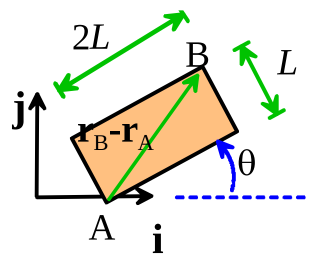
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 7: Rigid Body Kinematics, Inertial properties of rigid bodies
Due Friday April 20, 2018

1. The rigid body shown in the figure is at rest at time t=0, and rotates counterclockwise with constant angular acceleration vector $2\alpha\mathbf{k}$ Find



1.1 The angular velocity vector as a function of time

We can just integrate the angular acceleration $\boldsymbol{\omega} = 2\alpha t\mathbf{k}$

[1 POINT]

1.2 The spin tensor **W** (as a 2x2 matrix, also a function of time)

Using the formula $\mathbf{W} = \begin{bmatrix} 0 & -2\alpha t \\ 2\alpha t & 0 \end{bmatrix}$

[1 POINT]

1.3 The rotation tensor (a 2x2 matrix for a 2D problem) **R** that rotates the rectangle from its initial to its final position shown in the figure. Check your answer by computing

$$\frac{d\mathbf{R}}{dt}\mathbf{R}^T$$

$$\mathbf{R} = \begin{bmatrix} \cos\alpha t^2 & -\sin\alpha t^2 \\ \sin\alpha t^2 & \cos\alpha t^2 \end{bmatrix}$$

$$\frac{d\mathbf{R}}{dt}\mathbf{R}^T = \begin{bmatrix} -2\alpha t\sin\alpha t^2 & -2\alpha t\cos\alpha t^2 \\ 2\alpha t\cos\alpha t^2 & -2\alpha t\sin\alpha t^2 \end{bmatrix}\begin{bmatrix} \cos\alpha t^2 & \sin\alpha t^2 \\ -\sin\alpha t^2 & \cos\alpha t^2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2\alpha t \\ 2\alpha t & 0 \end{bmatrix}$$

This reduces to **W** as expected.

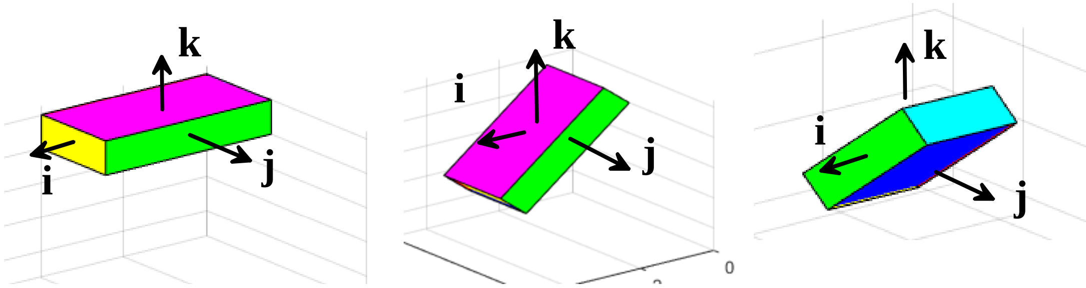
[2 POINTS]

1.4 Hence, express the rotated vector $\mathbf{r}_B - \mathbf{r}_A$ in (**i**,**j**) components.

$\mathbf{r}_B - \mathbf{r}_A$ can be found using the mapping $\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$

$$\begin{bmatrix} \cos\alpha t^2 & -\sin\alpha t^2 \\ \sin\alpha t^2 & \cos\alpha t^2 \end{bmatrix}\begin{bmatrix} 2L \\ L \end{bmatrix} = \begin{bmatrix} 2L\cos\alpha t^2 - L\sin\alpha t^2 \\ 2L\sin\alpha t^2 + L\cos\alpha t^2 \end{bmatrix}$$

[1 POINT]



2. The rectangular prism shown in the figure is subjected to two sequential rotations:
(1) A 45 degree rotation about the **j** axis
(2) A -90 degree rotation about the **k** axis

2.1 Write down the rotation tensor (matrix) for each rotation

We can use the general formula

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

For the first rotation $\theta = 45$, $n_x = n_z = 0$ $n_y = 1$

$$\mathbf{R}^{(1)} = \begin{bmatrix} \cos 45 & 0 & \sin 45 \\ 0 & \cos 45 + (1-\cos 45) & 0 \\ -\sin 45 & 0 & \cos 45 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

For the second rotation $\theta = -90$

$$\mathbf{R}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[3 POINTS]

2.2 Find the rotation matrix $\mathbf{R} = \mathbf{R}^{(2)}\mathbf{R}^{(1)}$ that describes the combined effects of both rotations (you can use matlab to do the matrix multiplication, but there is no need to submit the code or script).

Matlab gives

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

[2 POINTS]

2.3 Find the axis **n** and rotation angle θ that will complete the rotation $\mathbf{R}$ directly.

We can use the formula

$$1+2\cos\theta = R_{xx}+R_{yy}+R_{zz}$$

$$\mathbf{n}=\frac{1}{2\sin\theta}\left[\left(R_{zy}-R_{yz}\right)\mathbf{i}+\left(R_{xz}-R_{zx}\right)\mathbf{j}+\left(R_{yx}-R_{xy}\right)\mathbf{k}\right]$$

This gives $\cos\theta=\frac{1}{2}\left(\frac{\sqrt{2}}{2}-1\right)\Rightarrow\theta=1.71777\text{rad}$

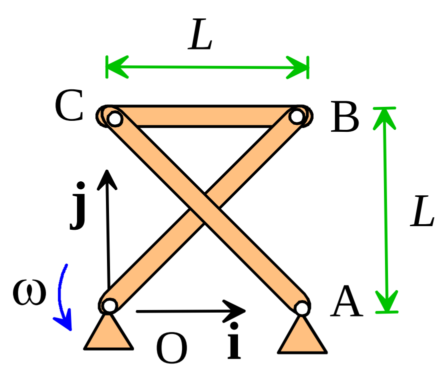
$$\mathbf{n}=\frac{1}{2\sin(1.71777)}\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}+-(1+\frac{1}{\sqrt{2}})\mathbf{k}\right)=0.3574\mathbf{i}+0.3574\mathbf{j}-0.8628\mathbf{k}$$

[3 POINTS]

Optional: You can check your answer by downloading a matlab script from the homework page of the course website that will animate a rigid rotation through an angle θ about an axis parallel to a unit vector **n**. You can use the code by navigating to the directory storing the file in the Matlab command window, and then typing

Animate_rotation(*angle*,[n_x, n_y, n_z])

Where *angle* is your solution for the rotation angle θ (in radians), and n_x, n_y, n_z are the components of your solution for the unit vector **n**.



3. The figure shows a four-bar chain mechanism that appears inside both natural and artificial knee joints. Member OB rotates counterclockwise at constant angular speed ω. Calculate the angular velocities and angular accelerations of members CB and AC.

At the instant shown, calculate the angular velocity and acceleration of links AB and BC.

The rigid body formulas for the three members give

$$\mathbf{v}_B-\mathbf{v}_O=\omega\mathbf{k}\times(\mathbf{r}_B-\mathbf{r}_O)=\omega\mathbf{k}\times(L\mathbf{i}+L\mathbf{j})=L\omega(-\mathbf{i}+\mathbf{j})$$

$$\mathbf{v}_C-\mathbf{v}_B=\omega_{BC}\mathbf{k}\times(\mathbf{r}_C-\mathbf{r}_B)=\omega_{BC}\mathbf{k}\times(-L\mathbf{i})=-L\omega_{BC}\mathbf{j}$$

$$\mathbf{v}_A-\mathbf{v}_C=\omega_{AC}\mathbf{k}\times(\mathbf{r}_A-\mathbf{r}_C)=\omega_{AC}\mathbf{k}\times L(\mathbf{i}-\mathbf{j})=L\omega_{AC}(\mathbf{i}+\mathbf{j})$$

We know that $\mathbf{v}_O=\mathbf{v}_C=\mathbf{0}$ so if we add all the equations listed above we get

$$L\omega(-\mathbf{i}+\mathbf{j})-L\omega_{BC}\mathbf{j}+L\omega_{AC}(\mathbf{i}+\mathbf{j})=\mathbf{0}$$

The **i** component of this equation gives $-\omega+\omega_{AC}=0$

The **j** component gives $\omega-\omega_{BC}+\omega_{AC}=0$

Solve these to get $\omega_{AC}=\omega \qquad \omega_{BC}=2\omega$

[3 POINTS]

For accelerations, use

$$\mathbf{a}_B - \mathbf{a}_O = -\omega^2(\mathbf{r}_B - \mathbf{r}_O) = -L\omega^2(\mathbf{i}+\mathbf{j})$$

$$\mathbf{a}_C - \mathbf{a}_B = \alpha_{BC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_B) - \omega_{BC}^2(\mathbf{r}_C - \mathbf{r}_B) = -L\alpha_{BC}\mathbf{j} + L4\omega^2\mathbf{i}$$

$$\mathbf{a}_A - \mathbf{a}_C = \alpha_{AC}\mathbf{k}\times(\mathbf{r}_A - \mathbf{r}_C) - \omega_{AC}^2(\mathbf{r}_A - \mathbf{r}_C) = \alpha_{AC}L(\mathbf{i}+\mathbf{j}) + L\omega^2(-\mathbf{i}+\mathbf{j})$$

At the stationary points $\mathbf{a}_O = \mathbf{a}_C = \mathbf{0}$, and add everything:

$$-L\omega^2(\mathbf{i}+\mathbf{j}) - L\alpha_{BC}\mathbf{j} + L4\omega^2\mathbf{i} + L\alpha_{AC}(\mathbf{i}+\mathbf{j}) + L\omega^2(-\mathbf{i}+\mathbf{j}) = \mathbf{0}$$

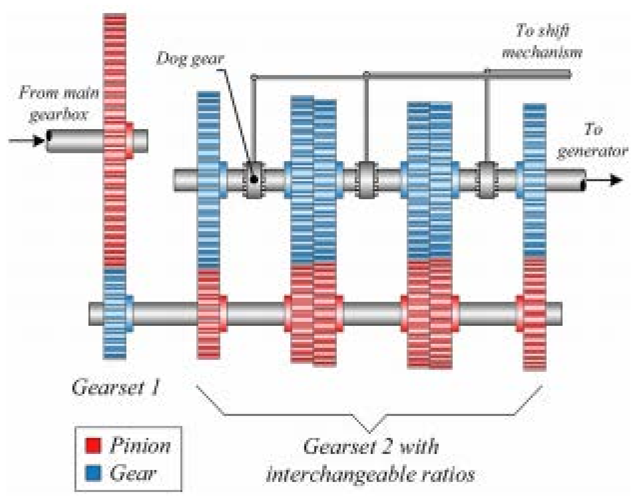
The **i** component gives $L\alpha_{AC} + 2L\omega^2 = 0$. The **j** component gives

$$-\alpha_{BC} + \alpha_{AC} = 0$$

Thus $\alpha_{AC} = -2\omega^2 \qquad \alpha_{BC} = -2\omega^2$

[3 POINTS]

4. This publication describes a variable speed transmission for a wind turbine[1]. The transmission has 6 possible gear ratios, which are achieved by connecting one of 6 possible (blue) gears to the output shaft (connected to the generator).



The numbers of teeth on each gear or pinion are shown in the table (in each case the red 'gear' drives a blue 'pinion'.

Calculate the 6 possible transmission ratios $\omega_{generator} / \omega_{gearbox}$

Table 4 Optimal gearsets selected through multi-objectives function for low–medium power density design at sites 1 through 18

		Gearset 2					
	Gearset 1	Gear 1	Gear 2	Gear 3	Gear 4	Gear 5	Gear 6
Pinion teeth	135	49	53	56	58	59	60
Gear teeth	50	101	97	94	92	91	90

Let N_A, N_B be the numbers of teeth on the main gearbox and gearset 1.

Let N_C, N_D be the numbers of teeth on Gearset 2 and the generator gear. Then use the gear formula

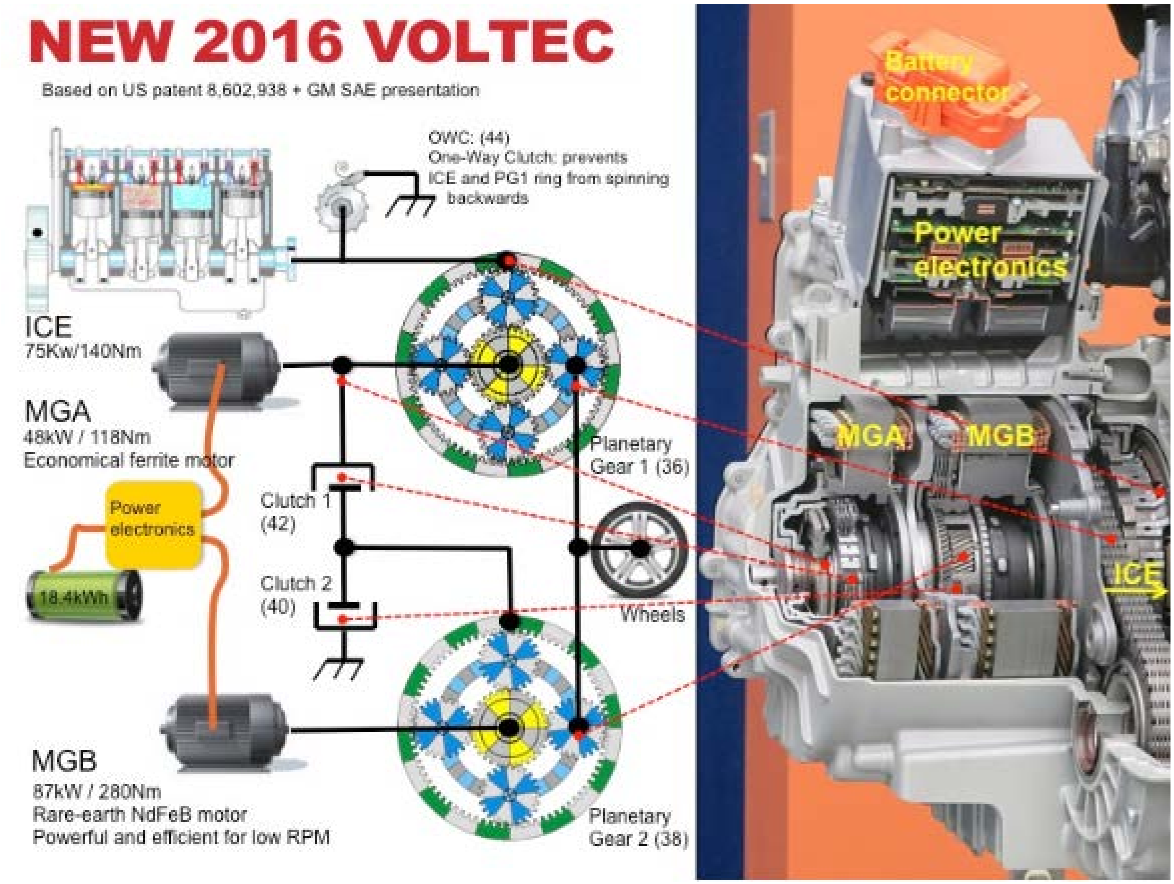
[1] In practice most modern turbines are direct drive: it is very difficult to design a gearbox capable of transmitting the necessary power for 20 years without failure.

$$\omega_{generator} / \omega_{gearbox} = \frac{N_A}{N_B}\frac{N_C}{N_D}$$

The resulting numbers (from matlab) are listed below

```
Ratio1 = 1.3099
Ratio2 = 1.4753
Ratio3 = 1.6085
Ratio4 = 1.7022
Ratio5 = 1.7505
Ratio6 = 1.8000
```

[2 POINTS]



5. The figure (from this publication) shows a schematic of the split-power transmission system for the second generation Chevy Volt. The transmission contains two epicyclic gears: the wheels are always connected to the planet carriers through the 'final drive gearing'. The sun gears are driven by two electric 'traction motor', while the ring gear can either be locked (close clutch C1 and open C2), to make the car an electric vehicle, or connected to the internal combustion engine (open C1 and close C2,C3) to make the car a hybrid. We can find the following specifications for the vehicle:

- Tire diameter 16.5 inches
- The gear ratio between the planet carriers and wheels is $\omega_{PC} / \omega_W = 2.64$
- Planetary gear 1 has 60 teeth on the sun and 112 teeth on the ring gear
- Planetary gear 2 has 52 teeth on the sun and 108 on the ring gear.

5.1 In 'EV' mode the car drives with the internal combustion engine off, and ring gears of both planetary gearsets stationary (clutch 2 is closed). For a vehicle driving at 30mph in EV mode, calculate the angular speeds of the two sun gears.

The angular speed of the wheel is $\omega = v / r = 13.4 / 0.20955 = 63.95\ rad / s$

The planet carriers spin at $\omega_{PC} = 2.64 \times 63.95 = 168.8\ rad / s$

The general expression relating angular speeds in an epicyclic gear is

$$\frac{\omega_R - \omega_{PC}}{\omega_S - \omega_{PC}} = -\frac{N_S}{N_R}$$

The ring gear is locked in EV mode so $\omega_R = 0$ giving

$$\omega_{PC}\left(1+\frac{N_R}{N_S}\right) = \omega_S$$

Substituting numbers gives

- $\omega = 484 rad / s$ (4622rpm) for PG1
- $\omega = 519 rad / s$ (4956rpm) for PG2

[3 POINTS]

5.2 Another mode used at higher speeds is 'Fixed Ratio Extended Range' mode, in which the IC engine is running; both clutches are closed (so the sun gear on PG1 is stationary and the ring gear in PG2 is stationary). For a vehicle driving at 35mph in this mode, calculate the angular speeds of MGB and the internal combustion engine.

The angular speed of the wheel is $\omega = v / r = 15.65 / 0.20955 = 74.68\ rad / s$

The planet carriers spin at $\omega_{PC} = 2.64 \times 74.68 = 197.17\ rad / s$

With the clutches closed the sun gear of PG1 is stationary, the ring gear of PG2 is stationary.

For PG1 the general formula gives

$$\omega_R = \left(1+\frac{N_S}{N_R}\right)\omega_{PC} = 302.79\ \ rad / s\ \text{(2891 rpm)}$$

For PG2 we have

$$\omega_{PC}\left(1+\frac{N_R}{N_S}\right) = \omega_S = 606.6\ rad / s\ \text{(5793 rpm)}$$

[2 POINTS]

5.3 At high speeds the car drives in 'High Extended Range Mode' with clutch 1 closed, clutch 2 open, and the IC engine running. In this mode the sun gear of PG1 has the same speed as the ring gear of PG2, and MGB is run slowly or backwards. For a car running at 53 mph, calculate the speeds of the IC engine and MGA in this mode, assuming MGB is stationary.

The angular speed of the wheel is $\omega = v / r = 23.69 / 0.20955 = 113.1\ rad / s$

The planet carriers spin at $\omega_{PC} = 2.64 \times 113.1 = 298.5\ rad / s$

The sun gear of PG2 is stationary, so its ring gear has speed

$$\omega_R = \left(1+\frac{N_S}{N_R}\right)\omega_{PC} = 442.16\ \ rad / s$$

The sun on PG1 spins at the same speed. The general epicyclic gear formula can be rearranged to give

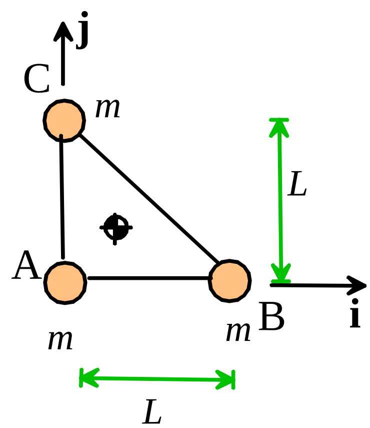
$$\omega_R = \omega_{PC}(1+\frac{N_S}{N_R}) - \frac{N_S}{N_R}\omega_S$$

Substituting numbers gives

$$\omega_R = 221.47 \ \ rad / s$$

[3 POINTS]

6. The figure shows three particles with equal mass m connected by rigid massless links.



6.1 Calculate the position of the center of mass of the assembly

$$\mathbf{r}_G = \frac{1}{M}\sum_i m_i \mathbf{r}_i = \frac{1}{3m}(mL\mathbf{i} + mL\mathbf{j}) = \frac{L}{3}(\mathbf{i}+\mathbf{j})$$

[1 POINT]

6.2 Calculate the 2D mass moment of inertia of the system about the center of mass

$$I_{Gzz} = \sum_i m_i\left(d_{xi}^2 + d_{yi}^2\right)$$

where $\mathbf{d}_i = d_{xi}\mathbf{i} + d_{yi}\mathbf{j} = \mathbf{r}_i - \mathbf{r}_G$ is the position vector of the ith particle with respect to the center of mass.

Use the formula $I_{Gzz} = \sum_i m_i\left(d_{xi}^2 + d_{yi}^2\right) = m\left[\left(\frac{L}{3}\right)^2 + \left(\frac{L}{3}\right)^2\right] + 2m\left[\left(\frac{L}{3}\right)^2 + \left(\frac{2L}{3}\right)^2\right] = \frac{4}{3}mL^2$

[1 POINT]

6.3 Suppose that the assembly rotates about its center of mass with angular velocity $\omega\mathbf{k}$ (the center of mass is stationary). What are the speeds of the particles A,B and C?

Use the circular motion formula. Particles B and C are distances $\sqrt{5}L/3$ from the COM, and particle A is a distance $\sqrt{2}L/3$ from the COM. Therefore

$$V_B = V_C = \sqrt{5}L\omega/3 \qquad V_A = \sqrt{2}L\omega/3$$

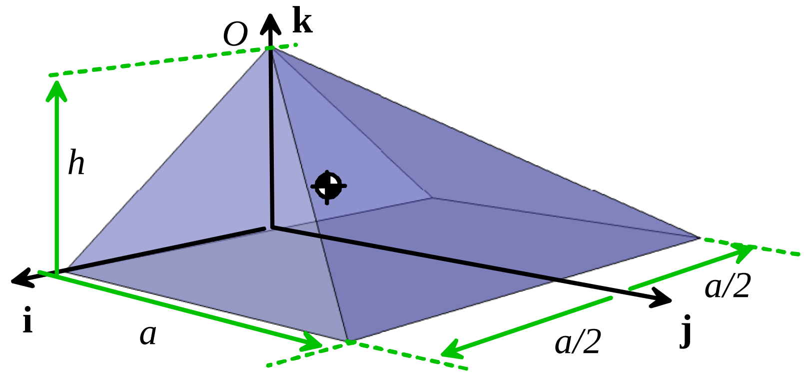
[1 POINT]

6.4 Calculate the total kinetic energy of the system (a) using your answer to 6.2; and (b) using your answer to 6.3

(a) The kinetic energy formula in terms of mass moment of inertia is $T = \frac{1}{2}I_{Gzz}\omega_z^2 = 2mL^2\omega^2/3$

(b) Summing the kinetic energies of the masses directly gives $T = \frac{1}{2}\left\{2m\left(\sqrt{5}L\omega/3\right)^2 + m(\sqrt{2}L/3)^2\right\} = 3mL^2\omega^2/3$ **[2 POINTS]**

7. The figure shows a pyramid with height h and base axa and uniform mass density ρ Using a Matlab 'Live Script', calculate



7.1 The total mass M

$$M = \mathbf{r}_{COM} = \frac{1}{M}\int_0^h \int_0^{a(1-z/h)} \int_{-a(1-z/h)/2}^{a(1-z/h)/2} \rho dxdydz = \rho a^2 h/3$$ (see below for Live Script)

[1 POINT]

7.2 The position vector of the center of mass (with respect to the origin shown in the figure)

$$\mathbf{r}_{COM} = \frac{1}{M}\int_0^h \int_0^{a(1-z/h)} \int_{-a(1-z/h)/2}^{a(1-z/h)/2} \rho(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})dxdydz$$

(see below for Live Script)

$$= \frac{3a}{8h}\mathbf{j} + \frac{h}{4}\mathbf{k}$$

[2 POINTS]

7.3 The inertia tensor (matrix) about the center of mass, in the basis shown

The inertia matrix is (see below for Live Script)

$$\mathbf{I} = \int_0^h \int_0^{a(1-z/h)} \int_{-a(1-z/h)/2}^{a(1-z/h)/2} \rho \begin{bmatrix} d_y^2+d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2+d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2+d_y^2 \end{bmatrix} dxdydz \quad d_x = x \quad d_y = y - 3a/8 \quad d_z = z - h/4$$

$$= M\begin{bmatrix} \frac{19}{320}a^2 + \frac{3}{80}h^2 & 0 & 0 \\ 0 & \frac{3}{80}h^2 & \frac{3}{160}ah \\ 0 & \frac{3}{160}ah & \frac{7}{64}a^2 \end{bmatrix}$$

[3 POINTS]

Graders – equivalent solutions that include the density should get credit too

8.4 Using the parallel axis theorem, calculate the mass moment of inertia about the tip O.

The general formula is

$$\mathbf{I}_O = \mathbf{I}_G + M\begin{bmatrix} d_y^2+d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2+d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2+d_y^2 \end{bmatrix}$$

In the problem here $d_x = d_y = 0 \;\; d_z = 3h/4$ so

$$= M\begin{bmatrix} \frac{19}{320}a^2 + \frac{3}{5}h^2 & 0 & 0 \\ 0 & \frac{3}{5}h^2 & \frac{3}{160}ah \\ 0 & \frac{3}{160}ah & \frac{7}{64}a^2 \end{bmatrix}$$

[2 POINTS]

Equivalent solutions that include the density should get credit (see below for numbers)

The ‘Live Script’ for this problem is shown.

Problem 7: Calculate inertia matrix for a pyramid

```
clear all
syms x y z a h rho M Ig Io rG mass real
M = rho*int(int(int(1,[-(1-z/h)*a/2,(1-z/h)*a/2]),y,[0,(1-z/h)*a]),z,[0,h])
rG = simplify(int(int(int(rho*[x,y,z],[-(1-z/h)*a/2,(1-z/h)*a/2]),y,[0,(1-z/h)*a]),z,[0,h])/M)
dx = x-rG(1);  dy = y-rG(2);  dz = z-rG(3);
IG = simplify(rho*int(int(int([dy^2+dz^2,-dx*dy,-dx*dz;...
                            -dx*dy, dx*2+dz^2,-dy*dz;...
                            -dx*dz, -dy*dz, dx^2+dy^2],[-(1-z/h)*a/2,(1-z/h)*a/2]),y,[0,(1-z/h)*a]),z,[0,h]))
IGwithmass = mass*IG/M
dx = 0; dy = 0; dz = h-rG(3);
Io = simplify(IG + M*[dy^2+dz^2,-dx*dy,-dx*dz;...
                            -dx*dy, dx*2+dz^2,-dy*dz;...
                            -dx*dz, -dy*dz, dx^2+dy^2]  )
Iowithmass = Io*mass/M
```

```
rG =
```

$$\begin{pmatrix} 0 & \frac{3\,a}{8} & \frac{h}{4} \end{pmatrix}$$

```
IG =
```

$$\begin{pmatrix} \frac{a^2\,h\,\rho\,\left(19\,a^2 + 12\,h^2\right)}{960} & 0 & 0 \\ 0 & \frac{a^2\,h^3\,\rho}{80} & \frac{a^3\,h^2\,\rho}{160} \\ 0 & \frac{a^3\,h^2\,\rho}{160} & \frac{7\,a^4\,h\,\rho}{192} \end{pmatrix}$$

```
IGwithmass =
```

$$\begin{pmatrix} \frac{\mathrm{mass}\,\left(19\,a^2 + 12\,h^2\right)}{320} & 0 & 0 \\ 0 & \frac{3\,h^2\,\mathrm{mass}}{80} & \frac{3\,a\,h\,\mathrm{mass}}{160} \\ 0 & \frac{3\,a\,h\,\mathrm{mass}}{160} & \frac{7\,a^2\,\mathrm{mass}}{64} \end{pmatrix}$$

```
Io =
```

$$\begin{pmatrix} \frac{a^2\,h\,\rho\,\left(19\,a^2 + 192\,h^2\right)}{960} & 0 & 0 \\ 0 & \frac{a^2\,h^3\,\rho}{5} & \frac{a^3\,h^2\,\rho}{160} \\ 0 & \frac{a^3\,h^2\,\rho}{160} & \frac{7\,a^4\,h\,\rho}{192} \end{pmatrix}$$

```
Iowithmass =
```

$$\begin{pmatrix} \frac{\mathrm{mass}\,\left(19\,a^2 + 192\,h^2\right)}{320} & 0 & 0 \\ 0 & \frac{3\,h^2\,\mathrm{mass}}{5} & \frac{3\,a\,h\,\mathrm{mass}}{160} \\ 0 & \frac{3\,a\,h\,\mathrm{mass}}{160} & \frac{7\,a^2\,\mathrm{mass}}{64} \end{pmatrix}$$