



EN40: Dynamics and Vibrations

Homework 3: Kinematics and Dynamics of Particles Due Friday Feb 15, 2019

School of Engineering
Brown University

Please submit your solutions to the MATLAB coding problems 4, 5, 6 by uploading a SINGLE file, with a .m extension, to Canvas.

1. Simple Newton's law problem. The European standard EN12195-1:2010 regulates [safe securing of loads for transportation](#). For a cargo secured by a single top-over lashing (see the figure) they give the following formula for the largest mass that can be transported without slipping on the truck-bed

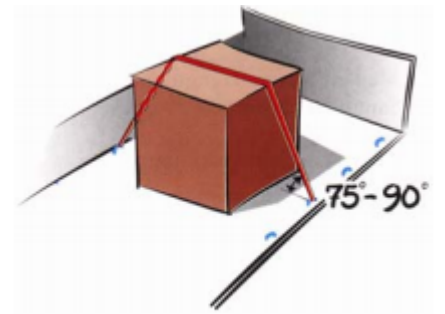
$$m = \frac{2\mu F \sin \alpha}{a_x - \mu g}$$

where μ is the coefficient of friction between the load and truck-bed; a_x is the vehicle's acceleration (the formula assumes $a_x > \mu g$); F is the tension force in the lashing, α is the angle

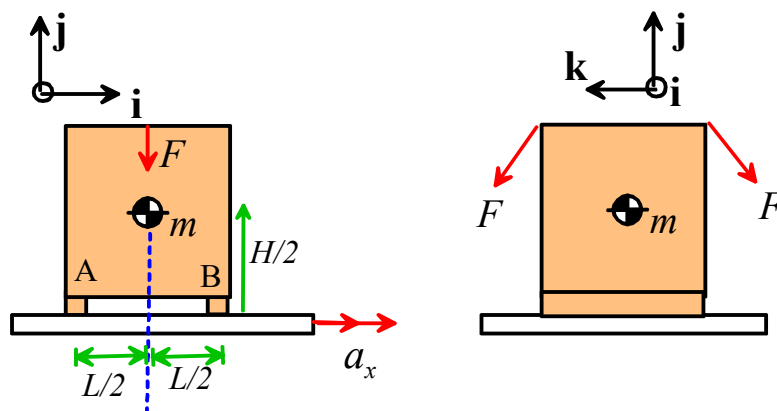
between the cable and the truck-bed (indicated as 75-90 degrees in the sketch), and g is the gravitational acceleration. The standard also states that for a load with COM at its center, the maximum mass that can be transported without tipping can be calculated from the formula

$$m = \frac{2F \sin \alpha}{a_x (H/L) - g}$$

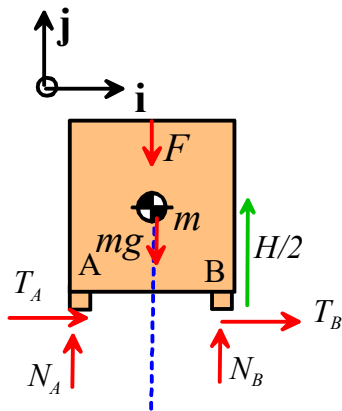
The goal of this problem is to derive these two formulas.



Top-over lashing



1.1 Draw a (2D) free body diagram showing the forces acting on the mass. Assume no slip occurs at the contact between the crate and truckbed.



[3 POINTS]

1.2 Write down $\mathbf{F} = m\mathbf{a}$ and $\sum \mathbf{M}_{COM} = \mathbf{0}$ for the mass

$$\text{Newtons law } (T_A + T_B)\mathbf{i} + (N_A + N_B - mg - 2F \sin \alpha)\mathbf{j} = ma_x\mathbf{i}$$

Sum of moments about COM is zero (no rotation)

$$[(T_A + T_B)H/2 + (N_B - N_A)L/2]\mathbf{k} = 0$$

[2 POINTS]

1.3 Hence, find formulas for the reaction forces acting at A and B (you will only be able to calculate the sum of the horizontal forces acting at A and B – see the ‘people mover’ example from class)

We have 3 equations

$$(T_A + T_B) = ma_x$$

$$N_A + N_B - mg - 2F \sin \alpha = 0$$

$$(T_A + T_B)H/2 + (N_B - N_A)L/2 = 0$$

We can solve these to see that

$$N_A + N_B = mg + 2F \sin \alpha$$

$$T_A + T_B = ma_x$$

$$N_B = (mg + 2F \sin \alpha - ma_x(H/L))/2$$

$$N_A = (mg + 2F \sin \alpha + ma_x(H/L))/2$$

[3 POINTS]

1.4 Use your solution to 1.3 to derive the two formulas in the European standard.

At the onset of slip

$$T_A = \mu N_A \quad T_B = \mu N_B \Rightarrow T_A + T_B = \mu(N_A + N_B) = \mu(mg + 2F \sin \alpha)$$

$$\Rightarrow \mu(mg + 2F \sin \alpha) = ma_x$$

$$\Rightarrow m = \frac{2\mu F \sin \alpha}{a_x - \mu g}$$

At the onset of tipping

$$N_B = 0$$

$$\Rightarrow (mg + 2F \sin \alpha - ma_x(H/L)) = 0$$

$$\Rightarrow m = \frac{2F \sin \alpha}{a_x(H/L) - g}$$

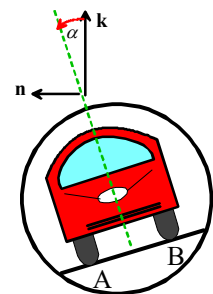
[2 POINTS]

2. Newtons laws in Normal-Tangential Coordinates. Suppose that the Virgin Hyperloop vehicle (see HW2) travels at constant speed 1080 km/hr around a curve with constant radius R .

2.1 If the magnitude of the vehicle's acceleration cannot exceed $0.2g$ ($g=9.81 \text{ m/s}^2$ is gravitational acceleration) what is the smallest allowable value for the curve radius R ?

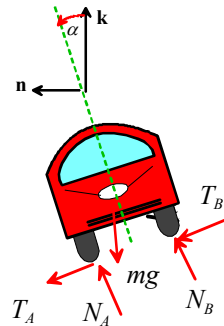
The vehicle is in constant speed circular motion around the track. The magnitude of the acceleration is therefore V^2 / R . If $V^2 / R = 0.2g$ it follows that

$$R_{\min} = \frac{(1080/3.6)^2}{(0.2 \times 9.81)} = 45.9 \text{ km}$$



[1 POINT]

2.2 Curved track will be banked at an angle α to minimize the lateral forces acting on the vehicle (and passengers inside the vehicle). Draw a free body diagram showing the forces acting on the vehicle (you can neglect air resistance).



[The friction forces can go upwards as well - since there is no slip either direction is fine.

[2 POINTS]

2.3 Assume that the vehicle travels at 1080 km/hr around a curve with the minimum allowable radius calculated in 7.1. Find formulas for the reaction forces acting on the vehicle normal and tangent to the banked track, in terms of the mass m of the vehicle and the bank angle α . (You only need to find the sum of the forces acting on the two wheels; there is no need to calculate the forces at A and B separately)

$\mathbf{F}=\mathbf{ma}$ gives

$$[(T_A + T_B) \cos \alpha + (N_A + N_B) \sin \alpha] \mathbf{n} + [-(T_A + T_B) \sin \alpha + (N_A + N_B) \cos \alpha - mg] \mathbf{k} = \frac{mV^2}{R} \mathbf{n} = 0.2mg \mathbf{n}$$

The \mathbf{n} and \mathbf{k} components give

$$(T_A + T_B) \cos \alpha + (N_A + N_B) \sin \alpha = 0.2mg$$

$$-(T_A + T_B) \sin \alpha + (N_A + N_B) \cos \alpha = mg$$

$$\Rightarrow N_A + N_B = mg(0.2 \sin \alpha + \cos \alpha)$$

$$T_A + T_B = mg(0.2 \cos \alpha - \sin \alpha)$$

[GRADERS: it is not necessary to use n-k coordinates to solve this problem; coordinates normal and tangent to the track are also OK as long as the gravity and acceleration vectors are correctly expressed as components in the new basis vectors. People will get a – sign in front of the tangential forces if they chose to make them act up the slope instead of down]

[2 POINTS]

2.4 Find the bank angle that will ensure that the force acting tangent to the track is zero.

The force is zero if $\tan \alpha = 0.2 \Rightarrow \alpha = 11.3^\circ$

[1 POINT]

3. In this problem we will analyze the motion of the first stage of the Falcon-9 launch vehicle

To keep the calculation simple, we will assume that

- The mass of the rocket decreases with time as propellant is ejected, and can be calculated from the formula $m = M - \mu t$ where M and μ are constants.
- Assume a vertical trajectory and include gravity
- Neglect air drag

Assume that the distance traveled by the vehicle $x(t)$ satisfies the differential equation

$$(M - \mu t) \frac{d^2 x}{dt^2} = T_0 - (M - \mu t)g$$

(this looks like $\mathbf{F}=\mathbf{ma}$, but it is derived by considering the rate of change of momentum of the rocket and propellant – applying Newton's law to an object that does not have constant mass is a bit tricky).



3.1 Write a MATLAB ‘Live Script’ to solve the differential equation for x , (using the ‘dsolve’ function) with initial conditions $x = 0$ $\frac{dx}{dt} = 0$ at time $t=0$. Please upload your script to CANVAS.

```
syms M t mu T0 g real
syms x(t)
IC = x(0)==0;
IC2 = subs(diff(x(t),t),t,0)==0;
diffeq = (M-mu*t)*diff(x(t),t,2)==T0-(M-mu*t)*g;
x_sol = simplify(dsolve(diffeq,IC,IC2,symvar('x(t)')))
x_plot = subs(x_sol,[M,mu,T0,g],[575000,2500,7.6e06,9.81]);
fplot(diff(x_plot,t),[0,163])
axes1 = gca;
grid on
xlabel('Time (s)')
ylabel('Speed (m/s)')
title('Falcon 9 velocity')
set(axes1,'FontSize',12);
```

The live script gives

$x_sol =$

$$-\frac{2 M T_0 \log(-M) + g \mu^2 t^2 - 2 T_0 \mu t - 2 M T_0 \log(\mu t - M) + 2 T_0 \mu t \log(\mu t - M) - 2 T_0 \mu t \log(-M)}{2 \mu^2}$$

This can be simplified to

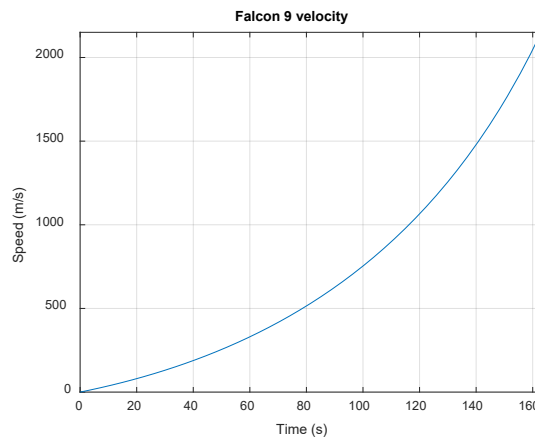
$$x = T_0 t / \mu - g t^2 / 2 + (T_0 / \mu^2)(M - \mu t) \log\{(M - \mu t) / M\}$$

[3 POINTS]

3.2 Hence (using your ‘Live Script’) plot the speed as a function of time. Use the following values for parameters:

- Total initial mass $M=575000\text{kg}$
- Propellant mass flow rate $\mu = 2500\text{kg} / \text{s}$
- Thrust $T_0 = 7.6 \times 10^6 \text{N}$
- Time interval $0 < t < 162$ sec.

You can compare your prediction with the data from HW2, if you are curious. For example, the velocity at the end of the first stage burn was measured to be 1848 m/s.



[2 POINTS]

4. We can set up a more sophisticated model of the Falcon-9 by using the MATLAB ode solver to integrate the equations of motion. For the complete calculation, assume that

- The mass of the rocket can be calculated from the formula

$$m = \begin{cases} 550000 - 2500t & t < 165s \\ 137500 - 263(t - 165) & 165 < t < 584s \\ 27000 & t > 584s \end{cases}$$

The mass is in kg, and the formula includes both stages of the rocket

- The thrust varies with time as

$$T_0 = \begin{cases} 7.6 \times 10^6 & t < 165s \\ 1.6 \times 10^6 & 165 < t < 584s \\ 0 & t > 584s \end{cases}$$

- A drag force

$$F_d = 50\rho v^2$$

(in Newtons) acts on the rocket, where v is the rocket's speed in m/s, and ρ is the air density, which varies with altitude as $\rho = 1.22 \exp(-x / 8.5 \times 10^3)$ kg m⁻³, where x is in m .

- Assume as an approximation that the trajectory is vertical, and neglect variations in gravitational force.

4.1 Show that the equation of motion

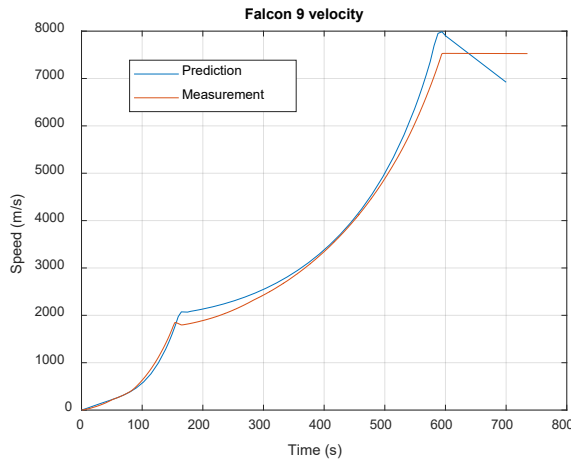
$$(M - \mu t) \frac{d^2 x}{dt^2} = T_0 - (M - \mu t)g - 50\rho v^2$$

can be written in the form required by the MATLAB ode solver as follows

$$\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ (T_0 - 50\rho v^2) / m - g \end{bmatrix}$$

Using $a = \frac{dv}{dt}$ and $v = \frac{dx}{dt}$ and rearranging the result gives the equations stated. [1 POINT]

4.2 Write a MATLAB script that will calculate x and v for a time interval $0 < t < 700s$. Plot the variation of speed v with time, and compare the prediction with the experimental data (use the data file from HW2, and plot the prediction and experiment on the same graph)



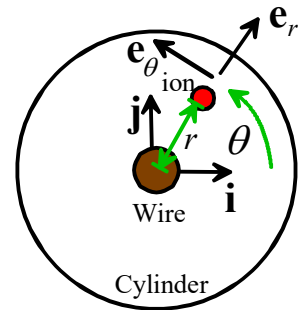
[4 POINTS]

5. A ‘[Dynamic Kingdon trap](#)’ is a device for storing charged particles (see e.g. this [patent](#)). It consists of a wire that runs down the axis of a cylinder. A time varying electric field is applied to the wire, which subjects the particle to a radial force

$$\mathbf{F} = q(V_0 + \Delta V \cos \Omega t) \frac{1}{2\pi\epsilon_0 r} \mathbf{e}_r$$

Here, $V_0, \Delta V$ are a static and fluctuating voltage applied between the wire and cylinder; Ω is its frequency, q is the charge of the trapped ion, and ϵ_0 is the permittivity of free space.

The goal of this problem is to analyze the motion of an ion in the trap. We will assume that the particle moves in the r, θ plane, and so must solve for $[r, \theta]$ as a function of time



5.1 Write down Newton’s laws for the particle (use polar coordinates) and show that the $\mathbf{e}_r, \mathbf{e}_\theta$ components of the equation can be re-arranged into the form required by matlab by re-writing them as

$$\frac{d}{dt} \begin{bmatrix} r \\ \theta \\ v \\ \omega \end{bmatrix} = \begin{bmatrix} v \\ \omega \\ r\omega^2 + (\lambda + \mu \cos \Omega t) / r \\ -2v\omega / r \end{bmatrix}$$

Find formulas for λ, μ in terms of $V_0, \Delta V, \epsilon_0, q, m$.

Newton’s law in polar coordinates gives

$$q(V_0 + \Delta V \cos \Omega t) \frac{1}{2\pi\epsilon_0 r} \mathbf{e}_r = m \left(\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right) \mathbf{e}_r + m \left(r \frac{d^2 \theta}{dt^2} + 2 \frac{dr}{dt} \frac{d\theta}{dt} \right) \mathbf{e}_\theta$$

$$\Rightarrow \frac{d^2 r}{dt^2} = \frac{dv}{dt} = \left(\frac{qV_0}{2\pi m \epsilon_0} + \frac{q\Delta V}{2\pi m \epsilon_0} \cos \Omega t \right) \frac{1}{r} + r\omega^2 \quad \frac{d^2 \theta}{dt^2} = \frac{d\omega}{dt} = -\frac{2v\omega}{r}$$

So if we define $\lambda = qV_0 / (2\pi m \epsilon_0)$ $\mu = q\Delta V / (2\pi m \epsilon_0)$ we obtain the formulas stated.

[2 POINTS]

5.2 Write a MATLAB code that will calculate $[r, \theta, v, \omega]$. For the parameter values listed below, plot the path of the particle, for a time period $0 < t < 50$ time units. You can use the following plot function to do the polar plot for you

```
polarplot(theta, r, 'Displayname', '(Problem a)', 'Color', [1, 0, 0]);
```

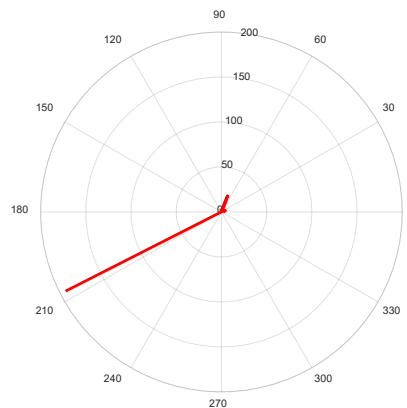
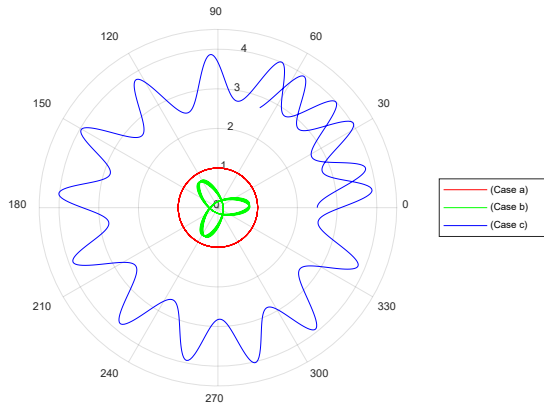
where theta is a vector of values of θ and r is a vector of values of r at successive time intervals (these can be extracted from the matrix that stores values of $[r, \theta, v, \omega]$ returned by ode45).

- (a) $\lambda = -1, \mu = 0, \Omega = 2$ (this is a static trap – the electric field does not vary with time), with initial conditions $r = 1, \theta = 0, v = 0, \omega = 1$. For this case you will need to specify the time intervals you would like in the solution, otherwise ode45 will take huge time steps and the trajectory will look funny. You can do this, eg, with

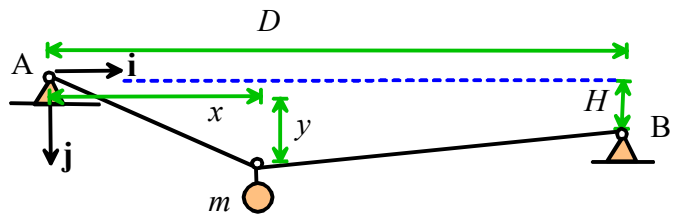
```
[time, sols] = ode45(@ (t,w) trap(t,w,lambda,mu,Omega), [0:0.1:50], initial_w);
```

Here, the 0:0.1:50 tells MATLAB you would like solution values every 0.1 time intervals.

- (b) $\lambda = -1, \mu = 0, \Omega = 2$ with initial conditions $r = 0.8, \theta = 0, v = 0, \omega = 0.4$. (use ‘hold on’ to put this on the same plot as (a)). The static trap is stable – a particle remains trapped, as long as nothing slows it down.
- (c) $\lambda = -1, \mu = 8, \Omega = 2$ (this is a dynamic trap – the electric field varies with time) with initial conditions $r = 2.5, \theta = 0, v = 0, \omega = 0.25$. Plot this case on the same graph as (a) and (b). Particles with the right radius and velocity are trapped.
- (d) $\lambda = -1, \mu = 8, \Omega = 2$ with initial conditions $x = 1.5, y = 0, v_x = 0, v_y = 0.25$. Run this case for 100 time units, and plot it on a new figure. A dynamic trap will expel particles that are too fast or slow, and are too close or too far from the wire

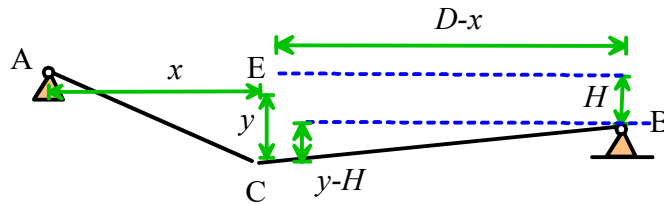


[4 POINTS]



6. The figure shows an idealization of a person with mass m on a zip-line. The person starts at end A of the line and travels to end B , a horizontal distance D and vertical distance H below A . The cable is idealized as a spring with stiffness k and un-stretched length L_0 , with $L_0 > \sqrt{D^2 + H^2}$. The tension in the cable is related to its stretched length by $T = k(L - L_0)$.

6.1 Show that the length of the cable is $L = \sqrt{x^2 + y^2} + \sqrt{(D-x)^2 + (y-H)^2}$ (this is just geometry)

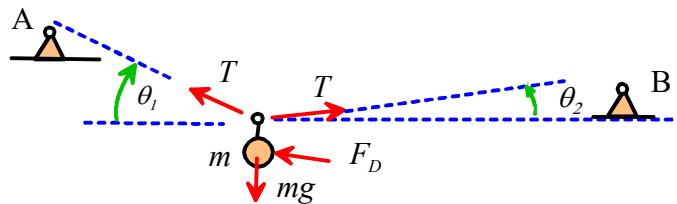


Apply Pythagoras to triangles ACE and CEB to get the two answers.

[2 POINTS]

6.2 Draw a free body diagram showing the forces acting on the mass (include the cable tension, which is the same on both sides of the mass). Assume that an air drag force with magnitude $F_D = \frac{1}{2} \rho A C_D V^2$ and direction opposite to the velocity vector acts on the rider. Here, ρ is the air density, A is the frontal area of the rider; C_D is the drag coefficient and $V = \sqrt{v_x^2 + v_y^2}$ is the rider's speed.

The FBD is shown below



[2 POINTS]

6.3 Use Newton's law to derive equations of motion for the mass, and show that they can be re-arranged in the following form (suitable for the MATLAB ODE solver)

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ (T/m)(D-x)/d_2 - (T/m)x/d_1 - cVv_x \\ g - (T/m)(y-H)/d_2 - (T/m)y/d_1 - cVv_y \end{bmatrix} \quad \begin{aligned} d_1 &= \sqrt{x^2 + y^2} \\ d_2 &= \sqrt{(D-x)^2 + (y-H)^2} \\ c &= \frac{1}{2m} \rho A C_D \end{aligned}$$

(i) Note that the drag force can be expressed as a vector as $\mathbf{F}_D = \left[\frac{1}{2} \rho A C_D V^2 \right] (v_x \mathbf{i} + v_y \mathbf{j}) / V$

(ii) Apply Newton's law

$$(-T \cos \theta_1 + T \cos \theta_2)\mathbf{i} + (mg - T \sin \theta_1 - T \sin \theta_2)\mathbf{j} + \mathbf{F}_D = m \left(\frac{dv_x}{dt} \mathbf{i} + \frac{dv_y}{dt} \mathbf{j} \right)$$

(iii) Trig shows that

$$\begin{aligned} \cos \theta_1 &= x / \sqrt{x^2 + y^2} & \cos \theta_2 &= (D - x) / \sqrt{(D - x)^2 + (y - H)^2} \\ \sin \theta_1 &= y / \sqrt{x^2 + y^2} & \sin \theta_2 &= (y - H) / \sqrt{(D - x)^2 + (y - H)^2} \end{aligned}$$

Collecting (1,2,3) and using $v_x = dx/dt$ $v_y = dy/dt$ gives the solution

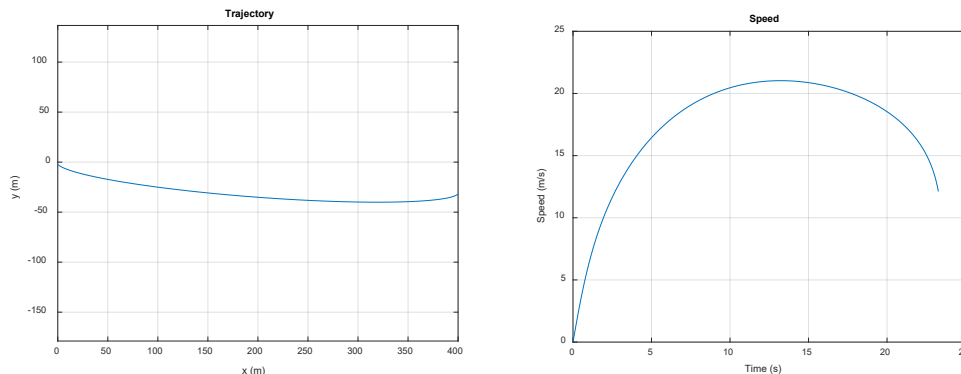
[3 POINTS]

Use ode45 to solve the equation. Use the following values for parameters

- Rider mass m 80kg
- Dimensions $D=400\text{m}$, $H=30\text{m}$
- Drag parameters: $A=1\text{m}^2$, $\rho = 1.22 \text{ kg m}^{-3}$ $C_D = 0.2$
- Line stiffness $k=10^6\text{Nm}^{-1}$ and unstretched length $L_0 = \sqrt{D^2 + H^2} + 2 \text{ m}$
- Initial position $x = 0, y = (L_0^2 - D^2 - H^2) / 2(L_0 - H), v_x = v_y = 0$

Add an 'event' function to your script that will stop the calculation when the rider reaches $x=D$.

Hence, plot (i) The trajectory of the rider; and (ii) a graph of the riders speed as a function of time. You can compare your predictions to data for a number of ziplines [here](#)

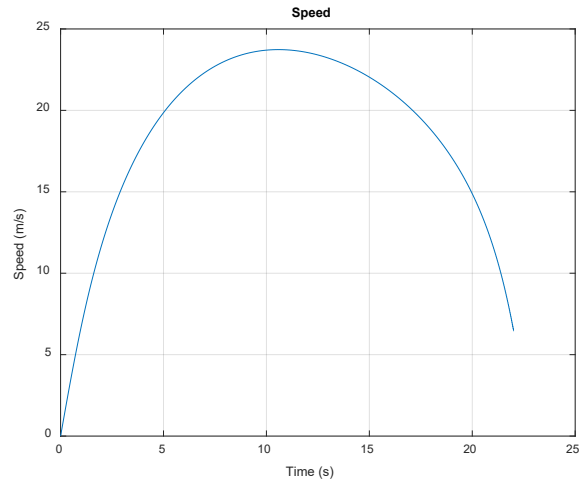


[4 POINTS]

6.4 The rider arrives at $x=D$ at over 10 m/s. How would you change the design to reduce the speed to below 7 m/s (but without changing H or D)?

Experimenting with the MATLAB (that's what this sort of design simulation is useful for – you don't have to design your ride by trial and error) shows that the rider slows down if the unstretched length of the cable is increased. An unstretched length

$L_0 = \sqrt{D^2 + H^2} + 5.5 \text{ m}$ would work. Making the cable too slack is risky, because riders might not make it to the end....



Other slightly less straightforward solutions include reducing the stiffness of the cable k (but it has to be reduced a lot, and the resulting cable stretch is a problem); restricting the ride to people with very low mass or large frontal area A to increase air drag. You can of course also add a brake to the system but that's a lot more expensive than just adding a couple of meters of cable....

[2 POINTS]