School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Conservation Laws for Particles
Due Friday March 2, 2018

1. This publication describes calibrated 'Mie' potentials for pseudoatoms[1] in hydrocarbon molecules. The potential specifies the energy of the bond between two pseudoatoms as a function of the distance r between them

$$U(r)=\varepsilon\frac{n}{n-m}\left(\frac{n}{m}\right)^{m/(n-m)}\left\{\left(\frac{\sigma}{r}\right)^{n}-\left(\frac{\sigma}{r}\right)^{m}\right\}$$

Where ε,σ,n,m are constants. For a C-H pseudoatom they give $\varepsilon=60k_B$ (k_B is the Boltzmann constant, $\sigma=3.81\overset{o}{\mathrm{A}}$ (angstroms – 10^{-10}m), n=16, m=6.

1.1 Find an expression for the force of attraction for the Mie Potential (give a general formula. You can use a Live Script, but a hand calculation is probably easier).

Power-law potentials are easy to differentiate….

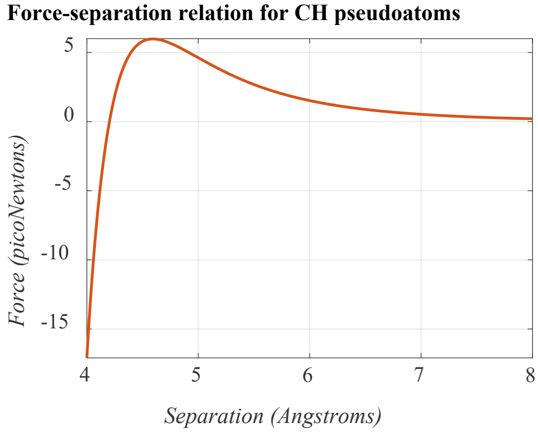
$$F(r)=-\frac{dU}{dr}=\frac{\varepsilon}{r}\frac{n}{n-m}\left(\frac{n}{m}\right)^{m/(n-m)}\left\{n\left(\frac{\sigma}{r}\right)^{n}-m\left(\frac{\sigma}{r}\right)^{m}\right\}$$

Graders – MATLAB is fine; also check for algebraically equivalent versions (eg the powers could be n-1 and m-1).

[2 POINTS]

1.2 Plot a graph of the force for C-H pseudoatoms as a function of r, for $4.\overset{o}{\mathrm{A}}<r<8.\overset{o}{\mathrm{A}}$.

The plot is shown. See also the Live Script. OK to make the graph negative; also other unit systems for the axes are fine as long as they are clear.



[2 POINTS]

[1] A pseudoatom includes a carbon and one or more hydrogen atoms

1.3 For the special case of n=16, m=6, find a formula for the force required to break the bond (i.e. the maximum value of the force), and calculate its value for the bond between C-H pseudoatoms.

Using a Live-Script:

$$F_{\max} = \frac{483^{1/10} 14^{7/10} 17^{3/10} \varepsilon}{289\sigma}$$

Substituting numbers: $F_{\max} = 5.98\,pN$

[2 POINTS]

1.4 For the special case of n=16, m=6, find a formula for the stiffness of the bond (i.e. the slope of the force-v-separation curve at zero force) , and calculate its value for C-H pseudoatoms.

The expression for the stiffness is

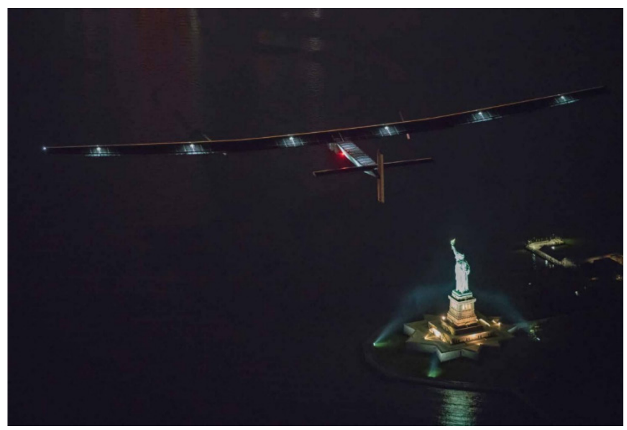
$$k = \frac{48 2^{2/5} 3^{1/5} \varepsilon}{\sigma^2}$$

For the C-H bond we get 0.45 N/m

[2 POINTS]

2. The 'Solar Impulse 2' is a solar powered aircraft that circumnavigated the world. Its designers provide the following specifications

- **Propulsion system:** 4 brushless electric motors rated at 17.4 horsepower (each)
- **Total battery mass** 633kg
- **Battery energy density** 260 Wh/kg
- **Solar panel capacity** 340kWh per day
- **Mass** 2300 kg
- **Airspeed**[2] 60 km/hr
- **Overall power-plant efficiency 94%**



2.1 The aircraft takes 4 hours to descend from a 9000m altitude to 1500m with negligible engine power. Calculate the change in potential energy of the vehicle during the descent, and determine the drag force acting on the aircraft (assume a constant flight speed of 60km/hr).

The change in PE is $\Delta W = mg\Delta h = 2300 \times 9.81 \times (9000 - 1500) = 169.2MJ$
In 4 hrs at 60 km/hr the aircraft travels d=240 km.
The work done by the drag force is $F_D d$
The work done by the drag force must equal the change in potential energy, so the drag force is $169.2 \times 10^6 / (240 \times 10^3) = 705N$
Doing the problem with power instead of work is also fine.

[2 POINTS]

[2] The aircraft varies speed to maximize its endurance – because of variations in air density the optimal speed varies with altitude.

2.2 During the night, the aircraft flies for 8 hours at 60 km/hr under battery power only. Calculate the total energy expended during this phase of flight, and determine the margin of spare battery capacity in the design.

The energy expended must be twice that in 2.1… This is $338.4MJ$. Or you can do the problem indirectly by calculating the drag power using the drag force and multiplying by the time.

The total battery capacity is $633 \times 260 \times 3600 = 592.5MJ$
With a 94% efficiency this gives a safety factor of about 1.65.
(Calculating the spare capacity is also OK – any sensible way to report the numbers is fine).

[2 POINTS]

2.3 Estimate the total energy that must be generated by the solar panels for one complete flight cycle. Compare your estimate with the panel capacity quoted by the designers.

From 2.1 it costs 169.2 MJ to stay aloft for 4 hours. Over 24 hours this comes to 169.2*24/4=1015.2MJ. Since the efficiency is 94% at least 1080 MJ of solar power must be generated per day

The designers specification gives 340kWh – this is 1224 MJ. The margin of error is extremely small, but our estimate of energy consumption is rather crude – variations in air density with altitude have a big effect on drag (air density at 9000m is less than half that at 1500m).

[2 POINTS]

2.4 Estimate the maximum rate of climb of the aircraft (assume a 60 km/hr airspeed). How long does it take to reach 9000m at the max climb rate?

The max engine power is 4*17.4 hp = 51.9 kW. The power expended in overcoming drag (using numbers from 2.1) is $169.2MJ / (4 \times 3600s) = 11.75kW$. The power available to climb is 40.15kW. The power required to climb is mgv_y so the vertical velocity is

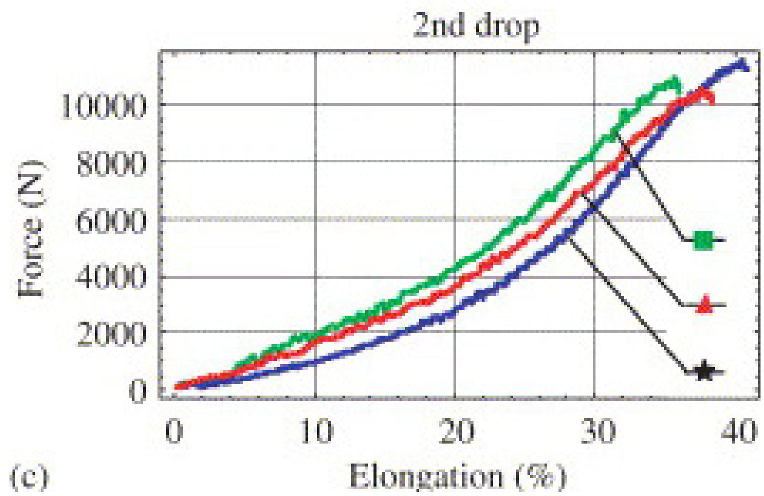
$$v_y = 40.15 \times 10^3 / (2300 \times 9.81) = 1.78m/s$$.

This may be an overestimate – it's not clear whether the engine power reported includes the 0.94% efficiency. If we want to give a safer estimate then

$$v_y = 0.94 \times 40.15 \times 10^3 / (2300 \times 9.81) = 1.67m/s$$

It takes about 1.2 hrs to reach 9000m (starting at 1500m) (a bit longer if we include the 94%).
In practice a much slower rate of climb is used – the aircraft spends all day reaching 9000m.

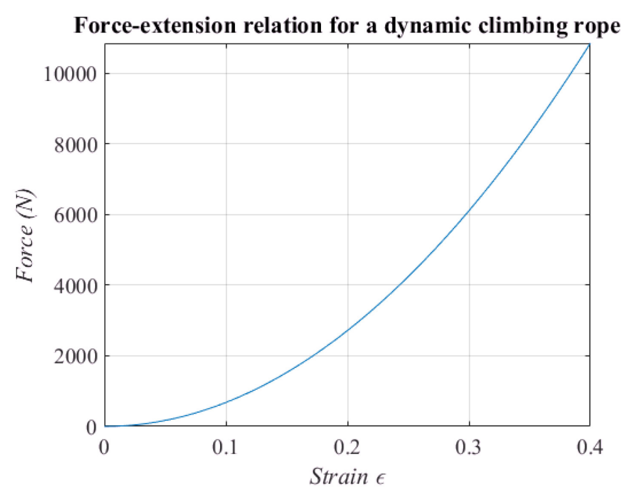
[2 POINTS]



3. The figure (from this publication) shows force-v-elongation curves for a 'dynamic' climbing rope[3] (the three curves are for three different levels of moisture content in the rope). The 'elongation' represents $\varepsilon = \Delta l / l$ where l is the unstretched length of the rope and Δl is its change in length (the horizontal axis shows 100ε)

3.1 Estimate values for the constant F_0 in a quadratic fit ($F = F_0 \varepsilon^2$) to the force *(F)*-v-elongation (ε) curve (use the blue, starred curve – this is for a rope with the greatest moisture content. You don't have to be too precise – just find a point on the curve somewhere sensible and use it to fit; we aren't getting paid to do this. Well, one of us actually is, but *you* aren't…).

The figure shows a fit with $F_0 = 6.77 \times 10^4$



[2 POINTS]

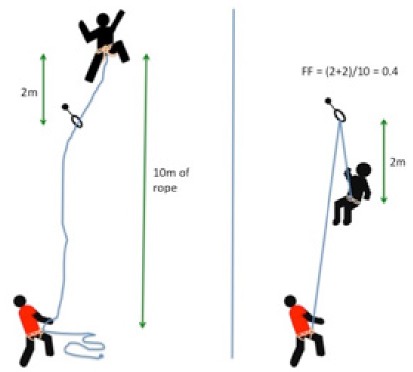
3.2 Find a formula for the work done to stretch a rope with initial length L_0 to a deformed length L_1

$$W = \int_{L_0}^{L_1} F(x)dx = \int_{L_0}^{L_1} \frac{F_0}{L_0^2}(x - L_0)^2 \, dx = \frac{1}{3}\frac{F_0}{L_0^2}(L_1 - L_0)^3$$

[2 POINTS]

[3] Dynamic rope is designed to stretch under load so as to break a climber's fall.

3.3 Consider a climber with mass *m* who is tethered by a length of rope *L*. Suppose the climber falls through a distance *h* before the rope begins to stretch, and is then brought to rest as the rope stretches (assume the rope and fall are both vertical – see the figure from this website).



Use energy conservation to show that the fractional change in length of the rope ε at the instant of maximum stretch satisfies

$$f = \frac{F_0}{3mg}\varepsilon^3 - \varepsilon$$

Where $f = h/L$ is the 'fall factor'

Let x be the change in length (in m) of the rope. The kinetic energy before the drop and at the bottom of the drop are both zero. Taking the datum for the gravitational potential energy at the height of the climber before the fall, we get

$$0 = -mg(h+x) + \frac{F_0}{3L^2}x^3$$

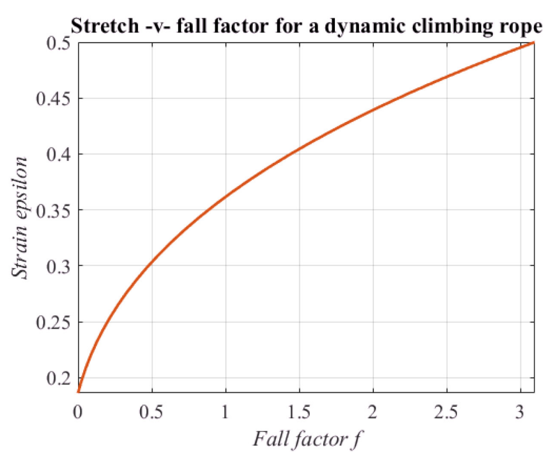
Divide through by L and mg and let $f = h/L \quad \varepsilon = x/L$ to see the answer.

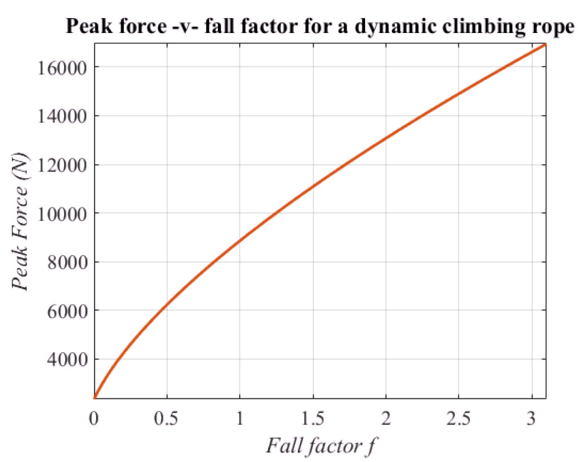
[2 POINTS]

3.4 Find a formula for the stretch resulting from a zero fall factor. Plot graphs of (i) the expected rope stretch ε for an 80kg climber as a function of fall factor f; and (ii) the expected maximum force in the rope, as a function of f.

If $f=0$ the stretch is $\varepsilon = \sqrt{\frac{3mg}{F_0}}$

This helps do the plots parametrically so we don't have to solve any equations.





(other limits for the axes are fine – going up to 50% strain is a bit higher than the rope can withstand.

[2 POINTS]

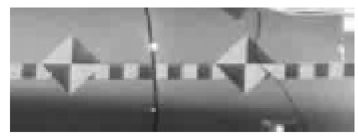
4. The NHTSA publishes extensive raw data taken from vehicle crash tests. This includes a set of high-speed videos were taken during a recent test on a 2018 Chevrolet Camaro. In this problem you will use their data to estimate the forces acting on the vehicle during the crash, the restitution coefficient of the frontal impact, and the impulse exerted on the vehicle during the impact.

The vehicle test weight was 1875kg

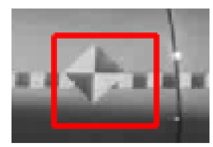
You will need to

1. Download the movie file crash.wmv from the EN40 Homework webpage
2. Download the Matlab script track_crash.m from the EN40 Homework webpage.

Save both files in the same directory. Then run the script to create a graph and a csv file of position –v- time for the body of the vehicle (it will take Matlab quite a long time to read the image – be patient!). The script will ask you to



1. Click on two adjacent tracking points on the vehicle body. The script counts the pixels between these points and uses the known distance between them (30.5cm) to determine the number of pixels per cm.
2. Select one of the tracking markers on the vehicle body (pick one that's not in the crumple zone)
3. Click on a point inside the rectangle that you would like to track in subsequent images.



The script will plot a graph of the horizontal displacement of your reference point (in cm) as a function of time. The data will be saved in a csv file that you can read in your own code for further analysis (you can read a csv file using `data = csvread('filename.csv');`)

4.1 Write a MATLAB script to calculate and plot graphs of the velocity and acceleration of the car during the test. You will need to differentiate the position –v- time data: you can do this by calculating the change in position between two successive readings, and dividing by the time difference between them, e.g. if $x(i)$ denotes the ith value of x, then

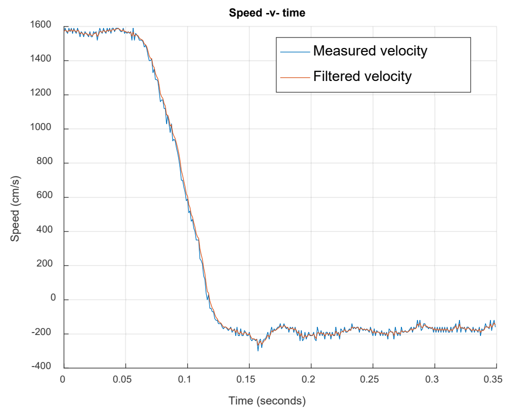
$$v_x(i) = \left(x(i) - x(i-1)\right) / (t(i) - t(i-1))$$

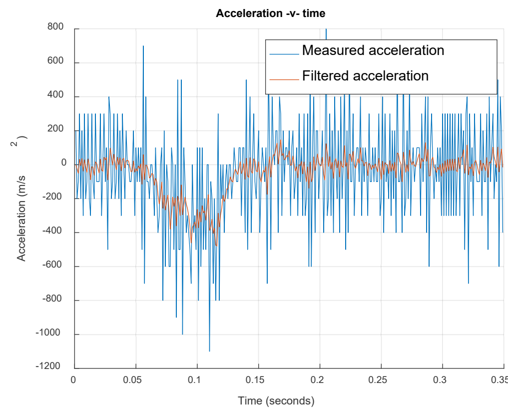
(you did a similar calculation for HW2). The data will be noisy, particularly the accelerations: you can use a simple first-order filter to smooth it, as follows: Let **y** be a vector (a list of numbers, eg velocity) that needs to be filtered. A vector **z** containing the filtered signal (a second list of numbers) can be constructed as follows:

$$z_1 = y_1$$

$$z_i = \alpha y_i + (1-\alpha) z_{i-1} \qquad i = 2,3,4....n$$

where $0 < \alpha < 1$ is a parameter that controls the cutoff frequency of the filter ($\alpha = 0.5$ works for the data in this problem, but you can try other values), and y_i, z_i denotes the *ith* value of **y** and **z**, respectively, and n is the length of the vector **y**. Please upload your Matlab script to Canvas as a solution to this problem. Please hand in a separate copy of your plots with your HW assignment.





[4 POINTS]

4.2 Use the data to estimate the restitution coefficient for the crash.

We can just read off the velocities before (1570 cm/s) and after (-200 cm/s) impact. The restitution formula gives

$$e \approx -(-200)/1570 = 0.13$$

(there will be some variability in these numbers depending on what velocity points are chosen)

[1 POINT]

4.3 Calculate the impulse exerted on the car during the impact, by (i) using its change in momentum; and (ii) using your MATLAB code from 4.1 to calculate and integrate the force acting on the car during the collision (use the matlab 'trapz' function to integrate the force).

The impulse-momentum formula gives $I = mv_x^{(1)} - mv_x^{(0)} = 1875\left(-200-(1571)\right)\times 10^{-2} = 33200Ns$

Integrating the force (obtained from the filtered acceleration) gives I=-32349.5Ns

(these are in the negative **i** direction, of course)

[2 POINTS]

5. The attractive force between two atoms in a diatomic molecule is related to the distance r between them by

$$F = 8F_0\left(\left(\frac{a}{r}\right)^5 - \left(\frac{a}{r}\right)^9\right)$$

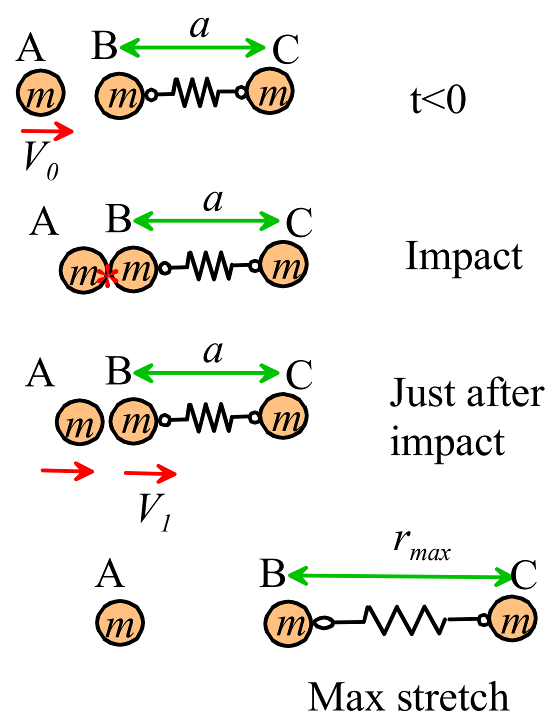
where F_0 and a are constants. Show that the potential energy of the interatomic bond is (to within an arbitrary constant)

$$V(r) = F_0 a\left(1 - 2\left(\frac{a}{r}\right)^4 + \left(\frac{a}{r}\right)^8\right)$$

By definition $V = -\int_a^r \mathbf{F}(r)\cdot d\mathbf{r} + C = -\int_a^r -8F_0\left(\left(\frac{a}{r}\right)^5 - \left(\frac{a}{r}\right)^9\right)\mathbf{i}\cdot dr\mathbf{i} = F_0 a\left(1 - 2\left(\frac{a}{r}\right)^4 + \left(\frac{a}{r}\right)^8\right)$

[3 POINTS]

6. The figure shows a straight-line collision between an ion (A) and a (neutral) diatomic molecule (B-C)



- The ion and each atom are idealized as hard spheres with mass m.
- **The bond between atoms B and C has potential energy given in problem 3**
- For time $t<0$ atoms the bond between B and C has zero force (i.e. the distance between atoms $r=a$); and the ion A is moving to the right with speed V_0 .
- The ion A collides with atom B at time $t=0$. The collision has restitution coefficient $e=1$.

6.1 Calculate the velocity V_1 of atom B just after the collision, in terms of V_0 (note that the bond between B and C exerts no forces during the collision)

This is a straight line perfectly elastic collision. The velocity of A is zero after impact, and the velocity of B is equal to V_0. Note that C is stationary at this instant (no force acts on C during the impulse because there is no force in the bond).

[2 POINTS]

6.2 What is the total linear momentum of the molecule with atoms (BC) just after the impact, in terms of m and V_0 ?

$$\mathbf{p} = mV_0\mathbf{i}$$

[1 POINT]

6.3 What is the total kinetic energy of the molecule with atoms (BC) just after the impact, in terms of m and V_0 ?

$$T = \frac{1}{2}mV_0^2$$

[1 POINT]

6.4 Using energy and momentum conservation, show that the maximum separation between atoms in molecule BC following the collision is (Hint: at the instant of maximum separation both atoms B and C have the same velocity).

$$r = a\frac{1}{\left(1-\frac{1}{2}\sqrt{\frac{mV_0^2}{F_0a}}\right)^{1/4}}$$

Note that (1) momentum is conserved; and (3) energy is conserved for the molecule with atoms BC

Let V_2 denote the velocity of C and D at the instant of maximum separation. Momentum and energy conservation give

$$mV_0 = 2mV_2$$

$$\frac{1}{2}mV_0^2 = F_0a\left(1-2\left(\frac{a}{r}\right)^4+\left(\frac{a}{r}\right)^8\right)+mV_2^2$$

Eliminating V_2

$$0 = \left(1 - \frac{1}{4}\frac{mV_0^2}{F_0 a} - 2\left(\frac{a}{r}\right)^4 + \left(\frac{a}{r}\right)^8\right)$$

This is a quadratic equation for $(a/r)^4$ with solution

$$\left(\frac{a}{r}\right)^4 = 1 \pm \frac{1}{2}\sqrt{4 - 4\left(1 - \frac{1}{4}\frac{mV_0^2}{F_0 a}\right)} = 1 - \frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}}$$

$$\Rightarrow r = a\frac{1}{\left(1 - \frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}}\right)^{1/4}}$$

(we took the negative square root in the quadratic because we are looking for solutions with $r>a$)

[4 POINTS]

6.5 Determine the critical value of V_0 that will just break the bond between the atoms (assume that the bond breaks if $r \to \infty$ after the collision).

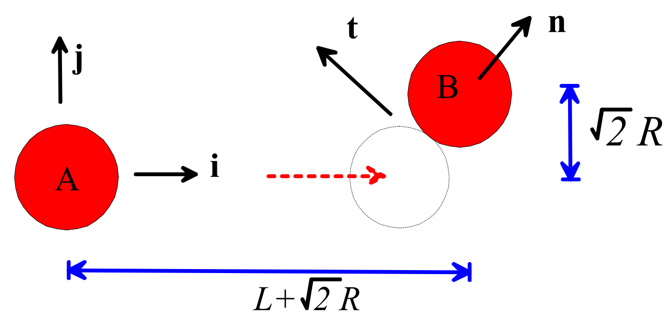
The bond breaks when $r \to \infty$ (so there is no more attractive force between the atoms in the molecule) which gives

$$1 - \frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}} = 0 \Rightarrow \frac{mV_0^2}{F_0 a} = 4 \Rightarrow V_0 = 2\sqrt{\frac{F_0 a}{m}}$$

[2 POINTS]

7. The figure illustrates a frictionless collision between two spheres. They both have radius R and mass m. The restitution coefficient for the collision is $e=1$.

At time $t=0$ the centers of the spheres have position vectors $\mathbf{r}^A = 0 \quad \mathbf{r}^B = L\mathbf{i} + \sqrt{2}R\mathbf{j}$ and velocity vectors $\mathbf{v}^A = V\mathbf{i} \quad \mathbf{v}^B = \mathbf{0}$.



7.1 Using basic geometry calculate unit vectors $\mathbf{n}, \mathbf{t}$ parallel and perpendicular to the line connecting the centers of the two spheres at the instant that they collide, as **i,j** components.

The distance between the centers is 2R, so **n** must be inclined at an angle
$\theta = \sin^{-1}(\sqrt{2}R/(2R)) = 45^o$
to the **i** direction.

The nomal and tangent vectors (which must be unit vectors) follow as
$\mathbf{n} = (\mathbf{i}+\mathbf{j})/\sqrt{2} \qquad \mathbf{t} = \pm(\mathbf{j}-\mathbf{i})/\sqrt{2}$

[2 POINTS]

7.2 Use the formulas derived in class to find expressions for the velocities of A and B after the collision, in **i,j** components.

From class:

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$
$$\mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

With the numbers given $\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n} = \left(-V\mathbf{i}\cdot(\mathbf{i}+\mathbf{j})/\sqrt{2}\right)(\mathbf{i}+\mathbf{j})/\sqrt{2} = -V(\mathbf{i}+\mathbf{j})/2$

Therefore

$$\mathbf{v}^{A1} = V(\mathbf{i}-\mathbf{j})/2 = \frac{V}{\sqrt{2}}(\mathbf{i}-\mathbf{j})/\sqrt{2}$$
$$\mathbf{v}^{B1} = V(\mathbf{i}+\mathbf{j})/2 = \frac{V}{\sqrt{2}}(\mathbf{i}+\mathbf{j})/\sqrt{2}$$

[2 POINTS]

7.3 What is the angle between the paths of A and B after the collision (i.e. the angle between their velocity vectors)? What is the angle between the path of A before collision and its path after collision?

$\mathbf{v}^{A1} \cdot \mathbf{v}^{B1} = 0$ so the angle between them is 90 degrees.

Similarly $\dfrac{\mathbf{v}^{A1} \cdot \mathbf{v}^{A0}}{\left|\mathbf{v}^{A1}\right|\left|\mathbf{v}^{A0}\right|} = \dfrac{1}{\sqrt{2}}$ so the angle between the path before and after collision is 45 degrees.

[2 POINTS]

7.4 What is the impulse exerted on A during the collision (in terms of *m* and *V*)? Give your answer in both **i,j** and $\mathbf{n}, \mathbf{t}$ components. What is the impulse on B? What is the total impulse on the system?

We can use the impulse-momentum formula:

$$\mathbf{I} = m_A(\mathbf{v}^{A1} - \mathbf{v}^{A0}) = -mV\left(\mathbf{i} + \mathbf{j}\right) = -m\frac{V}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) / \sqrt{2}$$

Note that the impulse is parallel to **n**, as expected for a frictionless collision.

The impulse on B is equal and opposite to that on A. $\mathbf{I} = mV\left(\mathbf{i} + \mathbf{j}\right) = m\frac{V}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) / \sqrt{2}$

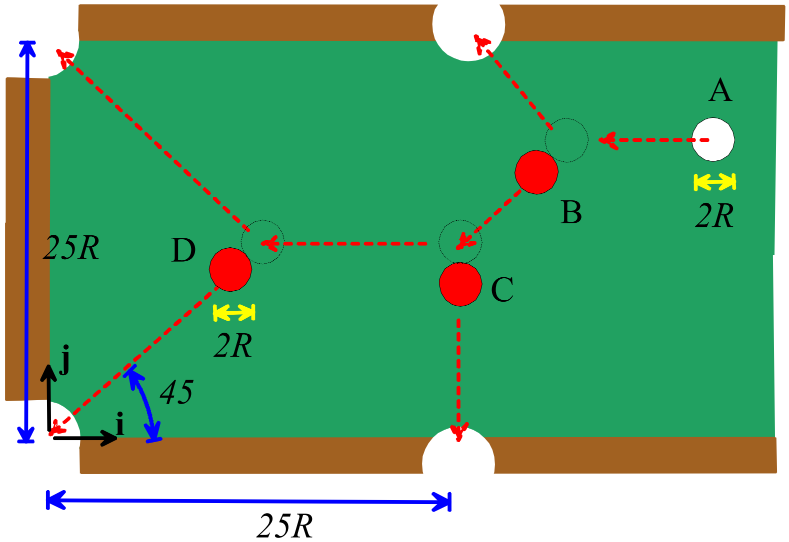
The total impulse is zero.

[2 POINTS]

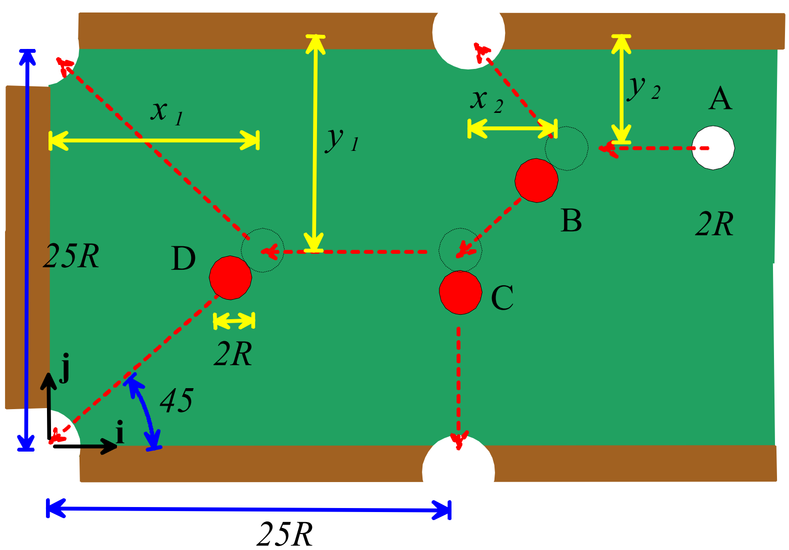
7.5 Please answer the following questions:
(a) Why is the total momentum of the system conserved during the collision?
(b) Why is the momentum of A and B conserved parallel to **t** during the collision?
© Why is the momentum of A and B *not* conserved parallel to **n**?

(a) No external impulse acts on A/B during the collision (any external forces exert a finite force, and we assume that the impact occurs over a vanishingly short period of time, so force*time from external forces is negligible). The impulse-momentum formula for a system of particles therefore shows that the momentum is conserved.
(b) The contact between A and B is frictionless. The force must therefore act parallel to **n**. Since no force acts on either A or B during the impact, the impulse-momentum formula for a single particle shows that momentum is conserved parallel to **t**
(c) A very large force acts on both A and B parallel to **n** during the collision. The force exerts a finite impulse on both A and B. The impulses on the two spheres are equal and opposite.

[3 POINTS]



7.6 The figure shows a trick pool shot. All the balls have the same radius R and mass m. Using the answers to 6.3, calculate initial positions of the four balls A,B,C,D that will sink all four balls, as indicated by the arrows (note that the origin is in the bottom left corner). The initial velocity of the cue ball (white) is $-V\mathbf{i}$.



We know the deflection angles are all 45 degrees, so geometry gives the dimensions shown in the figure as follows

$$x_1 = 25R/2 \qquad y_1 = 25R/2$$

$$x_2 = 25R/4 \qquad y_2 = 25R/4$$

From these we can calculate the coordinates (correcting for the ball radii) as follows

$$x_A = 25R\left(1+\frac{1}{4}\right)+\lambda \qquad y_A = 25R\left(1-\frac{1}{4}\right)$$

$$x_B = x_A - \lambda - \sqrt{2}R \qquad y_B = y_A - \sqrt{2}R$$

$$x_C = 25R \quad y_C = \frac{21}{2}R$$

$$x_D = \frac{25R}{2} - \sqrt{2}R \qquad y_D = \frac{25}{2}R - \sqrt{2}R$$

where λ is any sensible number.

[3 POINTS]

7.7 **Optional (no credit)** Check your answer by downloading the MATLAB p-code from the HW website. Run the code by

- opening Matlab,
- navigating to the directory with the downloaded code, and then
- typing check_hw4_p6(rA,rB,rC,rD), in the Matlab command window, w here rA, rB, rC, rD are four column vectors containing the coordinates of the four balls. Assume a ball radius *R=1*. You will see an animation of your shot.