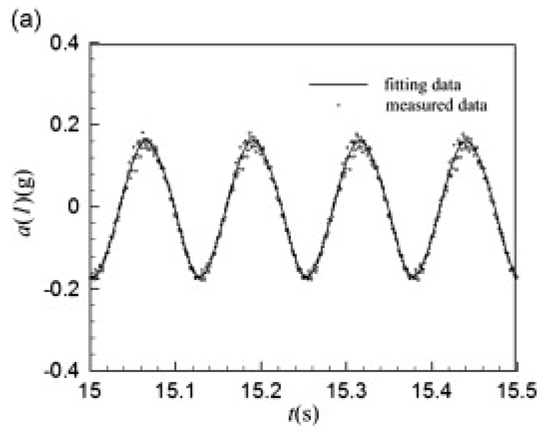
EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday March 23, 2018

School of Engineering
Brown University



1. The figure (from this publication) shows a vibration measurement from a vibration isolation system. Use the figure to estimate

1.1 The amplitude of the acceleration

The amplitude is about 0.18*9.81 m/s^2 peak-peak, which gives $A=1.7658m/s^2$ amplitude

[1 POINT]

1.2 The period of the vibration

There are 4 cycles in 0.5 sec so period is 0.125 sec

[1 POINT]

1.3 The frequency (in Hertz) and angular frequency (in rad/s)

Frequency is 8 Hertz (cycles per sec), or 16π rad/s

[1 POINT]

1.4 The amplitude of the velocity

The amplitude of the velocity follows as $\Delta v = \Delta A / \omega = 35.1 mm/s$

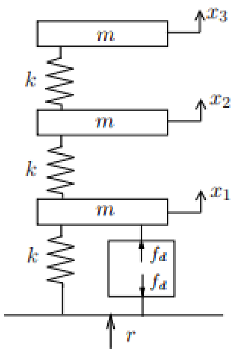
[1 POINT]

1.5 The amplitude of the displacement

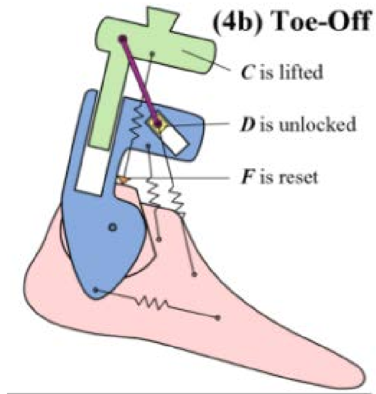
The amplitude of the displacement is $\Delta X = \Delta A / \omega^2 = 0.699 mm$

[1 POINT]

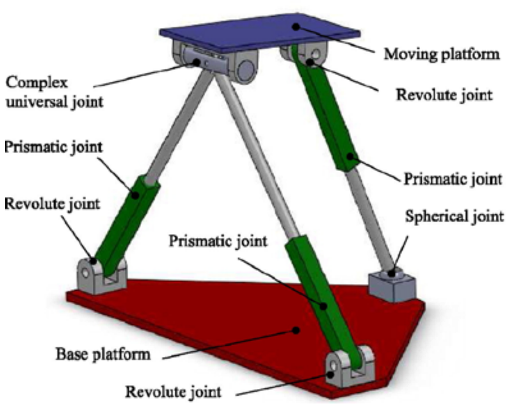
2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures



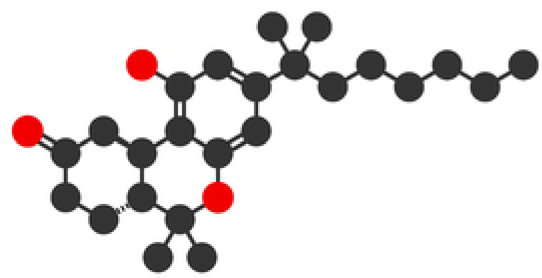
(b) Model of a 3 storey building with vibration suppression



(a) 2D Model of an artificial joint Assume C is fixed, and count only the foot (pink) and ankle (blue)



(d) Model of an articulated platform



(c) THC Molecule

For (a), there are three storeys idealized as particles that can only move vertically. This is 3DOF. No rigid body modes, so 3 vibration modes.

For (b) C is fixed so we don't count it at all. The two parts (foot and ankle) are two rigid bodies. There is one pin joint (2 constraints) and one slider joint (2 constraints) The formula gives

$$\#dof = 3r - c = 6 - 4 = 2$$

No rigid body modes because C is fixed; so 2 vibration modes.

For © each leg is 2 rigid bodies; the platform is a third.
There are 5 constraints at each revolute joint; 5 at each prismatic joint, 3 at the spherical joint, and 3 for each member joining at the complex universal joint. This is a total of 3x5+3x5+3x3=39 constraints. The formula gives

$$\#dof = 6r - c = 42 - 39 = 3$$

For (d) we have 27 atoms, so 81DOF. 6 rigid body modes, so 75 vibration modes.

[4 POINTS]

3. Solve the following differential equations

3.1 $\dfrac{d^2y}{dt^2}+25y=100 \qquad y=0 \qquad \dfrac{dy}{dt}=0 \qquad t=0$

We can re-write this as a case-I equation

$$\frac{1}{5^2}\frac{d^2y}{dt^2}+y=4$$

Comparing the equations shows that $C=4 \quad \omega_n=5 \quad \zeta=0$. The solution is

$$x(t)=C+(x_0-C)\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t$$

We are given $x_0=0 \quad v_0=0$ so

$$y(t)=4-4\cos 5t$$

[3 POINTS]

3.2 $\dfrac{d^2y}{dt^2}+100\dfrac{dy}{dt}+25y=50\sin(t) \qquad y=0 \qquad \dfrac{dy}{dt}=0 \qquad t=0$

We can rearrange this as a Case 5 equation

$$\frac{1}{5^2}\frac{d^2y}{dt^2}+\frac{2\times 10}{5}\frac{dy}{dt}+y=2\sin t$$
$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=C+K\sin t$$

It appears that $\omega=1,\ K=2,\ \omega_n=5,\ \zeta=10$.

The steady-state solution follows as

$$x_p(t)=X_0\sin\left(\omega t+\phi\right)$$

$$X_0=\frac{KF_0}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

$$\Rightarrow X_0=\frac{2}{\left\{\left(1-1/25\right)^2+\left(20/5\right)^2\right\}^{1/2}}=0.4862 \qquad \phi=\tan^{-1}\frac{-20/5}{1-1/25}=-1.3353$$

The homogeneous solution is

$$x_h(t)=\exp(-\varsigma\omega_n t)\left\{\frac{v_0^h+(\varsigma\omega_n+\omega_d)x_0^h}{2\omega_d}\exp(\omega_d t)-\frac{v_0^h+(\varsigma\omega_n-\omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$$

where $\omega_d = \omega_n \sqrt{\varsigma^2 - 1}$

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin\phi = -0.4862 \sin(-1.3353) = 0.4728$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0 \omega \cos\phi = -0.4862 \cos(-1.3353) = -0.1135$$

The total solution is therefore

$$y(t) = 0.4862 \sin(t - 1.3353)$$

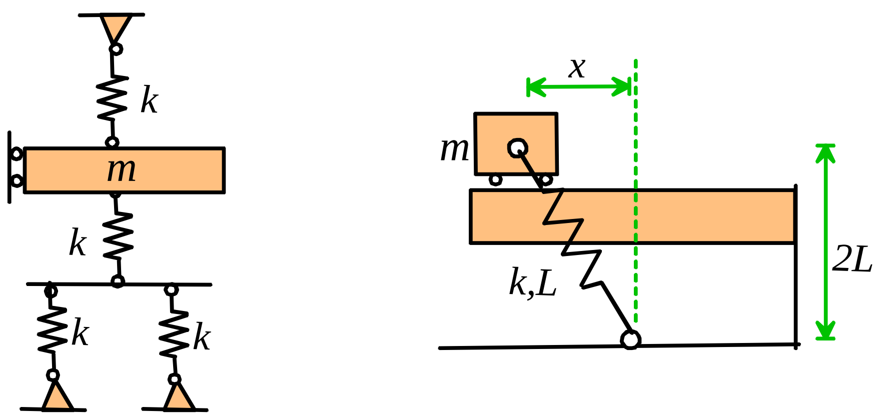
$$+\exp(-50t)\left\{\left(\frac{-0.1135 + (50 + 5\sqrt{99})0.4728}{10\sqrt{99}}\right)\exp(5\sqrt{99}t) - \left(\frac{-0.1135 + (50 - 5\sqrt{99})0.4728}{10\sqrt{99}}\right)\exp(-5\sqrt{99}t)\right\}$$

$$= 0.4862 \sin(t - 1.3353) + \exp(-50t)\left\{0.4728 \exp(5\sqrt{99}t) - 5.05 \times 10^{-5} \exp(-5\sqrt{99}t)\right\}$$

We can check that this is correct by substituting it into the differential equation, and by substituting *t=0* into *y* and *dy/dt* and checking that initial conditions are satisfied.

[3 POINTS]

4. Find formulas for the natural frequency of vibration for the systems shown in the figure



For the first system we can replace the springs with an equivalent single spring and use the standard result for a spring-mass system. The two springs in parallel have combined stiffness *2k* . This is in series with a spring with stiffness *k*, and can be replaced by *2k/3*. The 3 combined springs are in parallel with the one at the top, so the total stiffness is *5k/3*. The natural frequency is therefore

$$\omega_n = \sqrt{\frac{5k}{3m}}$$

[1 POINT]

For the second system we can use energy conservation to get the EOM.

The length of the spring is $\sqrt{4L^2 + x^2}$ so its potential energy is $\frac{1}{2}k\left(\sqrt{4L^2 + x^2} - L\right)^2$. Hence

$$T + V = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}k\left(\sqrt{4L^2 + x^2} - L\right)^2$$

We can take the time derivative:

$$\frac{d}{dt}(T + V) = m\left(\frac{dx}{dt}\right)\frac{d^2x}{dt^2} + k\left(\sqrt{4L^2 + x^2} - L\right)\frac{x}{\sqrt{4L^2 + x^2}}\left(\frac{dx}{dt}\right) = 0$$

$$\Rightarrow m\frac{d^2x}{dt^2} + k\left(\sqrt{4L^2 + x^2} - L\right)\frac{x}{\sqrt{4L^2 + x^2}} = 0$$

This is a nonlinear equation, so we have to linearize it. Note that $x=0$ is a solution to the EOM, so this is a static equilibrium position. For small values of x, therefore

$$\Rightarrow m\frac{d^2x}{dt^2} + k\left(\sqrt{4L^2} - L\right)\frac{x}{\sqrt{4L^2}} \approx 0$$

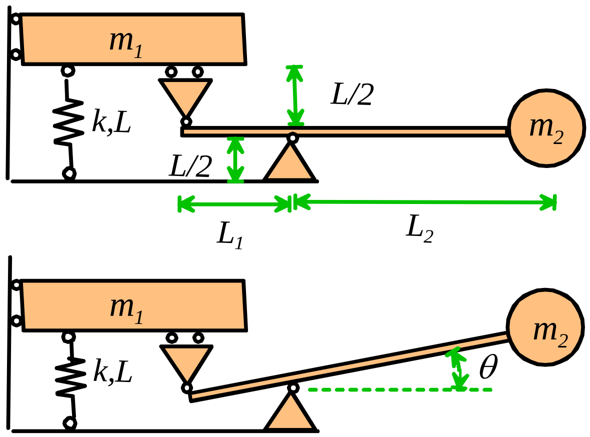
$$\Rightarrow m\frac{d^2x}{dt^2} + \frac{k}{2}x \approx 0$$

Rearranging this in standard form gives

$$\frac{2m}{k}\frac{d^2x}{dt^2} + x \approx 0$$

Thus (compare with the standard EOM) $\omega_n = \sqrt{k/2m}$

[3 POINTS]



5. The figure shows a design for an ‘anti-resonant’ vibration isolation system (see here for the patent; this reference analyzes the system in detail).

5.1 Find formulas for the kinetic and potential energy of the system (**neglect gravity**) in terms of θ and other relevant variables.

The potential energy in the spring is $\frac{1}{2}k(L_1\sin\theta)^2$

The vertical velocity of mass m_1 is $\frac{d}{dt}(L_1\sin\theta) = L_1\cos\theta\frac{d\theta}{dt}$. Its kinetic energy is therefore

$\frac{1}{2}m_1L_1^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2$

Mass m_2 is in circular motion, so its kinetic energy is $\frac{1}{2}m_2\left(L_2\frac{d\theta}{dt}\right)^2$

We have $T+V = \frac{1}{2}\left(m_1L_1^2\cos^2\theta + m_2L_2^2\right)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}kL_1^2\sin^2\theta$

[2 POINTS]

5.2 Find the equation of motion for the angle θ

Take the time derivative (you can use a Live script)

$$\frac{d}{dt}(T+V) = \left(m_1L_1^2\cos^2\theta + m_2L_2^2\right)\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + m_1L_1^2\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^3 + kL_1^2\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right) = 0$$

$$\Rightarrow \left(m_1L_1^2\cos^2\theta + m_2L_2^2\right)\frac{d^2\theta}{dt^2} + m_1L_1^2\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2 + kL_1^2\sin\theta\cos\theta = 0$$

[2 POINTS]

5.3 Hence, calculate a formula for its natural frequency.

For small θ (take Taylor expansions of all the nonlinear terms, e.g. using $\cos\theta = 1 \quad \sin\theta = \theta$ and neglect the second order terms in θ as well as products of $\theta, d\theta/dt$):

$$\Rightarrow \left(m_1 L_1^2 + m_2 L_2^2\right)\frac{d^2\theta}{dt^2} + kL_1^2\theta = 0$$

The natural frequency follows as

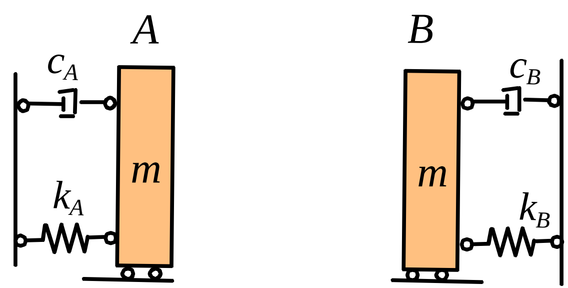
$$\Rightarrow \omega_n = \sqrt{\frac{kL_1^2}{\left(m_1 L_1^2 + m_2 L_2^2\right)}}$$

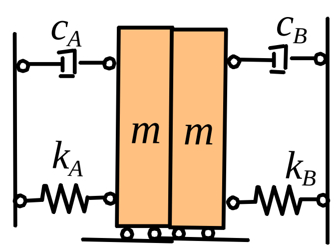
(For discussion: part of the point of the design is that the resonant frequency of the system can be made very low without needing a large mass.   But the dynamics of the system is a bit more subtle when the base is shaking.)

[2 POINTS]

6. When the two masses shown in the figure are un-coupled, systems A and B have natural frequency and damping factor $\zeta_A, \omega_{nA} \quad \zeta_B, \omega_{nB}$

If the two masses are connected together, what are the natural frequency and damping factor of the new system?





The formulas give

$$\omega_{nA} = \sqrt{\frac{k_A}{m}} \qquad \omega_{nA} = \sqrt{\frac{k_A}{m}} \qquad \omega_n = \sqrt{\frac{k_A + k_B}{2m}}$$

$$\zeta_A = \frac{c_A}{2\sqrt{k_A m}} \qquad \zeta_B = \frac{c_B}{2\sqrt{k_B m}} \qquad \zeta = \frac{c_A + c_B}{2\sqrt{2m(k_A + k_B)}}$$

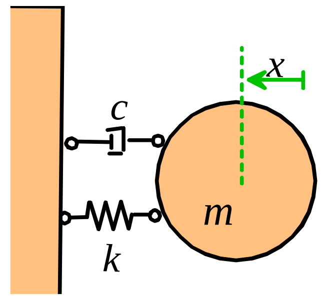
The first three equations show that

$$\omega_n^2 = \frac{k_A + k_B}{2m} = \frac{\omega_{nA}^2}{2} + \frac{\omega_{nB}^2}{2} \Rightarrow \omega_n = \sqrt{\frac{\omega_{nA}^2}{2} + \frac{\omega_{nB}^2}{2}}$$

The second three give

$$\omega_n\zeta = \frac{c_A + c_B}{4m} = \frac{\omega_{nA}\zeta_A}{2} + \frac{\omega_{nB}\zeta_B}{2}$$
$$\Rightarrow \zeta = \frac{\omega_{nA}\zeta_A + \omega_{nB}\zeta_B}{\sqrt{2(\omega_{nA}^2 + \omega_{nB}^2)}}$$

[4 POINTS]



7. The impact of a baseball with a flat rigid wall is idealized as a spring-mass system (this high-speed movie might help visualize the impact)

7.1 Assume that the center of the ball (i.e. the mass) is at position $x = 0$ and has velocity $dx/dt = -v_0$ just before impact. Write down an equation for the displacement *x(t)* and velocity $v(t)$ of the center of the ball (represented by the mass in the figure) during the impact, in terms of the natural frequency ω_n and damping factor ζ for the system (use the solution for the value of ζ that you think makes most sense, based on the behavior you see in the video).

From the solutions to vibration EOM (using the underdamped case – the other two could be given two, but of course they would not rebound from the wall)

$$x(t) = \frac{v_0}{\omega_d}\exp(-\zeta\omega_n t)\sin\omega_d t$$

$$v(t) = \frac{v_0}{\omega_d}\exp(-\zeta\omega_n t)\left(\omega_d\cos\omega_d t - \omega_n\zeta\sin\omega_d t\right)$$

[2 POINTS]

7.2 In a homework problem from 2017, the class estimated the stiffness, damping coefficient and mass of the baseball as $m = 0.145$kg, $k = 8.94 MN/m$ $c = 364 Ns/m$. Estimate the maximum value of x during the impact. Assume an impact velocity of 50 m/s.

Substituting numbers gives $\omega_n = \sqrt{\frac{k}{m}} = 7850$ $\quad \zeta = \frac{c}{2\sqrt{km}} = 0.1633$

The maximum deflection occurs when $v = 0$. It is straightforward to solve $\left(\omega_d \cos\omega_d t - \omega_n \zeta \sin\omega_d t\right) = 0$ (to do it by hand, simply use trig formulas to re-write the expression as

$$\sqrt{\omega_d{}^2 + (\omega_n \zeta)^2}\left(\frac{\omega_d}{\sqrt{\omega_d{}^2 + (\omega_n \zeta)^2}}\cos\omega_d t - \frac{\omega_n \zeta}{\sqrt{\omega_d{}^2 + (\omega_n \zeta)^2}}\sin\omega_d t\right) = 0$$

$$\Rightarrow \cos(\omega_d t + \phi) = 0 \qquad \phi = \tan^{-1}\frac{\omega_n \zeta}{\omega_d} \Rightarrow t = \frac{1}{\omega_d}\left(\frac{\pi}{2} - \phi\right)$$

Or just use a Live Script. The solution is $t = 0.18 \times 10^{-3} s$, which can be substituted back into the expression for x to get

$$\frac{v_0}{\omega_d}\exp(-\zeta\omega_n t)\sin\omega_d t = 5.1 mm$$

For discussion - It is possible to check this using the video…. You can download the movie from the 2017 website and use the image processing software on the website to plot a graph of the motion of the baseball. The estimate here turns out to be pretty good.

[3 POINTS]