School of Engineering
Brown University

# EN40: Dynamics and Vibrations

## Homework 8: Rigid Body Dynamics
## Due Friday April 27, 2018

**1.** The Fields Point wind turbines have the following specifications:

- Rated power: 1.5MW
- Rotor diameter: 92m
- Rated wind speed 13 m/s

1.1 Find a formula for the mass moment of inertia of the rotor, in terms of the total mass *M* of the three blades and the blade length *L* (approximate the blades as slender rods)

For one blade we can compute the inertia about one end as is

$$I_{Gzz} = \frac{1}{12}\frac{M}{3}L^2 + \frac{M}{3}\left(\frac{L}{2}\right)^2 = \frac{1}{3}\frac{M}{3}L^2 \ .$$

The total is therefore

$$I_{Gzz} = \frac{1}{3}ML^2$$

**[2 POINTS]**

1.2 The mass of a MW class rotor can be estimated from the empirical relation $M = 0.486d^{2.6}$ where *M* is the mass in kg and *d* is the diameter in m. Estimate the angular speed of the turbine on a day that they are operating (you can just look out the windows on the 7th floor of B&H) and hence determine the kinetic energy of the rotor. Also estimate the tip speed ratio (the ratio of the speed of a blade tip divided by the wind speed – just google the wind speed).

The turbine mass is $M = 62013kg$ ; the mass moment of inertia is $I_{Gzz} = 4.374 \times 10^7 \, kgm^2$

The kinetic energy is $T = \frac{1}{2} I_{Gzz} \omega^2$

A typical angular speed is 1-2 rad/s

Tip speed ratio is $TSR = \omega R / v$ , and should be somewhere between 6 and 8

**[2 POINTS]**

1.3 Assume that the turbine operates at a constant tip speed ratio of 7. Calculate the angular speed at the rated wind speed of 13 m/s.

A typical tip speed ratio is about 7.

For this value $\frac{\omega R}{v} = 7 \Rightarrow \omega = 7\frac{v}{R} = 1.97 rad / s$

**[1 POINT]**

1.4 Calculate the rated torque (i.e. the torque on the wind turbine when it is producing the rated power of 1.5MW at the angular speed in 1.3)

The formula for power is $P = Q\omega \Rightarrow Q = P / \omega = 1.5 \times 10^6 / 1.97 = 7.6 \times 10^5 Nm$

**[1 POINT]**

1.5 The 'natural time constant' of a wind turbine is defined as the time required to spin up the rotor from rest to its rated speed under the rated torque (assumed to be constant). Estimate the natural time constant for the Fields Point turbines.

The angular acceleration of the turbine is related to the torque by

$$Q = I_{Gzz} \alpha_z \Rightarrow \alpha_z = \frac{Q}{I_{Gzz}}$$

We can use the constant acceleration formula to calculate the time

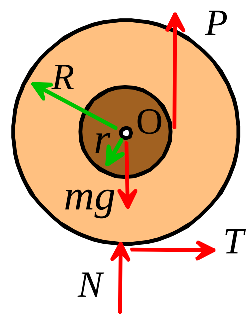
$$t = \frac{\omega}{\alpha_z} = \frac{\omega I_{Gzz}}{Q}$$

Substituting numbers gives *t=113*sec.

**[2 POINTS]**

**2.** The figure shows a spool (e.g. a yo-yo) with outer radius *R*, mass *m* and (2D) mass moment of inertia $I_{Gzz} = mR^2/2$ resting on a table. The hub has radius *r*. A constant vertical force *P* is applied to the yo-yo string. The goal of this problem is to (i) find a formula for the (horizontal) acceleration of the spool, and (ii) find a formula for the critical value of *P* that will cause slip at the contact between the spool and the table

2.1 Draw a free body diagram showing the forces acting on the spool. Assume that the spool remains in contact with the surface, and that no slip occurs at the contact.




(The friction force can go in either direction since there is no slip)

**[3 POINTS]**

2.2 Write down the equations of linear (Newton's law) and rotational (the moment-angular acceleration relation) motion. Your equation should include forces from 22.1, and the linear and angular acceleration of the spool. Please state which point you are taking moments about for the moment equation.

**F**=m**a**

$$T\mathbf{i} + (N + P - mg)\mathbf{j} = ma_{Gx}\mathbf{i}$$

Rotational motion (taking moments about the contact point)

$$rP\mathbf{k} = R\mathbf{j} \times a_{Gx}\mathbf{i} + \frac{1}{2}mR^2\alpha_z\mathbf{k} \Rightarrow rP = -Ra_{Gx} + \frac{1}{2}mR^2\alpha_z$$

**[2 POINTS]**

2.3 Write down a relationship between the angular acceleration $\alpha_z$ and linear acceleration $\mathbf{a}_G$ of the center of mass of the spool

The rolling wheel formula gives $\mathbf{a}_G = -R\alpha_z\mathbf{i}$

**[1 POINT]**

2.4 Use 2.2 and 2.3 to find formulas for (a) the angular acceleration and (b) the linear acceleration of the spool in terms of *P*, and other relevant variables.

Combining results from 2.2 and 2.3: $rP = mR^2\alpha_z + \frac{1}{2}mR^2\alpha_z \Rightarrow \alpha_z = \frac{2}{3}\frac{rP}{mR^2}$

The acceleration is therefore $\mathbf{a}_G = -\frac{2}{3}\frac{rP}{mR}\mathbf{i}$

**[2 POINTS]**

2.5 Find formulas for the reaction forces at the contact, in terms of *P*, *m*, *g*, *R* and *r*

$$\mathbf{F} = m\mathbf{a} \text{ gives } T = -\frac{2}{3}\frac{rP}{R}$$

$$N = mg - P$$

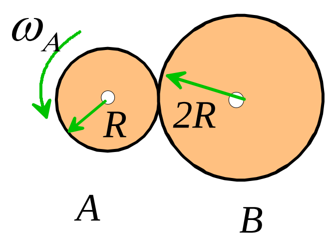
**[2 POINTS]**

2.6 The contact has a friction coefficient $\mu$ . Find a formula for the critical value of $P$ at the point where the contact begins to slip

$$|T| = \mu N \Rightarrow \frac{2}{3}P\frac{r}{R} = \mu(mg - P) \Rightarrow P = \frac{\mu mg}{\mu + 2r/(3R)}$$

**[2 POINTS]**

**3.** The two gears A and B in the figure have radii *R* and *2R*, and mass *m*. Their centers are stationary. Gear A rotates at angular speed $\omega_A$ . Find a formula for the total kinetic energy of the two gears in terms of *m*, *R* and $\omega_A$




The total KE is $(I_{Azz}\omega_A^2 + I_{Bzz}\omega_B^2)/2$;

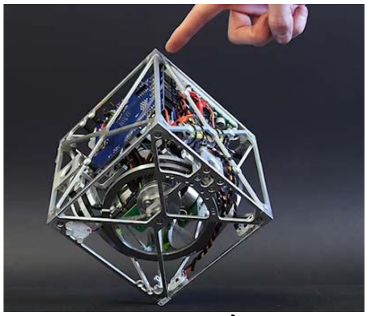
We know $\omega_B = -\omega_A R_A / R_B = -\omega_A / 2$

So the total KE is $\left((mR^2/2)\omega_A^2 + 2mR^2\omega_A^2/4\right)/2 = mR^2\omega_A^2/2$

**(Some people interpreted the question to mean that *m* is the combined mass of the two disks, which will halve this value – those solutions should get credit)**

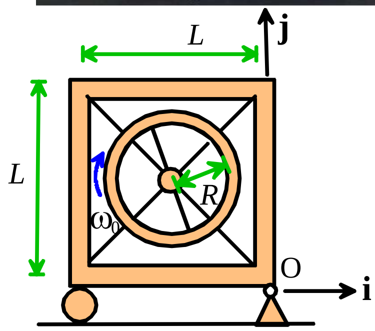
**[2 POINTS]**

**4.** The 'Cubli' is used to develop control algorithms used to stabilize aircraft and spacecraft. It consists of a cube whose attitude can be controlled by spinning a set of reaction wheels inside the cube.



This simplified 1-D version of the device is used to test the algorithm that stands the cube up on one edge. The goal of this problem is to do the preliminary design calculations needed to set up the system.

Idealize the rectangular frame as four rods with length $L$ and combined mass $M$ and the spinning wheel as a ring with radius $R$ and mass $m$. The corner at O is supported by a frictionless bearing.




4.1 Find formulas for the mass moments of inertia of the frame and the wheel (about the center of the wheel).

The ring is easy – we can use the formula $I_R = mR^2$

The frame is made up of four rods of mass *M/4*. The moment of inertia of one rod about its center of mass is $\frac{1}{12}\frac{M}{4}L^2$ . We need to shift the COM by a distance of *L/2* to the center of the frame. The total mass moment of inertia of the frame is therefore

$$I_F = 4\left(\frac{1}{12}\frac{M}{4}L^2 + \frac{M}{4}\left(\frac{L}{2}\right)^2\right) = \frac{1}{3}ML^2$$

**[2 POINTS]**

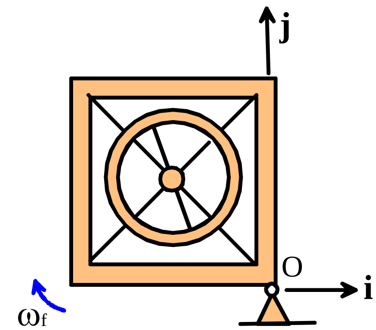
4.2 The frame is at rest and the wheel is spun up (clockwise) to an angular speed $\omega_0$ . Find the total angular momentum of the system about the corner at O.

The formula for angular momentum is $\mathbf{h}_O = \sum \mathbf{r} \times m\mathbf{v}_G + \sum \mathbf{I}\boldsymbol{\omega}$

Since the frame is not moving only the second term contributes and we get $\mathbf{h} = -mR^2\omega_0\mathbf{k}$

**[1 POINT]**

4.3 The wheel is then braked quickly, which causes the frame to rotate about the corner O at angular speed $\omega_f$ , while the motor driving the ring spins at (clockwise) angular speed $\omega_1$ (note that this is relative to the frame). Write down the angular momentum of the system about O.




Note that the frame rotates about O so the COM of the ring and frame are both in circular motion about O. We know the speed of their COMs are therefore $\omega_f L/\sqrt{2}$

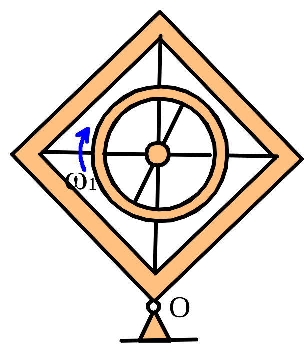
Use the formula again

$$
\begin{aligned}
\mathbf{h} &= \sum \mathbf{r}_G \times m\mathbf{v}_G + I_{Gzz}\omega_z \mathbf{k} \\
&= -\left(\frac{1}{3}ML^2\omega_f + \frac{L}{\sqrt{2}}M\frac{L}{\sqrt{2}}\omega_f + mR^2(\omega_1+\omega_f) + \frac{L}{\sqrt{2}}m\frac{L}{\sqrt{2}}\omega_f\right)\mathbf{k} \\
&= -\left(\frac{5}{6}ML^2 + m(R^2+\frac{L^2}{2})\right)\omega_f\mathbf{k} - mR^2\omega_1\mathbf{k}
\end{aligned}
$$

We could also use the fixed axis rotation formula for the frame (using the mass moment of inertia about O) but this would not work for the ring, because O is not a stationary point on the ring.

**[2 POINTS]**

4.4 Explain why angular momentum is conserved about O during the braking. Use momentum conservation to find an equation relating $\omega_f$ to $(\omega_1 - \omega_0)$




The external forces acting on the frame and ring together are (1) gravity and (2) reaction forces at O. We assume that the speed change of the rotor takes place over a very short time interval. The force of gravity is constant and exerts a negligible impulse on the system during this time interval. The reactions exert a finite impulse, but if we take moments about O the external angular impulse about O on the system vanishes. This means angular momentum must be conserved.

$$
\mathbf{h}_1 - \mathbf{h}_0 = 0 \Rightarrow -\left(\frac{5}{6}ML^2 + m(R^2+\frac{L^2}{2})\right)\omega_f\mathbf{k} - mR^2\omega_1\mathbf{k} + mR^2\omega_0\mathbf{k} = 0
$$

$$
\Rightarrow \omega_f = \frac{mR^2(\omega_0-\omega_1)}{\left(\frac{5}{6}ML^2 + m(R^2+\frac{L^2}{2})\right)}
$$

**[2 POINTS]**

4.5 For the special case $\omega_1 = 0$ show that the critical value of $\omega_0$ required to flip the frame (and ring) into the stationary vertical configuration is

$$\omega_0 = \sqrt{\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)(m+M)}\,\frac{\sqrt{gL}\sqrt{\left(\sqrt{2}-1\right)}}{mR^2}$$

Energy is conserved as the frame rotates up onto its edge.

The formula for the kinetic energy of a system of rigid bodies is

$$T = \sum \frac{1}{2} m \left|\mathbf{v}_G\right|^2 + \sum \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}$$

For 2D problems we can replace the last term by $\frac{1}{2} I_{Gzz} \omega_z^2$

Assume that the frame is at rest in the upright state. The total potential and kinetic energy in the upright state is therefore

$$T_1 + U_1 = \frac{1}{2} mR^2 \omega_1^2 + (m+M) g \frac{L}{\sqrt{2}}$$

In the initial state

$$\begin{aligned} T_0 + U_0 &= \frac{1}{2}(m+M)\left(\frac{L}{\sqrt{2}}\omega_f\right)^2 + \frac{1}{2} mR^2 \left(\omega_1 + \omega_f\right)^2 + \frac{1}{2}\frac{1}{3} ML^2 \omega_f^2 + (m+M) g \frac{L}{2} \\ &= \frac{1}{2}\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)\omega_f^2 + mR^2 \omega_1 \omega_f + \frac{1}{2} mR^2 \omega_1^2 + (m+M) g \frac{L}{2} \end{aligned}$$

Energy conservation gives

$$\begin{aligned} &\frac{1}{2}\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)\omega_f^2 + mR^2 \omega_1 \omega_f + \frac{1}{2} mR^2 \omega_1^2 + (m+M) g \frac{L}{2} = \frac{1}{2} mR^2 \omega_1^2 + (m+M) g \frac{L}{\sqrt{2}} \\ &\Rightarrow \frac{1}{2}\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)\omega_f^2 + mR^2 \omega_1 \omega_f - (m+M) gL \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = 0 \end{aligned}$$

For $\omega_1 = 0$ we get

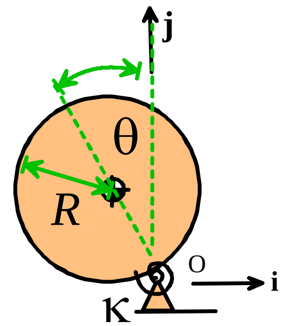
$$\omega_f = \sqrt{\frac{(m+M)}{\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)}}\sqrt{gL}\sqrt{\left(\sqrt{2}-1\right)}$$

From 5.4 we get

$$\omega_0 = \frac{\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)}{mR^2}\omega_f = \sqrt{\left(\frac{5}{6}ML^2 + m(R^2 + \frac{L^2}{2})\right)(m+M)}\frac{\sqrt{gL}\sqrt{\left(\sqrt{2}-1\right)}}{mR^2}$$

**[3 POINTS]**

**5.** The figure shows a simple idealization of a human head on a neck. The head is a sphere with radius *R* and mass *m*; the neck is a pivot with a torsional spring with (torsional) stiffness $\kappa$ .




5.1 Write down the total kinetic and potential energy of the system

The mass moment of inertia about O is $I_{Ozz} = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2$

The potential energy is $mgR\cos\theta + \frac{1}{2}\kappa\theta^2$

The kinetic energy is $\frac{1}{2}I_{Ozz}\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}\frac{7}{5}mR^2\left(\frac{d\theta}{dt}\right)^2$

**[2 POINTS]**

5.2 Use the energy method to derive an equation of motion for $\theta$

The system is conservative, therefore

$$\frac{d}{dt}(T+U) = \frac{7}{5}mR^2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} - mgR\sin\theta\frac{d\theta}{dt} + \kappa\theta\frac{d\theta}{dt} = 0$$

$$\Rightarrow \frac{7}{5}mR^2\frac{d^2\theta}{dt^2} + \kappa\theta - mgR\sin\theta = 0$$

**[2 POINTS]**

5.3 Find a formula for the natural frequency of vibration

Linearize the equation and put it in standard form

$$\frac{7}{5}mR^2\frac{d^2\theta}{dt^2}+(\kappa-mgR)\theta=0$$
$$\Rightarrow\frac{7}{5}\frac{mR^2}{(\kappa-mgR)}\frac{d^2\theta}{dt^2}+\theta=0$$

The natural frequency is therefore

$$\omega_n=\sqrt{\frac{5(\kappa-mgR)}{7mR^2}}$$

**[2 POINTS]**

5.4 Measurements report a natural frequency of 1.5 Hertz. Take the mass of a representative head to be 5kg and radius 11cm (detailed data is published here). Estimate the torsional stiffness of the neck.

We can rearrange the expression for natural frequency to read

$$\frac{7}{5}mR^2\omega_n^2+mgR=\kappa$$

Substituting numbers gives $\kappa=12.9Nm/rad$

**[2 POINTS]**