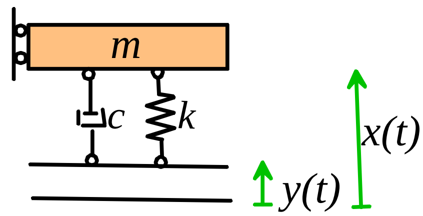
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 6, 2018



1. The vibration isolation system shown in the figure has

- m=20kg,
- $k = 19.8 kN / m$
- $c = 1.259 kNs / m$

If the base vibrates harmonically with an amplitude of 1mm and frequency of 100Hz, what is the steady-state amplitude of vibration of the platform (i.e. the mass *m*)?

We just need to find the right formulas to use and substitute numbers. We have that

$$\omega_n = \sqrt{k / m} = 31.46 \ rad / s$$

$$\zeta = c / (2\sqrt{km}) = 1$$

The magnification is

$$M = \frac{\sqrt{1 + (2\zeta\omega / \omega_n)^2}}{\sqrt{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2}} = 0.1$$

The vibration amplitude is therefore

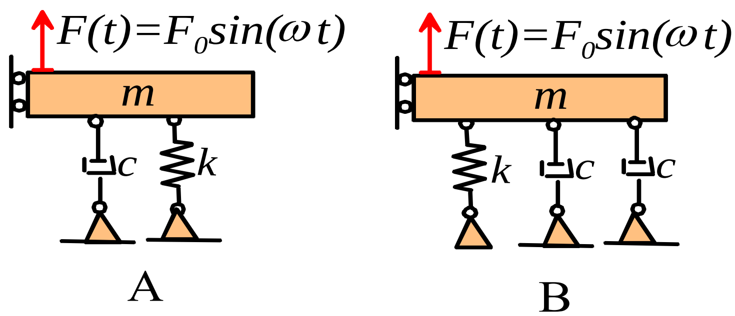
$$X_0 = KMY_0 = MY_0 = 0.1mm$$

[2 POINTS]

2. Both systems in the figure are subjected to a force with amplitude 1 kN and frequency equal to the undamped natural frequency of the spring-mass system ($\omega = \omega_n$).

The vibration amplitude of system B is measured to be 1mm.

What is the vibration amplitude of system A?



The formula for amplitude is $X_0 = KM(\omega/\omega_n, \zeta)F_0$.

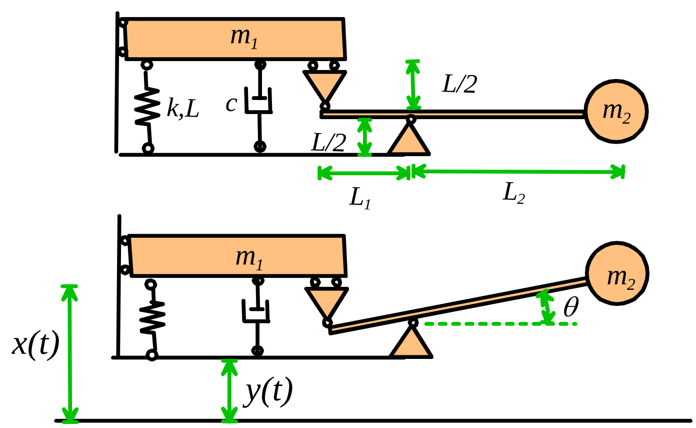
When $\omega = \omega_n$ the magnification $M = 1/(2\zeta)$ where $\zeta = 2c/(2\sqrt{km})$.

Removing a dashpot does not change the undamped natural frequency and halves ζ .

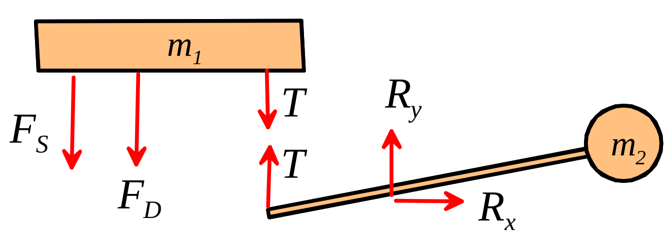
The amplitude will double.

[3 POINTS]

3. In this (hard!) problem we will analyze the behavior of the 'anti-resonant' vibration isolation system introduced in Homework 5. The system is illustrated in the figure. Assume that the base vibrates vertically with a displacement $y(t) = Y_0 \sin \omega t$. Our goal is to calculate a formula for the steady-state vertical motion $x(t)$ of the platform, and to compare the behavior of this system with the standard base excited spring-mass-damper design for an isolation system.



3.1 Draw free body diagrams showing the forces acting on the mass m_1 and the pendulum assembly (see the figure).



[3 POINTS]

3.2 Using geometry, find an expression for the acceleration of mass m_2 in terms of θ , y and their time derivatives (as well as relevant geometric constants) (e.g. by writing down a formula for the position vector relative to a fixed origin and differentiating it). Show that if θ and its time derivatives are small the result can be approximated by

$$\mathbf{a} \approx \left[\frac{d^2 y}{dt^2} + L_2 \frac{d^2\theta}{dt^2} \right] \mathbf{j}$$

Show also that

$$\frac{d^2 x}{dt^2} - \frac{d^2 y}{dt^2} \approx -L_1 \frac{d^2\theta}{dt^2}$$

The position vector of m_2 is

$$\mathbf{r} = L_2 \cos\theta \mathbf{i} + (y + L/2 + L_2 \sin\theta)\mathbf{j}$$

Differentiate twice with respect to time to get

$$\mathbf{a} = -L_2 \left\{ \left(\frac{d\theta}{dt} \right)^2 \cos\theta + \frac{d^2\theta}{dt^2} \sin\theta \right\} \mathbf{i} + \left[\frac{d^2 y}{dt^2} + L_2 \left\{ -\left(\frac{d\theta}{dt} \right)^2 \sin\theta + \frac{d^2\theta}{dt^2} \cos\theta \right\} \right] \mathbf{j}$$

Using the approximations $\cos\theta \approx 1 \qquad \sin\theta \approx \theta$ and neglecting squared or higher order products of θ and its time derivatives we get the result stated.

Similarly

$$x - y = L - L_1 \sin\theta$$

Differentiate this twice

$$\frac{d^2 x}{dt^2} - \frac{d^2 y}{dt^2} = -L_1 \left(-\left(\frac{d\theta}{dt} \right)^2 \sin\theta + \frac{d^2\theta}{dt^2} \cos\theta \right)$$

Use the same approximation to get the stated result.

[3 POINTS]

3.3 For the pendulum, write down $\mathbf{F} = m\mathbf{a}$ and $\mathbf{M} = 0$ about the center of mass, in terms of reaction forces shown in your FBD. Use the approximation in 3.2 for the acceleration.

For m_2 we have

$$R_x \mathbf{i} + (R_y + T)\mathbf{j} = +m_2 \left[\frac{d^2 y}{dt^2} + L_2 \frac{d^2\theta}{dt^2} \right] \mathbf{j}$$

$$R_x L_2 \sin\theta - R_y \mathrm{L}_2 \cos\theta - T(L_1 + L_2)\cos\theta = 0$$

[2 POINTS]

3.4 Write down $\mathbf{F}=m\mathbf{a}$ for mass m_1, and hence use 3.3 and the second of 3.2 to show that (if if θ and its time derivatives are small) then

$$\left(m_1+m_2\frac{L_2^2}{L_1^2}\right)\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=m_2\frac{L_2}{L_1}\left(1+\frac{L_2}{L_1}\right)\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky$$

F=m**a** gives

$$m_1\frac{d^2x}{dt^2}=-T-k(x-y-L)-c\frac{d}{dt}(x-y)$$

We can solve (3.3) for T to get

$$T=-\frac{L_2}{L_1}\left(\frac{d^2y}{dt^2}+L_2\frac{d^2\theta}{dt^2}\right)$$

From 2.2

$$T=-\frac{L_2}{L_1}\left(\frac{d^2y}{dt^2}\right)+\left(\frac{L_2}{L_1}\right)^2\left(\frac{d^2x}{dt^2}-\frac{d^2y}{dt^2}\right)$$

Hence substitute for T and rearrange to get

$$\left(m_1+m_2\frac{L_2^2}{L_1^2}\right)\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=kL+m_2\frac{L_2}{L_1}\left(1+\frac{L_2}{L_1}\right)\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky$$

[3 POINTS]

3.5 Show that the equation can be re-arranged into the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y$$

and show that

$$\omega_n=\sqrt{\left(\frac{kL_1^2}{L_1^2m_1+m_2L_2^2}\right)} \qquad \lambda=\sqrt{\left(\frac{m_2L_2\left(L_1+L_2\right)}{\left(L_1^2m_1+m_2L_2^2\right)}\right)}$$

$$\zeta=\frac{c}{2\sqrt{k(m_1L_1^2+m_2L_2^2)/L_1^2}}$$

We can re-write the equation as

$$\left(\frac{L_1^2 m_1 + m_2 L_2^2}{kL_1^2}\right)\frac{d^2x}{dt^2}+\frac{c}{k}\frac{dx}{dt}+x=\left(\frac{L_1^2 m_1 + m_2 L_2^2}{kL_1^2}\right)\left(\frac{m_2L_2\left(L_1+L_2\right)}{\left(L_1^2m_1+m_2L_2^2\right)}\right)\frac{d^2y}{dt^2}+\frac{c}{k}\frac{dy}{dt}+y$$

Hence

$$\omega_n=\sqrt{\left(\frac{kL_1^2}{L_1^2m_1+m_2L_2^2}\right)} \qquad \lambda=\sqrt{\left(\frac{m_2L_2\left(L_1+L_2\right)}{\left(L_1^2m_1+m_2L_2^2\right)}\right)}$$

$$\zeta=\frac{c}{2\sqrt{k(m_1L_1^2+m_2L_2^2)/L_1^2}}$$

[3 POINTS]

3.6 Suppose that the base is subjected to harmonic excitation $y=Y_0\sin\omega t$. Show (using calculus and the double-angle formula $\cos\psi\sin\omega t+\sin\psi\cos\omega t=\sin(\omega t+\psi)$) that

$$\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=Y_0\left\{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)\sin\omega t+\frac{2\zeta\omega}{\omega_n}\cos\omega t\right\}$$

$$=Y_0\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}\sin(\omega t+\psi) \quad \psi=\tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\lambda^2(\omega/\omega_n)^2}$$

Hence use the 'Case IV' solution to differential equations to show that the steady state solution for x has the form

$$x(t)=X_0\sin(\omega t+\phi) \qquad X_0=M(\omega/\omega_n,\zeta,\lambda)Y_0$$

and give a formula for the magnification factor M.

Substituting for y and evaluating the derivatives shows

$$\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=Y_0\left\{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)\sin\omega t+\frac{2\zeta\omega}{\omega_n}\cos\omega t\right\}$$

We can re-write this as

$$Y_0\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}\left\{\frac{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)}{\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}\sin\omega t+\frac{\frac{2\zeta\omega}{\omega_n}}{\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}\cos\omega t\right\}$$

Defining

$$\cos\psi = \frac{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)}{\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}} \qquad \sin\psi = \frac{\frac{2\zeta\omega}{\omega_n}}{\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

We can use the double angle formula $\cos\psi\sin\omega t+\sin\psi\cos\omega t=\sin(\omega t+\psi)$ to get the answer stated.

We can regard the equation

$$\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=Y_0\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}\sin(\omega t+\psi)$$

As a case IV EOM with

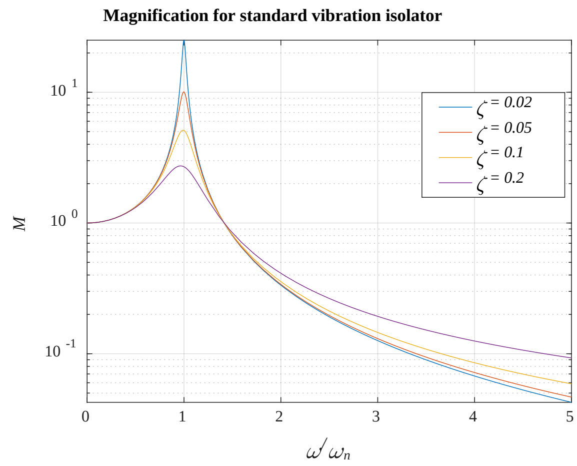
$$K=\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2} \qquad Y_0=F_0$$

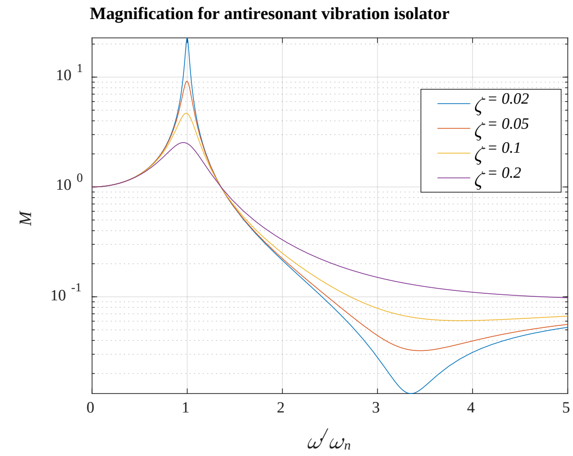
The solution follows from the formula sheet, and the magnification is

$$M=\frac{\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

[3 POINTS]

3.7. Plot a graph of M as a function of $0<\omega/\omega_n<6$ for $\lambda=0$,for values of $\zeta=0.02, 0.05, 0.1, 0.2$ (on the same plot). This graph shows the magnification for the 'standard' vibration isolation system, since $\lambda=0$ corresponds to a pendulum with zero mass – it should look the same as the 'Case V' magnification graph discussed in class. For comparison, plot a second graph of M as a function of ω/ω_n for $\lambda=0.6$ (an anti-resonant isolator),for values of $\zeta=0.02, 0.05, 0.1, 0.2$





[3 POINTS]

3.8 What is the frequency corresponding to the anti-resonance (the minimum value of *M*), in terms of λ, ω_n (give an approximate solution for $\zeta << 1$)? What is (approximately) the smallest vibration amplitude (in terms of λ, ζ)?

The minimum will occur when $\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)=0 \Rightarrow \omega=\omega_n/\lambda$. The corresponding vibration amplitude is

$$M=\frac{\left(\frac{2\zeta}{\lambda}\right)}{\sqrt{\left(1-\frac{1}{\lambda^2}\right)^2+\left(\frac{2\zeta}{\lambda}\right)^2}}$$

If λ is not close to 1, then

$$M\approx\frac{2\zeta}{\left|\left(\lambda-\frac{1}{\lambda}\right)\right|}$$

[2 POINTS]

3.9 For what range of frequency (in terms of λ, ω_n) does the pendulum system give better performance than the simpler spring-mass-damper system?

The magnification for the anti-resonant isolator is equal to that of the conventional system when

$$\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}=\sqrt{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

This gives

$$\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2 = 1$$

$$\Rightarrow \frac{\lambda^2\omega^2}{\omega_n^2} = 2 \Rightarrow \omega = \frac{\sqrt{2}}{\lambda}\omega_n$$

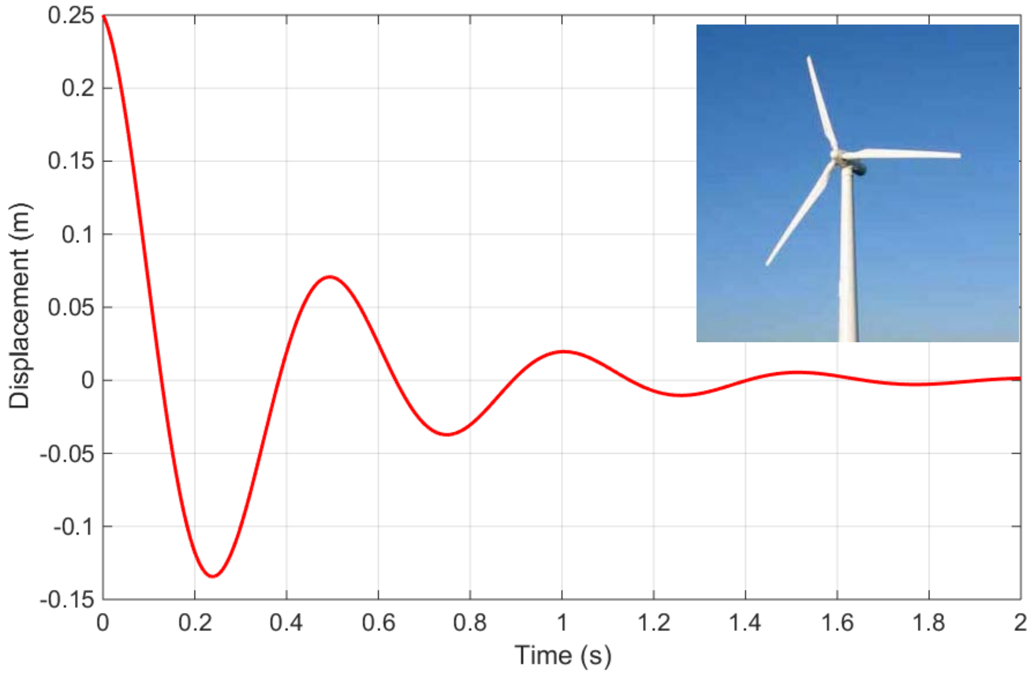
The anti-resonant system is better than the conventional system for ω below this value. It only isolates vibations if $\omega / \omega_n > \sqrt{2}$, however.

[2 POINTS]

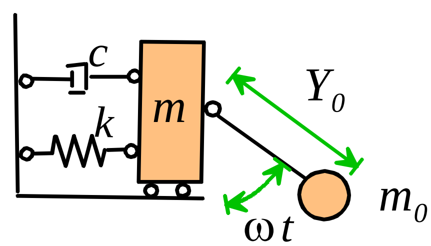
3.10 What sort of application would be best suited for an anti-resonant vibration isolator?

The isolator is only useful if the excitation is close to a harmonic motion at a fixed frequency ω - if the base will move with a range of frequencies, the isolator will block the ones close to the anti-resonance, but not the others. Antiresonant isolators are often used between the rotor blade assembly of a helicopter and the helicopter body, for example (because the excitation frequency is set by the rotor angular speed)

[2 POINTS]



4 An unbalanced wind-turbine is idealized as a rotor-excited spring-mass system as shown in the figure. The mass m represents the tower, and m_0 represents the combined mass of the three rotor blades. The spring and damper represent the stiffness and energy dissipation in the tower. The rotor is 'unbalanced' because its center of mass is a small distance Y_0 away from the axle. The total mass $(m+m_0)$ of the system is 25000kg.



The figure shows the results of a free vibration experiment on the turbine.

4.1 Use the data provided to determine the following quantities:

(a) The vibration period

The period is 0.5 sec

[1 POINT]

(b) The log decrement

The log decrement is $\frac{1}{2}\log\left(\frac{0.25}{0.02}\right)=1.26$

[1 POINT]

(c) The undamped natural frequency

The undamped natural frequency is $\omega_n = \frac{\sqrt{4\pi^2+\delta^2}}{T} = 12.8 \;\; rad/s$

[1 POINT]

(d) The damping factor

The damping factor is $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.2$

[1 POINT]

(e) The spring stiffness

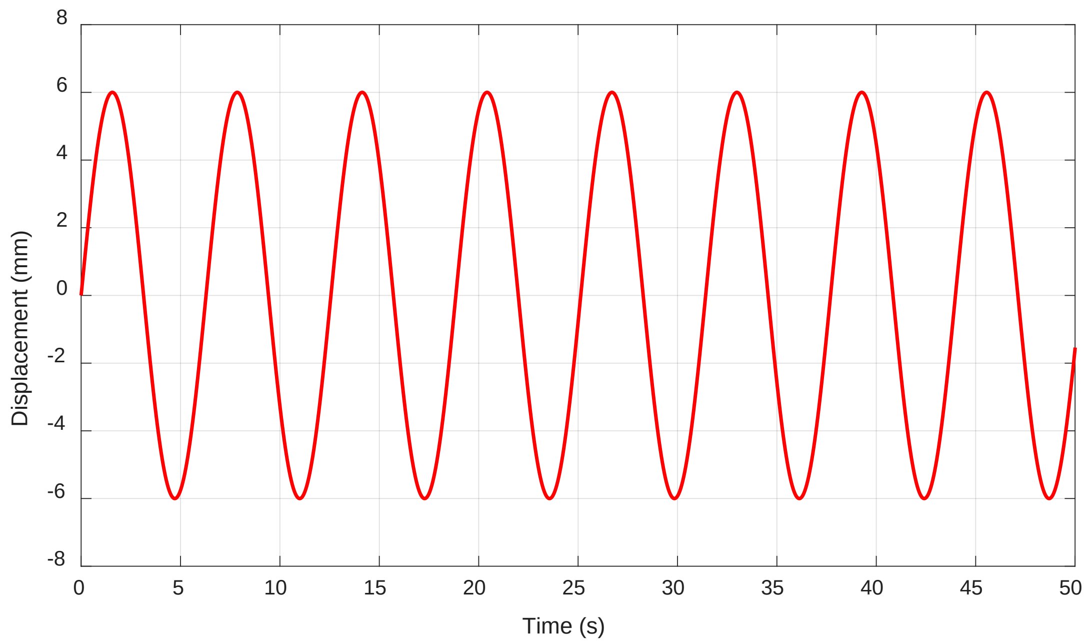
$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2 = 4100 \;\; kN / m$$

[1 POINT]

(f) The dashpot coefficient

$$\zeta = \frac{c}{2\sqrt{km}} \Rightarrow c = 2\zeta\sqrt{km} = 128000 Ns / m$$

[1 POINT]



4.2 The figure shows the measured displacement of the system during operation. The blades have a radius of 40m, and the total mass of the system $(m + m_0)$ is 25000kg . Assuming that the rotor can be balanced by adding mass to the tip of one blade, estimate the mass that must be added to balance the rotor.

We know that the vibration amplitude of the unbalanced rotor is

$$X_0 = \frac{m_0}{m + m_0} Y_0 \frac{\omega^2 / \omega_n^2}{\sqrt{(1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2}}$$

We can use the vibration measurement to estimate the product $m_0 Y_0$:

From the graph, we see that the period is about 6 sec so $\omega \approx 1$ rad/s, so using the numbers from 4.1

$$X_0 = \frac{m_0}{25000} Y_0 \frac{1/12.8^2}{\sqrt{(1-1/12.8^2)^2 + (2\times 0.2/12.8)^2}} = \frac{0.0061}{25000} m_0 Y_0$$

Since the measured amplitude is about 6mm, we conclude that $m_0 Y_0 \approx 25000 kgm$. We want to move the COM back to the center of the rotor – recall that the COM is $(1/M)\sum \mathbf{r}_i m_i$ so the required mass at the blade tip is 25000/40 = 625 kg.

[4 POINTS]