

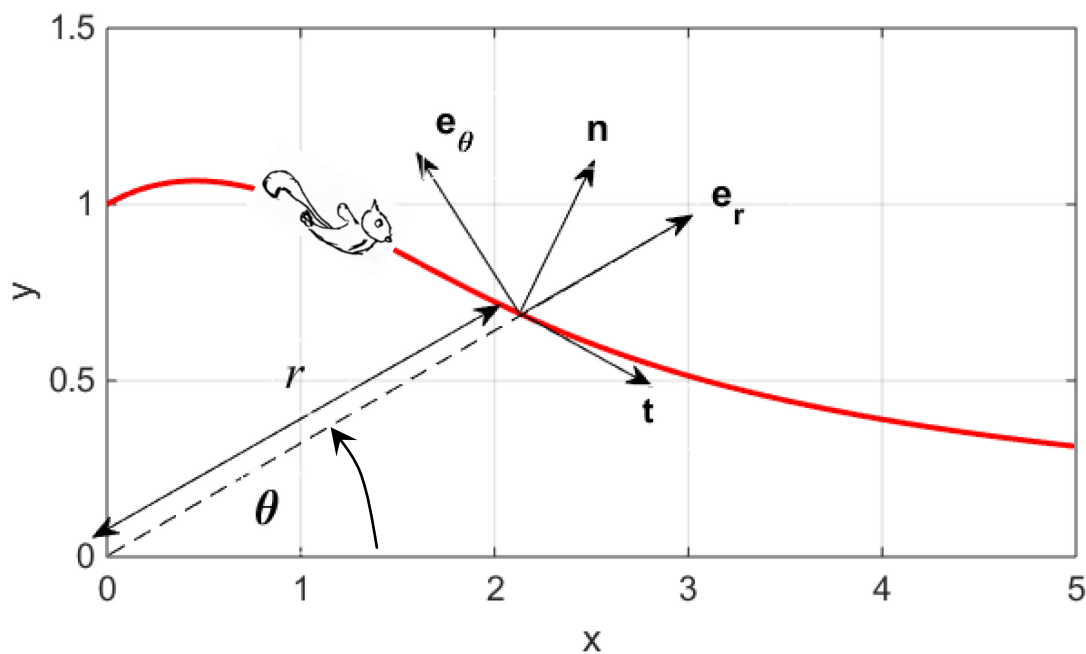


School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 3: Kinematics and Dynamics of Particles Due Friday Feb 16, 2018

Please submit your solutions to the MATLAB coding problems by uploading a SINGLE file, with a .m extension, to Canvas.



1. The figure shows the trajectory of a flying squirrel, which is measured (in polar coordinates) to be

$$r = (1+t) \text{ (meters)} \quad \theta = \frac{\pi}{2(1+t)^2} \text{ (radians)}$$

(Note that $\theta = \pi/2$ at time $t=0$, and decreases as t increases).

1.1 Find a formulas for the velocity vector and the acceleration vector (in polar coordinates $\{\mathbf{e}_r, \mathbf{e}_\theta\}$) in terms of time

The velocity is

$$\mathbf{v} = \frac{dr}{dt} \mathbf{e}_r + r \frac{d\theta}{dt} \mathbf{e}_\theta = \mathbf{e}_r - \frac{\pi}{(1+t)^2} \mathbf{e}_\theta$$

and the acceleration is

$$\begin{aligned}
\mathbf{a} &= \left\{ \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right\} \mathbf{e}_r + \left\{ r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right\} \mathbf{e}_\theta \\
&= -\frac{\pi^2}{(1+t)^5} \mathbf{e}_r + \left\{ (1+t) \cdot \frac{2.3 \cdot \pi}{2(1+t)^4} - 2 \cdot \frac{\pi}{(1+t)^3} \right\} \mathbf{e}_\theta \\
&= -\frac{\pi^2}{(1+t)^5} \mathbf{e}_r + \frac{\pi}{(1+t)^3} \mathbf{e}_\theta = \frac{\pi}{(1+t)^5} \left\{ -\pi \mathbf{e}_r + (1+t)^2 \mathbf{e}_\theta \right\}
\end{aligned}$$

[3 POINTS]

1.2 Find unit vectors normal and tangent to the path of the squirrel **AT TIME $t=0$** , in the $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ basis. Choose the sign of \mathbf{n} so that the \mathbf{n} has a positive \mathbf{e}_r component.

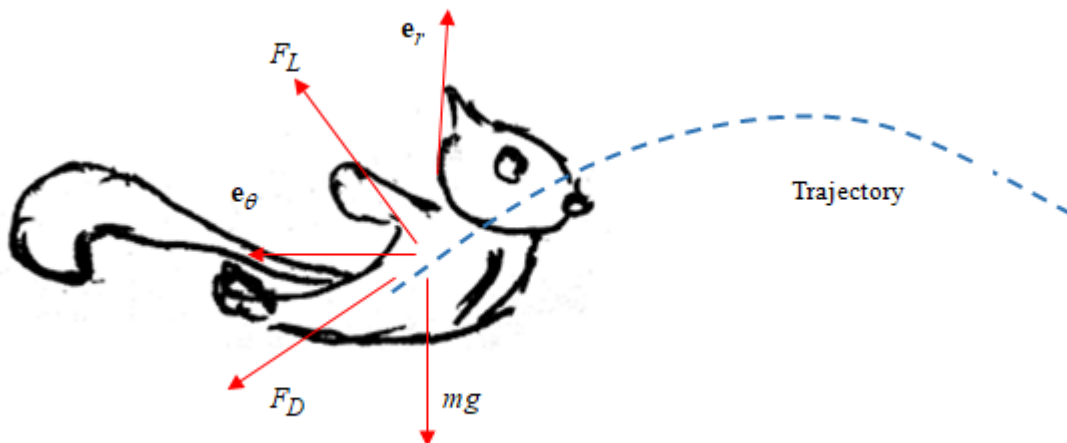
The tangent vector is parallel to \mathbf{v} , and the normal vector can be found as $\mathbf{k} \times \mathbf{t}$, which gives

$$\begin{aligned}
\mathbf{t} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+\pi^2}} (\mathbf{e}_r - \pi \mathbf{e}_\theta) \\
\mathbf{n} &= \pm \mathbf{k} \times \mathbf{t} = \pm \frac{1}{\sqrt{1+\pi^2}} (\pi \mathbf{e}_r + \mathbf{e}_\theta)
\end{aligned}$$

Take the positive sign to give a positive \mathbf{e}_r component.

[2 POINTS]

1.3 Draw a free body diagram showing the forces acting on the squirrel **at time $t=0$** on a copy of the figure provided. Include lift and drag forces, and any other forces you think are relevant. For reference, also draw the unit vectors $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ **AT TIME $t=0$** on your copy of the figure



[3 POINTS]

1.4 If the squirrel has mass 100 grams, calculate the magnitude of the lift and drag force acting on the squirrel **at time $t=0$** (you can use the approximation $g \approx 10 \text{ m/s}^2$ and leave your answer in terms of π - there is no need to get a number for the answer)

Newton's law $\mathbf{F}=\mathbf{ma}$ gives

$$-F_D \mathbf{t} + F_L \mathbf{n} - mg \mathbf{j} = m \mathbf{a}$$

(we use the positive sign for \mathbf{n} in . We can solve for F_D, F_L by taking the dot product with \mathbf{t}, \mathbf{n} and recalling $\mathbf{t} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{n} = 1$ $\mathbf{n} \cdot \mathbf{t} = 0$

$$F_D = -m \mathbf{a} \cdot \mathbf{t} - mg \mathbf{j} \cdot \mathbf{t}$$

$$F_L = m \mathbf{a} \cdot \mathbf{n} + mg \mathbf{j} \cdot \mathbf{n}$$

Note that at time $t=0$ $\mathbf{j} = \mathbf{e}_r$, and we can substitute for the other vectors from previous parts of the problem

$$F_D = -m\pi(-\pi\mathbf{e}_r + \mathbf{e}_\theta) \cdot \frac{1}{\sqrt{1+\pi^2}}(\mathbf{e}_r - \pi\mathbf{e}_\theta) - mg\mathbf{e}_r \cdot \frac{1}{\sqrt{1+\pi^2}}(\mathbf{e}_r - \pi\mathbf{e}_\theta) = m \frac{2\pi^2 - 10}{\sqrt{1+\pi^2}}$$

$$F_L = m\pi(-\pi\mathbf{e}_r + \mathbf{e}_\theta) \cdot \frac{1}{\sqrt{1+\pi^2}}(\pi\mathbf{e}_r + \mathbf{e}_\theta) + mg\mathbf{e}_r \cdot \frac{1}{\sqrt{1+\pi^2}}(\pi\mathbf{e}_r + \mathbf{e}_\theta) = m \frac{\pi(11 - \pi^2)}{\sqrt{1+\pi^2}}$$

[4 POINTS]

2. In HW2 we used straight-line motion formulas and elementary calculus to analyze the motion of the 'Bloodhound' vehicle. In this problem we will do the calculus by setting up and solving the differential equations of motion in MATLAB.



To keep the calculation simple, assume that

- The vehicle has mass 7500kg
- The vehicle is subjected to a constant thrust $T= 190 \text{ kN}$
- The vehicle is subjected to air drag force $F_D = cV^2$, where V is the speed and $c=0.7 \text{ N s}^2/\text{m}$

2.1 Show that the distance traveled by the vehicle $x(t)$ (in meters) satisfies the differential equation

$$\frac{d^2x}{dt^2} = 25.3 - \frac{0.7}{7500} \left(\frac{dx}{dt} \right)^2$$

The acceleration is $\mathbf{a} = \frac{d^2x}{dt^2} \mathbf{i}$; the velocity is $\mathbf{v} = \frac{dx}{dt} \mathbf{i}$, the force is $\mathbf{F} = (T - cV^2) \mathbf{i}$ and substituting numbers into $\mathbf{F}=\mathbf{ma}$ gives the differential equation stated.

[3 POINTS]

2.2 Write a MATLAB ‘Live Script’ to solve the differential equation for x , with initial conditions

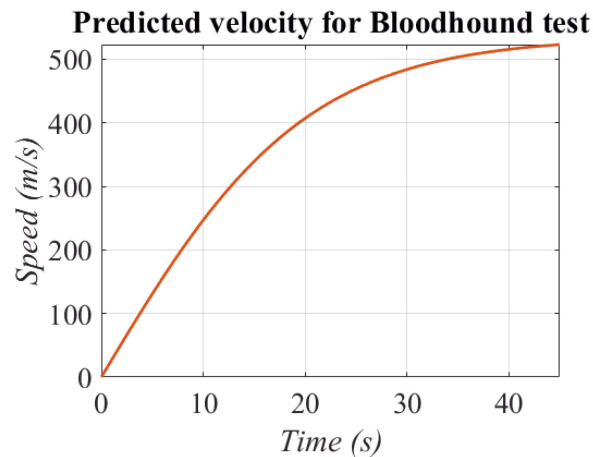
$$x = 0 \quad \frac{dx}{dt} = 0 \quad \text{at time } t=0. \quad \text{Please upload your script to CANVAS.}$$

See the live script for the solution.

[3 POINTS]

2.3 Hence (using your ‘Live Script’) calculate and plot the speed as a function of time, for $0 < t < 50$.

The plot is shown.



[2 POINTS]

3. [This publication](#) describes an ‘electrodynamical trap’ for neutral atoms. The atoms are trapped by a time dependent electric field between 6 electrodes, which exerts the following forces on an atom¹:

$$\mathbf{F} = \frac{3\alpha U_{30} \sin \omega t}{4z_0^4} \left\{ \left(\frac{3U_{30} \sin \omega t}{2z_0^2} (x^2 + y^2) - U_1 \right) (x\mathbf{i} + y\mathbf{j}) + 2 \left(\frac{3U_{30} \sin \omega t}{z_0^2} z^2 + 3U_1 \right) z\mathbf{k} \right\}$$

Here, (x,y,z) is the position of the atom; z_0 is the gap between the electrodes; U_1 and U_{30} are the magnitudes of the voltages applied to the electrodes, ω is the angular frequency of the fluctuating voltage, and α is a constant. The goal of this problem is to calculate the trajectory of an atom in the trap.

¹ Intro physics courses usually discuss forces on *charged* particles. It is less well known that *neutral* atoms also experience (weaker) forces in an electric or magnetic field. The forces arise because of an interaction between the electric or magnetic moment of an atom with the external field. [This paper](#) has a short review.

3.1 Show that the equations $\mathbf{v} = d\mathbf{r} / dt$ and $\mathbf{F} = m d\mathbf{v} / dt$ (m is the atom's mass) can be expressed as

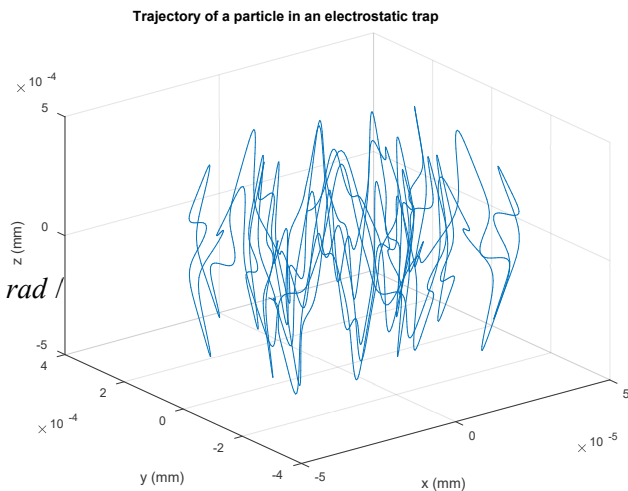
$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ C_1(C_2(x^2 + y^2) - U_1)x \\ C_1(C_2(x^2 + y^2) - U_1)y \\ 2C_1(2C_2z^2 + 3U_1)z \end{bmatrix} \quad C_1 = \frac{3\alpha U_{30} \sin \omega t}{4mz_0^4} \quad C_2 = \frac{3U_{30} \sin \omega t}{2z_0^2}$$

The $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of the position – velocity relation gives the first 3 rows of this equation; the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components of $\mathbf{F} = m\mathbf{a}$ gives the last 3 rows. This problem is trivial – it's included just so people can check their equations before coding them in MATLAB...

[2 POINTS]

3.2 Write a MATLAB script to solve the equations in part 2.1, with the following parameters.

- Initial position: $(x = 0, y = 0.1, z = 0.1)$ mm; initial velocity $(v_x = 0.01, v_y = v_z = 0)$ m/s
- Trap parameters: $z_0 = 1$ mm, $U_1 = 8000V$ $U_2 = 4000V$ $\omega = 1600\pi$ rad/s
- Atom properties (representing Li) $\alpha = 10^{-39} Jm^2 / V^2$, $m = 1.15 \times 10^{-26}$ kg
- Run the simulation for $0 < t < 0.1$ sec.



Plot the 3D trajectory of the atom. Be careful with units – don't forget to convert mm to m.

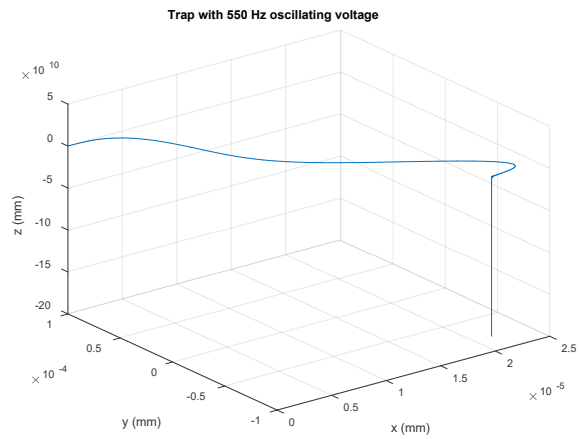
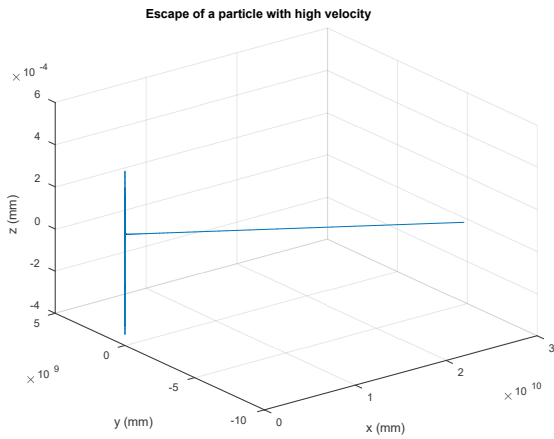
The trajectory is shown.

[4 POINTS]

3.3 Use your simulations to show that:

- Atoms with a high velocity will escape the trap (try initial velocity $v_x = 0.1v_y = v_z = 0$)
- The frequency of the fluctuating voltage must exceed a critical value to trap the particle (try, eg a simulation with $\omega = 1100\pi \text{ rad / s}$)

Trajectories are shown for the two suggested cases.

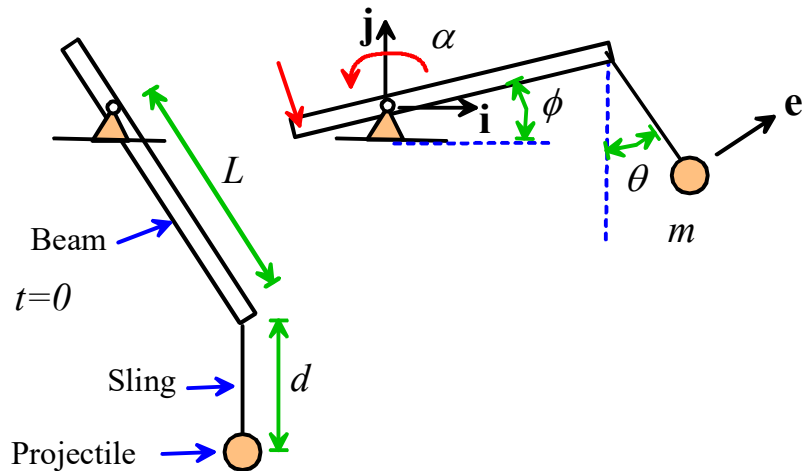


[2 POINTS]

4. The goal of this problem is to optimize a simple model of a trebuchet, illustrated in the figure. Assume that the beam (or throwing arm) starts at rest with $\phi = -\pi/3$, and for $t > 0$ has a constant angular acceleration $\frac{d^2\phi}{dt^2} = \alpha$ (See footnote²).

The sling starts with a vertical orientation $\theta = 0$, and rotates counterclockwise during the launch.

The sling is released to launch the projectile when the velocity vector of the projectile is at a 45 degree angle to the horizontal. The goal of this problem is to find the optimal ratio of sling length d to arm length L that maximizes the launch velocity.



4.1 Write down a formula for the angle ϕ in terms of α and time.

This is calculus – we can use the straight-line motion formulas to integrate angular accelerations just the same way as linear accelerations. So

$$\phi = -\frac{\pi}{3} + \frac{1}{2}\alpha t^2$$

[1 POINT]

4.2 Write down a formula for the position vector of the projectile, in terms of ϕ, θ, L, d

Using trig: $\mathbf{r} = (L \cos \phi + d \sin \theta)\mathbf{i} + (L \sin \phi - d \cos \theta)\mathbf{j}$

[1 POINT]

4.3 By differentiating the position vector, find formulas for the velocity and acceleration of the projectile (it is helpful to use a Matlab ‘Live Script’ to do all the algebra in this problem – doing the calculation by hand is quite painful)

Using the Live Script:

$$\mathbf{v} = \left[d \frac{d\theta}{dt} \cos(\theta) - L \alpha t \sin \phi \right] \mathbf{i} + \left[d \frac{d\theta}{dt} \sin(\theta) + L \alpha t \cos \phi \right] \mathbf{j}$$

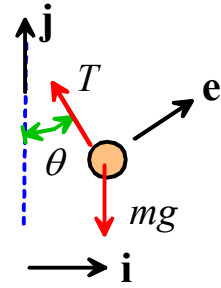
$$\mathbf{a} = \left[d \cos(\theta(t)) \frac{\partial^2}{\partial t^2} \theta(t) - d \sin(\theta(t)) \left(\frac{\partial}{\partial t} \theta(t) \right)^2 - \alpha \sin \left(\frac{\alpha t^2}{2} - \frac{\pi}{3} \right) - \alpha^2 t^2 \cos \left(\frac{\alpha t^2}{2} - \frac{\pi}{3} \right) \right] \mathbf{i}$$

$$+ \left[\alpha \cos \left(\frac{\alpha t^2}{2} - \frac{\pi}{3} \right) + d \cos(\theta(t)) \left(\frac{\partial}{\partial t} \theta(t) \right)^2 + d \sin(\theta(t)) \frac{\partial^2}{\partial t^2} \theta(t) - \alpha^2 t^2 \sin \left(\frac{\alpha t^2}{2} - \frac{\pi}{3} \right) \right] \mathbf{j}$$

[2 POINTS]

² This is not a very accurate assumption: a realistic model should solve a second equation of motion for the beam and counterweight, but that’s too much work for just a homework assignment....

4.4 Draw a free body diagram showing the forces acting on the projectile as it is launched (include gravity, and assume that the sling is a two-force member).



[2 POINTS]

4.5 Write down $\mathbf{F} = m\mathbf{a}$ for the projectile

$$\begin{aligned}
 & T(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - mg\mathbf{j} \\
 = m & \left[d \cos(\theta(t)) \frac{\partial^2}{\partial t^2} \theta(t) - d \sin(\theta(t)) \left(\frac{\partial}{\partial t} \theta(t) \right)^2 - \alpha \sin\left(\frac{\alpha t^2}{2} - \frac{\pi}{3}\right) - \alpha^2 t^2 \cos\left(\frac{\alpha t^2}{2} - \frac{\pi}{3}\right) \right] \mathbf{i} \\
 & + \left[\alpha \cos\left(\frac{\alpha t^2}{2} - \frac{\pi}{3}\right) + d \cos(\theta(t)) \left(\frac{\partial}{\partial t} \theta(t) \right)^2 + d \sin(\theta(t)) \frac{\partial^2}{\partial t^2} \theta(t) - \alpha^2 t^2 \sin\left(\frac{\alpha t^2}{2} - \frac{\pi}{3}\right) \right] \mathbf{j}
 \end{aligned}$$

[2 POINTS]

4.6 Write down a formula for a unit vector \mathbf{e} perpendicular to the sling, in terms of θ (see the figure)

$$\text{Simple trig gives } \mathbf{e} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

[1 POINT]

4.7 Note that the tension force in the sling acts perpendicular to \mathbf{e} . We can therefore eliminate the unknown tension in the sling from 3.5 by taking the dot product of both sides of $\mathbf{F} = m\mathbf{a}$ with \mathbf{e} (i.e. calculate $\mathbf{F} \cdot \mathbf{e} = m\mathbf{a} \cdot \mathbf{e}$). Use this trick to show that θ satisfies the equation of motion (use your 'Live Script' to do the simplification)

$$d \frac{d^2\theta}{dt^2} = -g \sin\theta + L\alpha^2 t^2 \cos(\theta - \phi) - L\alpha \sin(\theta - \phi)$$

See the Live Script for solution

[2 POINTS]

4.8 Rearrange the equation in 4.7 into a form that MATLAB can solve with 'ode45'

We have to turn the 2nd order equation into two first order equations. We do this by introducing $\omega = \frac{d\theta}{dt}$ as another unknown. Then we can write

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ -(g/d)\sin\theta + (L/d)\alpha^2 t^2 \cos(\theta - \phi) - L\alpha \sin(\theta - \phi) \end{bmatrix}$$

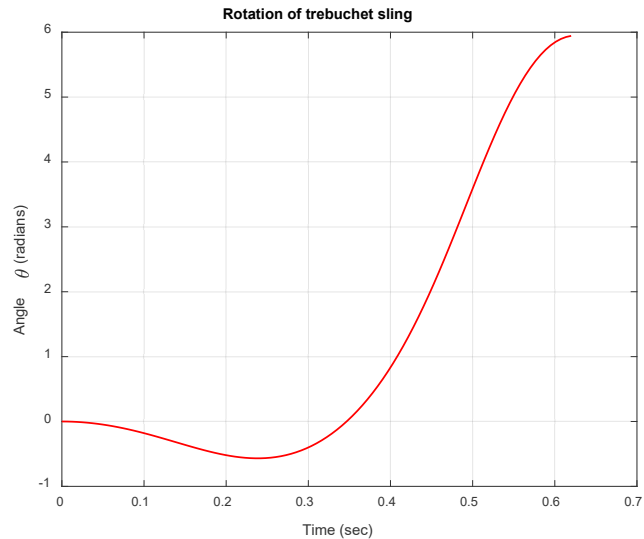
[1 POINT]

4.9 Write a MATLAB script to solve the differential equation, and plot a graph of θ as a function of time. Use the following values for parameters: $L = 2m$ $g = 9.81m/s^2$ $\alpha = 5g/L$ $d = 1.1m$. Assume that $\theta = d\theta/dt = 0$ at time $t=0$, and run your simulation for a time interval $0 < t < \sqrt{3\pi/\alpha}$

Optional: you can download a function [here](#) that will animate the motion of the system. Cut and paste the script into your MATLAB code (put it anywhere below the 'end' statement in your main code), and call the function after you use the 'ode45' function with

```
animate_trebuchet(times,sols
,alpha,L,d)
```

Here, `times,sols` are the (vector and matrix) outputs from `ode45`; and the other symbols are parameters in the system.



[4 POINTS]

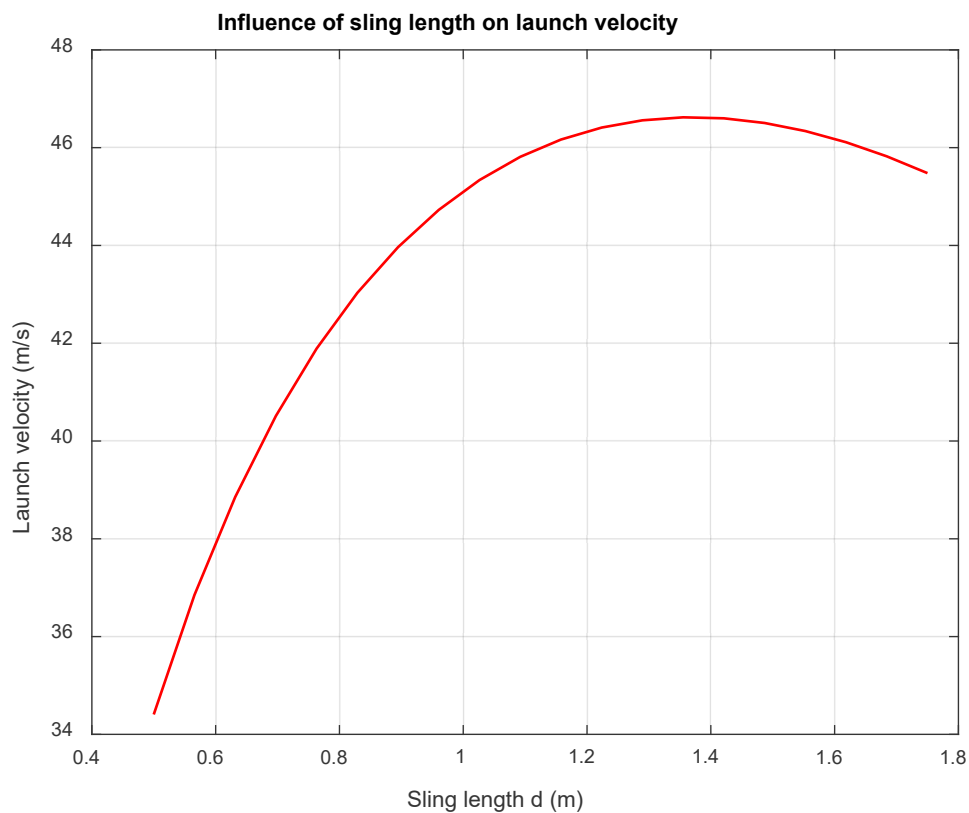
4.10 Add an ‘event’ function to your script that will stop the calculation when the components of the velocity vector satisfy $v_x + v_y = 0$ (see footnote³). Note that (see the animation in 3.9) $v_y > 0, v_x > 0$ near the start of the simulation, so we expect $v_x + v_y$ to cross zero from above.

Hence plot a graph showing the magnitude of the velocity of the projectile at the instant it is released from the trebuchet (i.e. at the end of the simulation) as a function of the sling length d in the range $0.5 < d < 1.75m$ (you will need to do this with a loop that (i) uses ode45 to solve the equations 3.8 for progressively increasing d values; and (ii) stores the values of d and the velocity at the end of the simulation in MATLAB vectors that you can plot. Make sure you comment out the animation when you do this, or the calculation will take forever.)

Hence, estimate (using your graph) the optimal value of d/L .

The optimal d/L is about $1.35/2=0.675$

[4 POINTS]



³ Hopefully you can see that this means the angle between the velocity vector and the horizontal is 45 degrees.