac{1}{2}at^2 = \frac{1}{2} \times 1.0 \times 9.81 \times (45.57)^2 = 10186m$  ; then use the same approach to find the distance traveled using deceleration

$$t_2 = \frac{447.04}{0.8 \times 9.81} = 56.96s \Rightarrow x_2 = \frac{1}{2} \times 0.8 \times 9.81 \times (56.96)^2 = 12732m$$

The total is 22918 m = 14.2 miles (!)

The other method is to relate the speed to the distance traveled  $v^2 = 2ax$  and apply this to the two portions.

[2 POINTS]

2. [In this article](#), a group from Brown led by Prof Mathiowitz tracked the motions of magnetic pills inside the stomachs of various animals (including humans) to estimate the forces acting on the pills. Your goal in this problem is to repeat their calculations. The raw data consists of  $(x,y,z)$  coordinates (in mm) of the position of the pill at 0.1 second intervals. You can download the data from the links below (store the files with a .csv extension) and you can look at the file with MS excel.

- [Position data for a fed rat](#)
- [Position data for a fasted rat](#)

The pills used in rats have mass 4.5 grams.

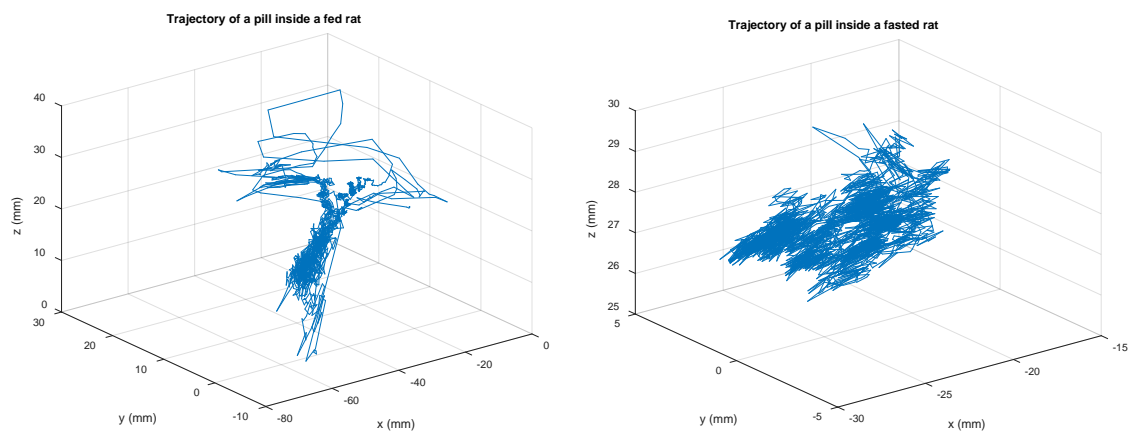
Write a MATLAB script that accomplishes the following tasks (Your MATLAB code should be uploaded to CANVAS as a submission to this problem):

2.1 Read the data into a matrix using the MATLAB 'csvread' command. Note that since the data has a header row you will need to use (eg)

```
data_Fed = csvread('Fed_Rat.csv',1,0);
```

to skip the first row.

2.2 Plot the trajectory (as a 3D curve) for both files (on separate plots)



**[2 POINTS]**

2.3 Calculate the velocity vector of the pill as a function of time. You can do this by calculating the change in position between two successive readings, and dividing by the time difference between them, e.g. if  $x(i)$  denotes the  $i$ th value of  $x$ , then the  $i$  component of acceleration is

$$v_x(i) = (x(i+1) - x(i)) / (t(i+1) - t(i))$$

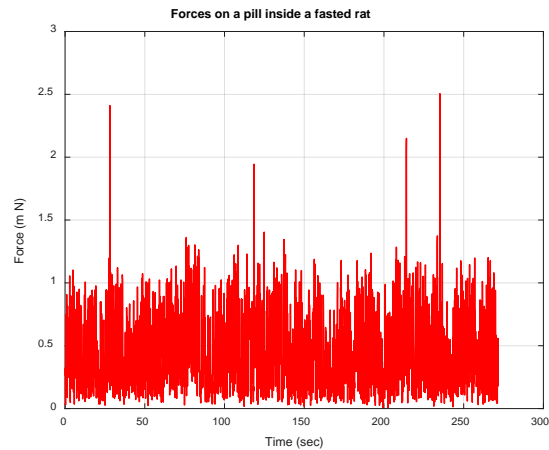
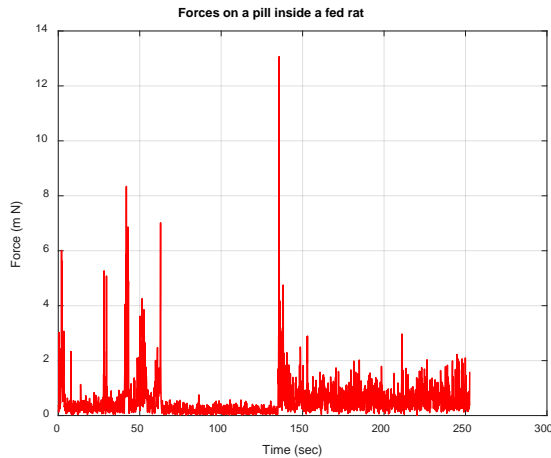
You can construct the vector  $v_x$  using a loop in MATLAB, e.g. to find the  $x$  component of velocity

```
for i=1:length(time)-1
    vx(i) = (x(i+1)-x(i))/(time(i+1)-time(i));
end
```

Here  $x$  is a vector containing the  $x$  component of position and  $vx$  is a vector containing  $x$  components of velocity (note that  $vx$  contains one point fewer than the vector  $x$ ). Alternatively, you could use the built-in MATLAB 'diff' function (see the MATLAB manuals for details).

2.4 Use the solution to 2.3 to calculate the acceleration vector of the pill as a function of time. You will need to do this with a second loop (or use the 'diff' function again).

Use your data to calculate and plot the magnitude of the force acting on the pill as a function of time (you will have to calculate the magnitudes with a loop). Plot the data for fed and fasted rats on separate figures.



**[4 POINTS]**

2.5 Use the MATLAB 'trapz' function to integrate the magnitude of the force with respect to time. Hence, compute and compare the average force magnitudes acting on a pill inside a fasted and a fed rat.

Average force on the pill in a fed rat 0.577176 (milliNewtons)  
Average force on the pill in a fasted rat 0.474331 (milliNewtons)

**[2 POINTS]**

3. The goal of this problem is to estimate more accurately the variations of acceleration, speed and distance traveled by the Bloodhound as functions of time. An attempt at the speed record will have 3 phases (i) the car will accelerate under jet engine power to 150 m/s; (ii) acceleration will continue for 20 secs under combined jet engine and rocket power; (iii) The jet engine and rocket will be shut off, and air brakes and a parachute will be deployed to slow the vehicle. Its designers [provide the following data](#)

- Vehicle mass 7500kg
- Total thrust during acceleration: 90kN for velocities between 0 and 150 m/s (under jet engine only) and 200kN for velocities above 150 m/s (under combined jet engine and rocket propulsion)
- The rocket fires for 20 seconds
- Drag during acceleration  $F_D = cV^2$ , where  $V$  is the speed and  $c=0.7 \text{ N s}^2/\text{m}$  (estimated by us)
- Drag (aerodynamic braking) during deceleration  $F_D = cV^2$  where  $c=1.1$  (estimated by us) (there is no engine thrust and brakes are not applied until low speeds)

3.1 To make the calculus simple, neglect the drag during acceleration. Use Newton's laws to obtain expressions for the acceleration of the vehicle during the three phases of the run (i) under jet engine power only; (ii) under combined rocket and jet propulsion; and (iii) under aerodynamic braking

(i)  $a = T / m = 12$  (simple case) or  $a = T / m - (c / m)V^2 = 12 - (0.7 / 7500)V^2$  (more accurate case)

(ii)  $a = T / m = 26.7$  (simple case) or  $a = T / m - (c / m)V^2 = 26.7 - (0.7 / 7500)V^2$  (accurate case)

(iii)  $a = -cV^2 / m$

[2 POINTS]

3.2 Hence, integrate the accelerations to calculate the speed as a function of time (**Optional** – if you can figure out how to do it, account for drag during phases (i) and (ii) - MATLAB can do the rather messy calculus, as long as you substitute numbers into the formulas before doing them). There is no need to submit Matlab scripts for this problem.

For the simple case we have

(i)  $\frac{dV}{dt} = \frac{T}{m} \Rightarrow V = \frac{T}{m}t$        $\frac{dx}{dt} = \frac{T}{m}t \Rightarrow x = \frac{T}{2m}t^2$

For the next part of the calculation we also have to find the time when the velocity reaches 150m/s, and the distance traveled during phase (i):

$$150 = 12t_{end} \Rightarrow t_{end} = 25 / 2s$$

$$x_1 = \frac{12}{2}t_{end}^2 = 1875 / 2m$$

(ii)  $\frac{dV}{dt} = \frac{T}{m} \Rightarrow \int_{150}^v dv = \int_{25/2}^t \frac{T}{m}t \Rightarrow v = 150 + 26.7(t - 25 / 2) \text{ m/s}$

(ii)  $\frac{dx}{dt} = \frac{T}{m}t \Rightarrow \int_{1875/2}^x dx = \int_{25/2}^t \frac{T}{m}t dt \Rightarrow x = \frac{1875}{2} + \frac{26.7}{2}(t - \frac{25}{2})^2 \text{ m}$

Again, for phase 3 we need the velocity and distance traveled at the end of phase 2

$$v = 150 + 26.7(20) = 683.3 \text{ m/s}$$

$$x = \frac{1875}{2} + \frac{26.7}{2}(20)^2 = 9271 \text{ m}$$

$$\frac{dV}{dt} = -\frac{cv^2}{m} \Rightarrow \int_{683.3}^v \frac{dv}{v^2} = \int_{25/2+20}^t \frac{c}{m} dt \Rightarrow -\frac{1}{v} + \frac{1}{683.3} = \frac{1.1}{7500}(t - 65/2)$$

$$(iii) \Rightarrow v = \frac{6150000}{902t - 20315} \text{ m/s}$$

$$\frac{dx}{dt} = v \Rightarrow \int_{9271}^x dx = \int_{65/2}^t \frac{6150000}{902t - 20315} dt \Rightarrow x = \frac{75000}{11} \log\left(\frac{451t - 4063}{4500 - 1800}\right) + \frac{55625}{6} \text{ m}$$

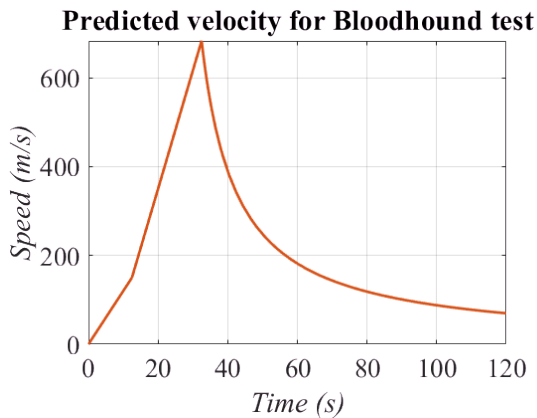
(the calculations are easier in a Live Script – see the solution script)

Note that there are other ways to handle the integrals (for example,  $t$  does not have to be measured from the start of the test; you could use  $t$  in the integrals to denote the time elapsed from the start of each phase. This makes the integrals a bit simpler but the plots in the next section a bit more tricky).

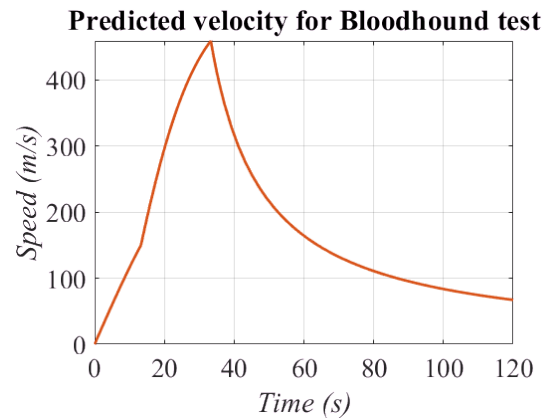
For the optional calculations see the Live Script solutions

[6 POINTS]

3.3 Use MATLAB to plot a graph of the velocity as a function of time. You can compare your prediction with the designer's data [here](#).



(a) Simple calculations



(b) Including drag during acceleration

[2 POINTS]

For discussion –

(1) The predictions agree quite well with the data on the Bloodhound site – except that our calculations predict that 150m/s is reached much more quickly than the actual data. This is because we assume that phase (i) is run with max thrust, but in practice the thrust builds more gradually. This doesn't change the predicted max velocity, since that's determined mostly by the rocket thrust and drag.

(2) Including drag has a huge effect on the predicted max velocity. Drag is unimportant during phase (1), however.

(3) Exploring the sensitivity of the predictions to the assumptions is interesting; for example a small increase in drag (eg change  $c$  from 0.7 to 0.8) will cause the maximum velocity to drop below 1000 mph. This puts a lot of pressure on the computational fluid dynamics team to get accurate numbers. Also the rocket is critical – with jet engine power only the vehicle will reach a terminal velocity well below 1000 mph. The problem with rocket power is that the driver has essentially no control over its power – once it is fired, it will continue to thrust until it burns out. Using a combination of jet and rocket gives the driver some level of control.

3.4 What do you think might cause the largest errors in this estimate? How would you go about estimating the magnitudes of the errors? What would you have to do to correct them?

Some possible causes of error:

- (i) The jet engine thrust is not constant (the driver regulates the power)
- (ii) The estimates neglects mass reduction caused by propellant ejected by the rocket
- (iii) The air drag equation is too simple – there is a big increase in drag coefficient near the speed of sound, and a subsequent drop in drag above the speed of sound.

We could estimate the errors by:

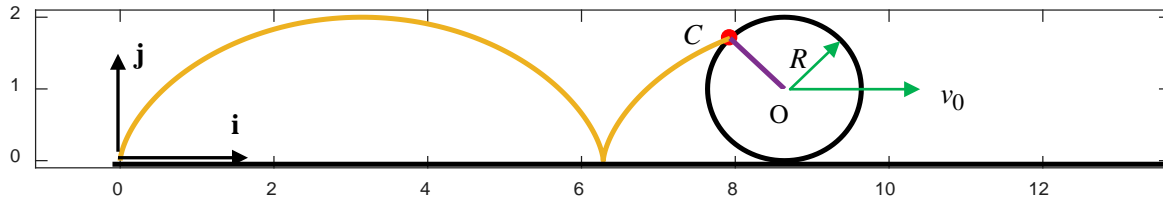
- (i) Estimate bounds on the max and min thrust, and run the calculation with both. This will give some error bars
- (ii) Do the calculation with the max and min mass – again, we get some error bars.
- (iii) Do the calculation with max and min drag coefficients – more error bars.

Notice that the error checking is should be quick and easy – just assume worst and best cases in your calculations and check the difference. If it's small, you don't need to worry.

We could improve the estimates by

- (i) Estimate the time variation of the thrust and put it into the differential equations.
- (ii) Use the proper rocket equations instead of a constant flux. There is a discussion in [this section of the course notes](#)
- (iii) Do the calculation with a velocity dependent drag coefficient.

**[2 POINTS]**



4. A point C on the circumference of a wheel that rolls without slip on a flat surface follows the parametric curve

$$x = Vt - R \sin\left(\frac{v_0 t}{R}\right) \quad y = R - R \cos\left(\frac{v_0 t}{R}\right)$$

where  $R$  is the radius of the wheel,  $v_0$  is the speed of its center (at O), and  $t$  is time. **Assume that  $v_0$  is constant throughout this problem.** You might find the animation [here](#) helpful to visualize the motion of C.

4.1 Find formulas for the velocity and acceleration vectors of the point C on the wheel in  $\{\mathbf{i}, \mathbf{j}\}$  components, in terms of  $v_0$ ,  $t$ , and  $R$ .

This is just calculus

$$\mathbf{r} = \left( v_0 t - R \sin\left(\frac{v_0 t}{R}\right) \right) \mathbf{i} + \left( R - R \cos\left(\frac{v_0 t}{R}\right) \right) \mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left( v_0 - v_0 \cos\left(\frac{v_0 t}{R}\right) \right) \mathbf{i} + \left( v_0 \sin\left(\frac{v_0 t}{R}\right) \right) \mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left( \frac{v_0^2}{R} \sin\left(\frac{v_0 t}{R}\right) \right) \mathbf{i} + \left( \frac{v_0^2}{R} \cos\left(\frac{v_0 t}{R}\right) \right) \mathbf{j}$$

[3 POINTS]

4.2 What is the magnitude of the acceleration of point C, in terms of  $v_0$  and  $R$ ?<sup>1</sup>

The magnitude is  $v_0^2 / R$

The reason this looks like the circular motion formula is that

- (i) You can measure the position, velocity and acceleration of C relative to a fixed origin ( $\mathbf{r}$ ) or relative to an origin that moves at constant speed (eg relative to the center of the wheel the position vector is ( $\hat{\mathbf{r}} = \mathbf{r} - v_0 t \hat{\mathbf{i}}$ ))
- (ii) The acceleration is the same in both reference frames in (i) (to see this just differentiate both position vectors with respect to time twice)
- (iii) An observer traveling with the center of the wheel will see C in circular motion at constant speed  $v_0$ . So the acceleration magnitude can be calculated using the circular motion formula.
- (iv) My parents reside in that undiscovered country from whose bourn no traveler returns, so I have not been able to check whether they understand this explanation.

[2 POINTS]

4.3 Find a formula for the time(s) when C touches the ground (i.e.  $y=0$ ) in terms of  $V$  and  $R$ . What is the speed of C at these instants?

When  $y=0$  we know that  $\cos(v_0 t / R) = 1 \Rightarrow t = 2n\pi R / v_0 \quad n = 0, 1, 2, \dots$  At these instants the speed of C is zero.

[2 POINTS]

4.4 Find a formula for the time(s) when C is at its greatest height above the ground (i.e.  $y=2R$ ). What is the speed of C at these instants? What is the direction of the acceleration (i.e. give a unit vector parallel to the acceleration in  $\mathbf{i}, \mathbf{j}$  components)

When  $y=2R$  we know that  $\cos(Vt / R) = -1 \Rightarrow t = (2n + 1)\pi R / v_0 \quad n = 0, 1, 2, \dots$

At this instant  $\mathbf{v} = 2v_0 \hat{\mathbf{i}}$  so the speed is  $2v_0$ .

The acceleration vector is  $\mathbf{a} = -\frac{v_0^2}{R} \hat{\mathbf{j}}$  so the direction is parallel to  $-\hat{\mathbf{j}}$  (vertically down)

[2 POINTS]

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<sup>1</sup> (For discussion: notice that you get something that looks like the circular motion formula, but  $V$  is the speed of O, instead of C; and  $R$  is the radius of the wheel, not the radius of curvature of the path. Can you explain the result in language your parents would understand?)



4.5 Use 4.4 and 4.2 to calculate the radius of curvature of the path of C (in terms of  $R$ ) at the instants when C is at its greatest height above the surface.

We know that

- (i) in normal-tangential coordinates  $\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{\rho}\mathbf{n}$  where  $V$  is the speed and  $\rho$  is the radius of curvature of the path.
- (ii) We know the speed is  $2v_0$ .
- (iii) The tangent vector is parallel to the velocity, which is in the  $\mathbf{i}$  direction. The normal vector must therefore be either parallel to  $+\mathbf{j}$  or  $-\mathbf{j}$
- (iv) The acceleration vector is given in 4.4. Hence

$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{\rho}\mathbf{n} = \frac{dV}{dt}\mathbf{i} \pm \frac{(2v_0)^2}{\rho}\mathbf{j} = -\frac{v_0^2}{R}\mathbf{j}$$

This tells us that  $dV/dt = 0$  (you can check this using the formulas in 4.1), and also that  $\rho = \pm 4R$ . By convention we pick  $\rho$  to be positive, and  $\mathbf{n}$  is therefore parallel to  $-\mathbf{j}$ .

[3 POINTS]

4.6 Find formulas for unit vectors normal and tangent to the path of C at time  $t = \pi R / (3v_0)$  (express the unit vector in  $\mathbf{i}, \mathbf{j}$  components) (hint: think about the direction of the velocity vector, and note that  $\mathbf{n}$  must be perpendicular to the tangent).

The velocity is parallel to  $\mathbf{t}$ . Substitute the time into 4.1  $\mathbf{v} = \frac{1}{2}v_0\mathbf{i} + \frac{\sqrt{3}}{2}v_0\mathbf{j} = v_0\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$ .

Therefore  $\mathbf{t} = (\mathbf{i} + \sqrt{3}\mathbf{j})/2$

We can use  $\mathbf{n} = \pm \mathbf{k} \times \mathbf{t} = \pm(\mathbf{j} - \sqrt{3}\mathbf{i})/2$  to find  $\mathbf{n}$ . (By convention we usually pick  $\mathbf{n}$  to point towards the center of curvature, which would correspond to  $\mathbf{n} = -(\mathbf{j} - \sqrt{3}\mathbf{i})/2$ )

[2 POINTS]

4.7 Use 4.6 and 4.1 to calculate the normal and tangential components of acceleration at time  $t = \pi R / (3v_0)$

(4.1) gives the acceleration vector as  $\mathbf{a} = \left(\frac{v_0^2}{R}\right)\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$ . Therefore

$a_t = \mathbf{a} \cdot \mathbf{t} = \frac{\sqrt{3}}{2} \frac{v_0^2}{R}$      $a_n = \mathbf{a} \cdot \mathbf{n} = \frac{1}{2} \frac{v_0^2}{R}$  (answers with a negative  $a_n$  are OK, but by convention they should be positive)

[3 POINTS]

4.8 What is the radius of curvature of the path of C at time  $t = \pi R / (3v_0)$  ?

Use the same method as 4.5. The speed is  $v_0$  at the instant of interest (the magnitude of the velocity, from 4.6); therefore from 4.7

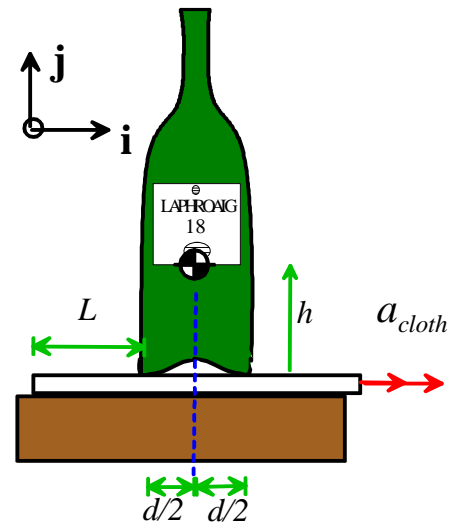
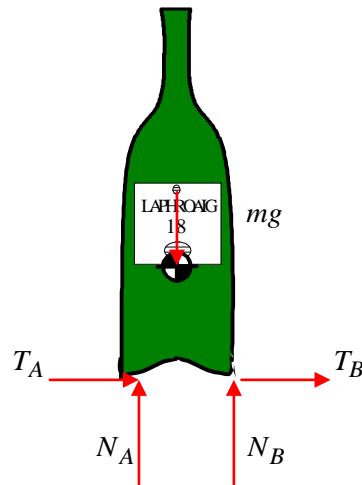
$$\frac{dV}{dt} \mathbf{t} + \frac{V^2}{\rho} \mathbf{n} = \frac{dV}{dt} \mathbf{t} + \frac{v_0^2}{\rho} \mathbf{n} = \frac{\sqrt{3} v_0^2}{2 R} \mathbf{t} + \frac{1}{2} \frac{v_0^2}{R} \mathbf{n}$$

Comparing the normal components of acceleration we see that  $\rho = 2R$

[2 POINTS]

5. A bottle with mass  $m$  rests on a table-cloth. The contact between them has friction coefficient  $\mu$ . At time  $t=0$  the object and cloth are both stationary. For time  $t>0$ , the cloth is pulled with constant acceleration  $a_{cloth} = 5\mu g$  the right. Note that since  $a_{cloth} > \mu g$ , slip must occur at the contact just after  $t=0$ .

5.1 Draw a free body diagram showing the forces acting on the object



[3 POINTS]

5.2 Find a formula for the horizontal acceleration of the bottle. Assume that the bottle does not tip over.

$$\mathbf{F} = m\mathbf{a} \text{ gives } (T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = m a_x \mathbf{i}$$

$$\text{The friction law (since there is slip) gives } T_A = \mu N_A \quad T_B = \mu N_B$$

$$\text{Hence } N_A + N_B = mg, \quad a_x = \frac{T_A + T_B}{m} = \mu \frac{N_A + N_B}{m} = \mu g$$

[2 POINTS]

5.3 What is the maximum possible value of the friction coefficient for the bottle not to tip (express your answer in terms of  $h, d$ ?)

We need to find the two reaction forces. Take moments about the COM, then use equations from 1.2

$$(T_A + T_B)h + \frac{d}{2}(N_B - N_A) = 0$$

$$\Rightarrow (N_B - N_A) = -\frac{2\mu mgh}{d}$$

We can add  $N_A + N_B = mg$  to this equation

$$\Rightarrow 2N_B = mg - \frac{2\mu mgh}{d} =$$

For the bottle not to tip we must ensure  $N_B > 0$ , so  $\mu < d / (2h)$ .

[2 POINTS]

5.4 Find a formula for the horizontal distance moved by the bottle as a function of time.

The straight line motion formula gives  $x = \frac{1}{2}\mu g t^2$

[1 POINT]

5.5 Find a formula for the distance moved by the cloth as a function of time

The straight line motion formula gives  $x_{cloth} = \frac{5}{2}\mu g t^2$

[1 POINTS]

5.6 At time  $t=0$  the bottle is a distance  $L$  from the edge of the cloth. Calculate the distance that the bottle has moved in the  $\mathbf{i}$  direction at the instant that the edge of the cloth reaches the edge of the bottle ( $L=0$  at this instant). Express your answer in terms of  $L$  and any other parameters you think are relevant.

The cloth must move a distance  $L$  further than the bottle, so

$$x_{cloth} - x = \frac{4}{2}\mu g t^2 = L \Rightarrow t = \sqrt{\frac{L}{2\mu g}}$$

The bottle therefore moves a distance

$$x = \frac{1}{2}\mu g \frac{L}{2\mu g} = \frac{1}{4}L$$

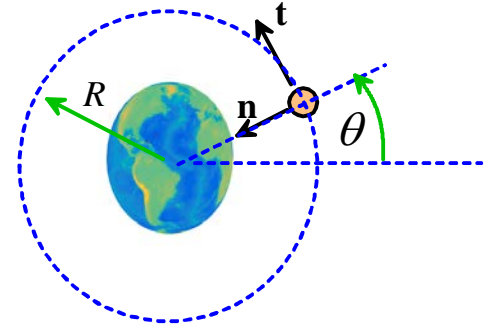
[3 POINTS]

For discussion – this tells us how to design the trick. Any objects we put on the cloth must be at least  $L/4$  away from the edge of the table, otherwise they will fall off. Notice that the critical distance doesn't depend on the shape, size, or mass of the object; and also doesn't depend on the friction coefficient. But tall objects on cloths with too much friction will tip over – to design against tipping we need to make sure the objects have a low COM. That's why we use things like plates and tumblers rather than bottles....

6. The figure shows a satellite with mass  $m$  in a circular orbit around the earth. The satellite is subjected to a radial force

$$\mathbf{F} = \frac{\mu m}{R^2} \mathbf{n}$$

where  $\mu = 3.986012 \times 10^5 \text{ km}^3 \text{ s}^{-2}$ . The satellite takes a time  $T$  to complete a full orbit.



6.1 Find a formula for the velocity vector of the satellite, in terms of  $R$  and  $T$ . Express your answer in the normal-tangential basis  $\mathbf{n}, \mathbf{t}$  shown.

The speed is the distance traveled divided by the orbit period, and the direction is parallel to  $\mathbf{t}$  so

$$\mathbf{v} = \frac{2\pi R}{T} \mathbf{t}$$

[1 POINT]

6.2 Find a formula for the acceleration vector of the satellite (normal-tangential coordinates), in terms of  $R$  and  $T$ .

This is just circular motion

$$\mathbf{a} = \frac{V^2}{R} \mathbf{n} = \frac{4\pi^2 R}{T^2} \mathbf{n}$$

[2 POINTS]

6.3 Use Newton's law to find a formula for  $R$  in terms of  $T$  and other relevant variables

$$\begin{aligned} \mathbf{F} = m\mathbf{a} &\Rightarrow \frac{\mu m}{R^2} \mathbf{n} = -mR \left( \frac{2\pi}{T} \right)^2 \mathbf{n} \\ \Rightarrow R &= \left( \frac{T^2 \mu}{4\pi^2} \right)^{1/3} \end{aligned}$$

[2 POINTS]

6.4 Find the radius (in km) and speed (in km/s) of (i) a satellite in low earth orbit with  $T=100$  minutes; (ii) the moon.

For the satellite in LEO: 7137 km and 7.47 km/s  
For the moon (period is 27 days):  $3.8 \times 10^5$  km and 1.02 km/s

[2 POINTS]