EN40: Dynamics and Vibrations

Homework 4: Conservation Laws for Particles
Due Friday March 1, 2019

School of Engineering
Brown University

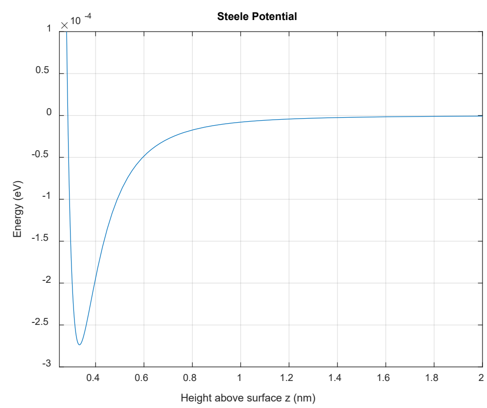
1. The 'Steele' potential is used in molecular dynamics to model the interaction of a gas molecule with a planar surface (e.g. in simulations of chemisorption). It gives the potential energy of a gas molecule that is a height z above a surface as

$$U = 2\pi\rho_s \Delta a^2 E_0 \left[\frac{2}{5}\left(\frac{a}{z}\right)^{10} - \left(\frac{a}{z}\right)^4 - \frac{a^4}{3\Delta(z+0.61\Delta)^3} \right]$$

Here, E_0 and a are the binding energy and equilibrium length of a bond between the molecule and one atom in the surface; ρ_s is the number of atoms per unit volume in the solid surface, and Δ is the spacing between planes of atoms in the solid material. This reference gives the following values for these parameters for interactions between Argon and Carbon.

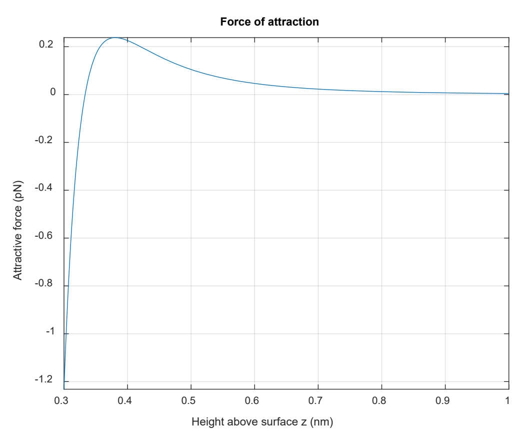
E_0	$7.938\text{x}10^{-22}$ J (0.005eV)
a	*0.33525* nm
ρ_S	0.34 nm^{-3}
Δ	0.335 nm

1.1 Plot the energy as a function of z (use units of eV for the energy, and nm for the distance z. $0.25<z<2$ nm gives a clear plot) (Just hand in your plot; there is no need to submit MATLAB code)



[2 POINTS]

1.2 Plot a graph showing the force of attraction between the molecule and the surface as a function of its distance z from the surface. (use units of picoNewtons for the force, and nm for the distance z. $0.3<z<1$ nm gives a clear plot) (Just hand in your plot; there is no need to submit MATLAB code)



[2 POINTS]

1.3 Find the (static) force required to detach the molecule from the surface (in picoNewtons). If you are using a MATLAB live script to do the calculation you will need to use the 'vpasolve' function to find the value of *z* that maximizes the force of attachment. See Sect 6.1 of the MATLAB tutorial for an example.

MATLAB gives $F=0.238$ pN.

[2 POINTS]

```
syms z real
E0 = 0.005; %
eV = 1.602e-19;   % Conversion from eV to J
nm = 1.e-09;  % Conversion from nm to m
pN = 1.e-12;  % conversion from pN to N
a = 0.33525;
rhos = 0.34;
Delta = 0.335;
U = 2*pi*rhos*Delta*a^2*E0*( (2/5)*(a/z)^10 - (a/z)^4 - a^4/(3*Delta*(z+0.61*Delta)^3) );
%
% Problem 1.1
%
fplot(U,[0.25,2])
ylim([-0.0003 0.0001])
grid on
xlabel('Height above surface z (nm)')
ylabel('Energy (eV)')
title('Steele Potential')
%
% Problem 1.2
%
F = diff(U*eV/nm/pN,z);
fplot(F,[0.3,1])
grid on
xlabel('Height above surface z (nm)')
ylabel('Attractive force (pN)')
title('Force of attraction')
%
% Problem 1.3
%
zatmax = vpasolve(diff(F,z)==0,[0.3,0.5]);
Fmax = double(subs(F,z,zatmax))
```

2. The Volkswagen I.D. R electric vehicle broke the Pike's Peak International Hill Climb record last summer, with a time of 7:57min.

The course is on a 12.42 mile public road, which climbs 1440m.

The vehicle has the following specifications:

- Mass 1100kg
- Max power output from the (combined) two electric motors $P_{\max}$ 500kW
- Top speed $v_{\max}$ 240 km/hr
- Acceleration (0-100 km/hr) 2.3s

2.1 Calculate the power needed to climb the 1440m at constant speed in 7.57 min. Why do you think the designers choose to exceed the required power by such a large margin?

You can do this calculation by either (1) calculating the change in energy, and dividing by the time $P = mgh/t = 1100\times 9.81\times 1440/(7\times 60+57) = 32.6kW$

or (2) using the energy equation $P_{motor} + P_{gravity} = 0 \Rightarrow P_{motor} - mg\mathbf{j}\cdot v_y\mathbf{j} = 0 \Rightarrow P_{motor} = mgv_y$, which gives the same answer.

[2 POINTS]

2.2 Suppose the power curve for the electric motors can be approximated by the function

$$P = 4P_{\max}\frac{v}{v_{\max}}\left(1-\frac{v}{v_{\max}}\right)$$

where v is the cars speed. Use the power-kinetic energy relation (neglect air drag) to find a formula for the acceleration of the car on a level road. Use the result to estimate the time required to reach 100 km/hr. Why is the actual time greater than this estimate?

The energy equation gives $P_{motor} = \dfrac{d}{dt}(KE) = \dfrac{d}{dt}\left(\dfrac{1}{2}mv^2\right) = mv\dfrac{dv}{dt}$

This gives a differential equation for v that can be solved by separating variables and integrating

$$4P_{\max}\frac{v}{v_{\max}}\left(1-\frac{v}{v_{\max}}\right) = mv\frac{dv}{dt} \Rightarrow \int_0^v \frac{dv}{v_{\max}-v} = \frac{4P_{\max}}{mv_{\max}^2}\int_0^t dt \quad \Rightarrow \left[-\log(v_{\max}-v)\right]_0^v = \frac{4P_{\max}}{mv_{\max}^2}t$$

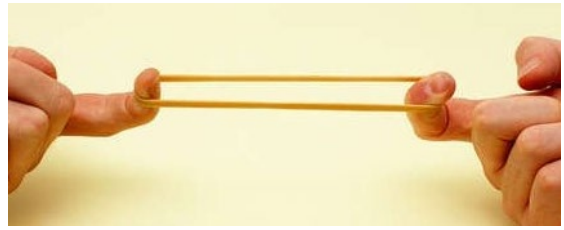
$$\Rightarrow -\log(v_{\max}-v)+\log(v_{\max}) = \frac{4P_{\max}}{mv_{\max}^2}t \quad \Rightarrow t = \frac{mv_{\max}^2}{4P_{\max}}\log\frac{v_{\max}}{v_{\max}-v} = 1.32\sec$$

[3 POINTS]

The average acceleration required to achieve this time is $a=v/t = 2.15$g. It would be very difficult to make tires that could produce this acceleration without spinning the wheels, particularly at low speeds when there is not much down-force on the car. A typical max acceleration is of order 1.5 to 1.75g. This is quite close to the 2.3s quoted in the spec. Air drag also would reduce the acceleration, but not by much (you can easily do the calculation with air drag to check, as discussed in class).

[2 POINTS]

3. A rubber band has cross-sectional area A_0 and unstretched length L_0, and is made from a rubber with mass density ρ. Suppose that the force required to stretch the band to a length L can be approximated by



$$F = 2A_0\mu\left\{\left(\left[\frac{L}{L_0}\right]^{1/2} - \left[\frac{L}{L_0}\right]^{-3}\right) + \frac{1}{50}\left(\left[\frac{L}{L_0}\right]^{4} - \left[\frac{L}{L_0}\right]^{-4}\right)\right\}$$

where μ is a constant (a property of the rubber). The factor of 2 assumes the band is a loop with 2 sides.

3.1 Find a formula for the work done on the rubber band when it is stretched from length L_0 to a length $4L_0$ (Matlab will do the integral for you).

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{x} = \int_{L_0}^{4L_0} 2\mu A_0\left[\left(\left[\frac{L}{L_0}\right]^{1/2} - \left[\frac{L}{L_0}\right]^{-3}\right) + \frac{1}{50}\left(\left[\frac{L}{L_0}\right]^{4} - \left[\frac{L}{L_0}\right]^{-4}\right)\right]dL$$

Matlab can do the integral, with the result

$$W = \frac{397601}{24000}\mu A_0 L_0$$

[3 POINTS]

3.2 Is the force exerted by the rubber band conservative?

Yes – the work done to stretch it to a particular length depends only on the final length, and does not depend on the path used to reach that length.

If you want a more mathematical demonstration, note that the force exerted by the band (with one end at the origin) can be expressed as

$$\mathbf{F} = -F(L)\frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{L}$$

where $L = \sqrt{x^2 + y^2 + z^2}$. You can have Matlab compute curl(**F**) (or if you enjoy such things do it by hand) and show that curl(**F**)=**0**

[2 POINTS]

3.3 Suppose that the rubber band is stretched to a length $4L_0$ and then fired vertically into the air. Find a formula for the maximum height it will reach. Does the height depend on the dimensions of the rubber band? (Aside – a recent publication in Physical Review Letters has some interesting high-speed movies of rubber bands being fired. One of the paper's authors is Jacy Bird, who graduated with and ScB in Engineering from Brown in 2003).

We can take the rubber band and earth together as a conservative system, and regard the agency that stretches the rubber band as external to our system. We take the initial state of the system at the instant before the band is stretched; and the final state at the instant when the band reaches its max height.

The work-energy relation for the system gives

$$W_{ext} = (T_1 + U_1) - (T_0 + U_0)$$

Note: Initial and final KE are zero; the band is unstretched in the initial and final state, so

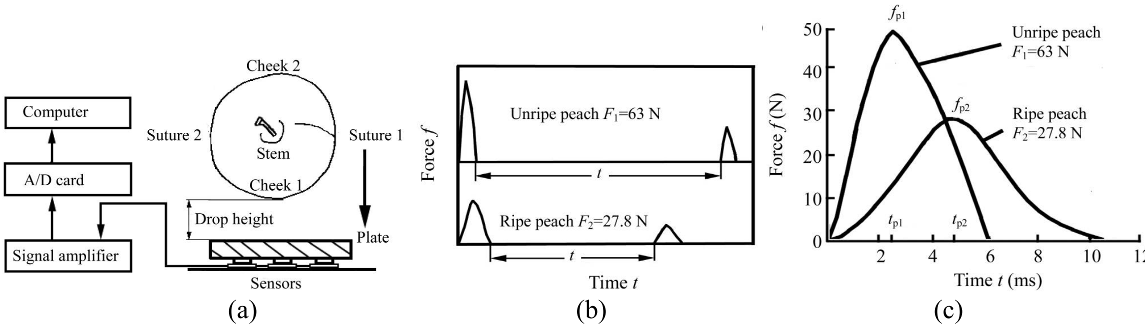
$$W_{ext} = mgh = 2\rho g A_0 L_0 h$$

(The factor of 2 is because the band has two sides) From 3.1 we see that

$$h = \frac{397601}{48000}\frac{\mu}{\rho g}$$

The height does not depend on the band dimensions, at least according to this simple calculation. Air resistance may change the conclusion.

[2 POINTS]



Problem 5: Measuring forces exerted on a peach during an impact test. (a) Apparatus; (b) Forces measured during two successive bounces; (c) A detailed view of the variation of force with time during the first bounce.

4. Bounce tests are often used to measure the properties of fruit (this paper is an example). The goal of this problem is to work through some of the calculations that are used to interpret the experimental data. A typical test setup and representative results (from the publication) are shown in the figure.

4.1 Suppose the fruit is dropped from rest from a height *h* above the force-plate. Use energy methods to find a formula for the speed of the fruit just before impact.

During free-fall the peach + earth is a conservative system, so $W_{ext} = (T_1 + U_1) - (T_0 + U_0)$. No external work is done on the system, so $0 = (T_1 + U_1) - (T_0 + U_0) = \frac{1}{2}mv_1^2 - mgh \Rightarrow v_1 = \sqrt{2gh}$

[2 POINTS]

4.2 The variation of the force acting on the fruit during impact can be approximated by the formula

$$F(t) = \left(\pi^2 + \zeta^2\right)\frac{mv_0}{\pi T}\exp(-\zeta t/T)\sin(\pi t/T) \qquad 0 < t < T$$

where exp denotes the exponential function, *T* is the time the fruit is in contact with the force-plate; *m* is the fruit mass, v_0 is the impact speed and ζ is a constant that quantifies how much energy is lost during the rebound (for $\zeta = 0$ the collision is elastic and no energy is lost; the energy loss increases with ζ). Find a formula for the impulse of the force, in terms of ζ, m, v_0 (Matlab will do the integral for you, but you will need to declare pi as a sym for the answer to come out in a sensible form).

The impulse is $I = \int_0^T F(t)dt = mv_0\left(\exp(-\zeta) + 1\right)$

[2 POINTS]

4.3 Hence, find a formula for the restitution coefficient for the collision in terms of ζ

We can use the impulse-momentum formula for a single particle to find the velocity of the fruit just after the bounce. Take the **j** direction to be vertical, then:

$$I\mathbf{j} = mv_1\mathbf{j} - (-mv_0\mathbf{j}) \Rightarrow mv_1 = I - mv_0 = mv_0 \exp(-\zeta) \Rightarrow v_1 = v_0 \exp(-\zeta)$$

The restitution coefficient is $e = -v_1 / (-v_0) = \exp(-\zeta)$

[2 POINTS]

4.4 In the publication, the authors find the restitution coefficient measuring the time *t* between the first and second impact of the fruit with the force-plate. They use the formula

$$e = t\sqrt{\frac{g}{8h}}$$

where *h* is the initial drop height. Derive this formula.

The restitution coefficient is $e = -v_1 / (-v_0)$. We have that $v_0 = \sqrt{2gh}$ from part 5.1. We can find v_1 from the time the fruit is airborne. At the start of the trajectory its velocity is $v_1\mathbf{j}$; at the end it is $-v_1\mathbf{j}$. The acceleration is $-g\mathbf{j}$, and using the constant acceleration formula

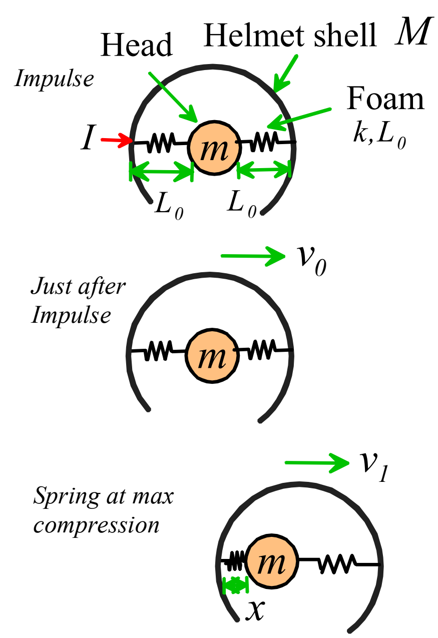
$$-v_1\mathbf{j} = v_1\mathbf{j} - gt \Rightarrow v_1 = \frac{1}{2}gt$$

Therefore

$$e = \frac{\frac{1}{2}gt}{\sqrt{2gh}} = t\sqrt{\frac{g}{8h}}$$

[2 POINTS]

5. The figure shows an idealization of a head inside a protective helmet. The helmet shell has mass *M,* and the protective foam inside the shell is idealized as springs with stiffness k and unstretched length L_0



At time *t=0* the helmet and head are both at rest, and the length of the springs is equal to their unstretched length.

The helmet shell is then subjected to a horizontal impulse *I*, which causes it to move at speed v_0 just after the impulse.

The springs exert no force on either the head or the helmet casing during the impulse. Gravity and vertical motion of the head or helmet may be neglected.

5.1. Find a formula for v_0 in terms of *I* and *M*.

$$v_0 = I / M$$

[1 POINT]

5.2. Find expressions for the total linear momentum and total kinetic energy of the system just after the impulse, in terms of I and the mass M of the shell

$$\mathbf{p}_0 = I$$

$$T_0 = \frac{1}{2}Mv_0^2 = \frac{I^2}{2M}$$

[2 POINTS]

5.3. Consider the system at the instant when the foam on the left of the head is compressed to its smallest thickness (i.e. x is a minimum). Using energy and/or momentum conservation, find a formula for x at this instant. Assume that the foam remains in contact with the head on both sides (so one spring is stretched), as indicated in the figure.

Energy and momentum are conserved, and the casing and head have the same speed at the instant of max compression, so

$$\frac{1}{2}(M+m)v_1^2 + k\left(L_0 - x\right)^2 = \frac{I^2}{2M}$$

$$(M+m)v_1 = I$$

$$\Rightarrow k\left(L_0 - x\right)^2 = \frac{I^2}{2M} - \frac{I^2}{2(M+m)} = \frac{mI^2}{2M(M+m)}$$

$$\Rightarrow x = L_0 - I\sqrt{\frac{m}{2kM(M+m)}}$$

[3 POINTS]

5.4. Hence, find a formula for the minimum foam thickness L_0 necessary to prevent the head from striking the helmet shell, in terms of m, M, I, k

$$L_0 = I\sqrt{\frac{m}{2kM(M+m)}}$$

[1 POINT]

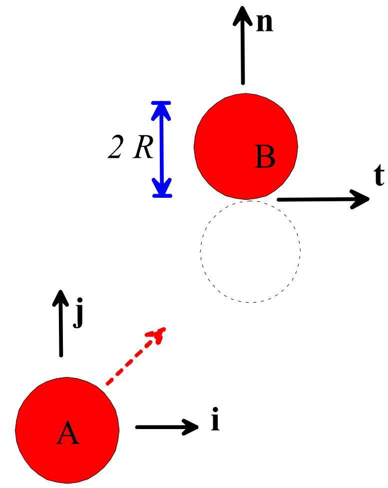
5.5. Find a formula for the total impulse exerted on the head between $t=0$ and the instant of maximum foam compression, in terms of I, M, m.

The impulse on the head is equal to its change in momentum. Thus

$$\mathbf{p} = mv_1\mathbf{i} = \frac{m}{M+m}I$$

[2 POINTS]

6. The figure illustrates a frictionless collision between two spheres. They both have radius R and mass m. The collision has a restitution coefficient e.



At time $t=0$ the spheres have velocity vectors $\mathbf{v}^A = v_x\mathbf{i} + v_y\mathbf{j}$ $\mathbf{v}^B = \mathbf{0}$. They collide so that the direction of the line connecting their centers is parallel to the **j** direction.

6.1 Use the formulas derived in class to find expressions for the velocities of A and B after the collision, in **i,j** components.

We have that

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

In the problem, $\mathbf{v}^{B0} = \mathbf{0}$ $\quad \mathbf{v}^{A0} = v_x\mathbf{i} + v_y\mathbf{j}$ $\quad \mathbf{n} = \mathbf{j}$ so substituting into the formulas gives

$$\mathbf{v}^{A1} = v_x\mathbf{i} + v_y\mathbf{j} + \frac{1}{2}(1+e)\left[-\left(v_x\mathbf{i} + v_y\mathbf{j}\right)\cdot\mathbf{j}\right]\mathbf{j} = v_x\mathbf{i} + \frac{v_y}{2}(1-e)\mathbf{j}$$

$$\mathbf{v}^{B1} = \mathbf{0} - \frac{1}{2}(1+e)\left[-\left(v_x\mathbf{i} + v_y\mathbf{j}\right)\cdot\mathbf{j}\right]\mathbf{j} = \frac{v_y}{2}(1+e)\mathbf{j}$$

[3 POINTS]

6.2 What is the impulse exerted on A during the collision (in terms of e, m and V)? What is the impulse on B? What is the total impulse on the system?

We can calculate the impulse on A using the impulse-momentum formula for a single particle, i.e. the impulse on A is equal to its change in momentum,

$$\mathbf{I} = m(\mathbf{v}^{A1} - \mathbf{v}^{A0}) = m\left(v_x\mathbf{i} + \frac{v_y}{2}(1-e)\mathbf{j}\right) - m\left(v_x\mathbf{i} + v_y\mathbf{j}\right) = -\frac{1}{2}(1+e)mv_y\mathbf{j}$$

The impulse on B is equal and opposite to that on A (or you can get the same answer using the impulse-momentum formula for B)

The total impulse is zero, since no external force acts on the system.

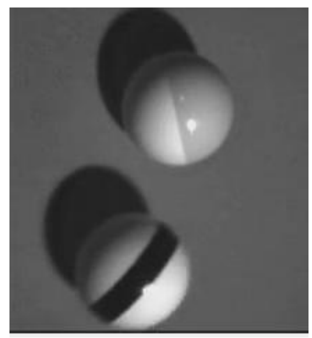
[2 POINTS]

6.3 Please answer the following questions:
(a) Why is the total momentum of the system conserved during the collision?
(b) Why is the momentum of A and B conserved parallel to **t** during the collision?
(c) Why is the momentum of A and B *not* conserved parallel to **n**?

(a) No external impulse acts on A/B during the collision (any external forces exert a finite force, and we assume that the impact occurs over a vanishingly short period of time, so force*time from external forces is negligible). The impulse-momentum formula for a system of particles therefore shows that the momentum is conserved.
(b) The contact between A and B is frictionless. The force must therefore act parallel to **n**. Since no force acts on either A or B during the impact, the impulse-momentum formula for a single particle shows that momentum is conserved parallel to **t**
(c) A very large force acts on both A and B parallel to **n** during the collision. The force exerts a finite impulse on both A and B. The impulses on the two spheres are equal and opposite.

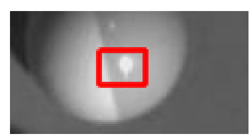
[3 POINTS]

7. In this problem you will compare your predictions from the previous problem to an experimental high speed video from [Dr. Alciatore's extensive website](#).



You will need to
1. Download the movie file billiard_impact.avi from Canvas
2. Download the Matlab script track_collision.m from the EN40 Homework webpage.

Save both files in the same directory. Then run the script to create a graph and a csv file of position –v- time for the body of the vehicle (it will take Matlab quite a long time to read the movie – be patient!). The script will ask you to

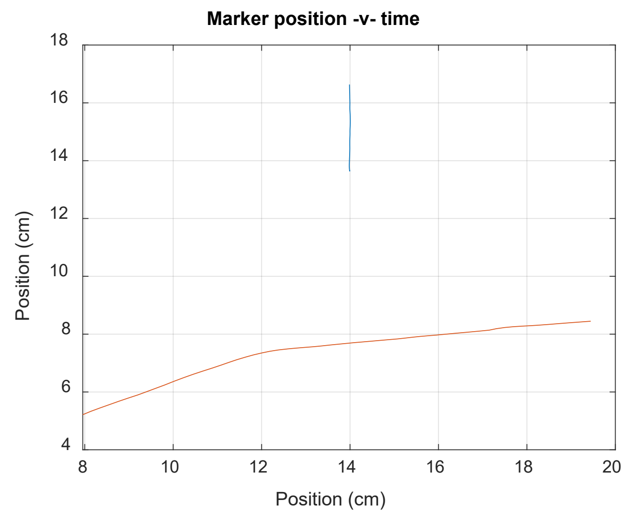


1. Click on two diametrically opposed points on either of the two balls. The script counts the pixels between these points and uses the known diameter (5.7cm) to determine the number of pixels per cm.

2. Select a distinctive region on one of the balls that will look identical in subsequent frames of the movie, by clicking the top left and bottom right corners of a rectangle. (Matlab will track the ball by searching for the same pattern of pixels in every image and finding the best match). For the object ball the reflection of the light works quite well.
3. Click on a point inside the rectangle that will be used as the tracked reference point. After you do this MATLAB will play the movie, and show a green x at the position on the image that has the best correlation with the initial reference image. If the x jumps off the ball, or jitters too much, you may need to start the process again and select a different reference image.
4. Repeat steps (2) and (3) for the second ball (for the cue ball a rectangle surrounding the bottom edge of the ball works but is a bit more sensitive to the precise region you choose; you may need to try more than once to get good data).

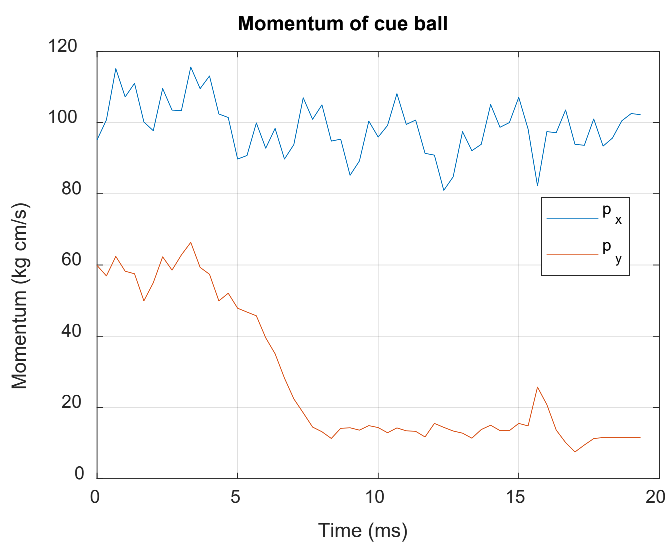
The script will plot a graph showing the trajectories of the two points you selected in the images (the coordinate system will be adjusted to make the trajectory of the object ball parallel to the **j** direction). The data will be saved in a csv file that you can read in your own code for further analysis (recall that you can read a csv file using `data = csvread('filename.csv');`)

A plot with a fairly good choice of reference images is shown below for reference



7.1 Write a MATLAB script to calculate and plot graphs of the following quantities as a function of time (you will need to differentiate the position with respect to time to find the velocity – see HW2 if you need a reminder of how to do this). Assume that the balls both have mass 169 grams. For each case discuss briefly whether the behavior you see is consistent with the predictions you made in the preceding problem. There is no need to submit MATLAB code for this problem

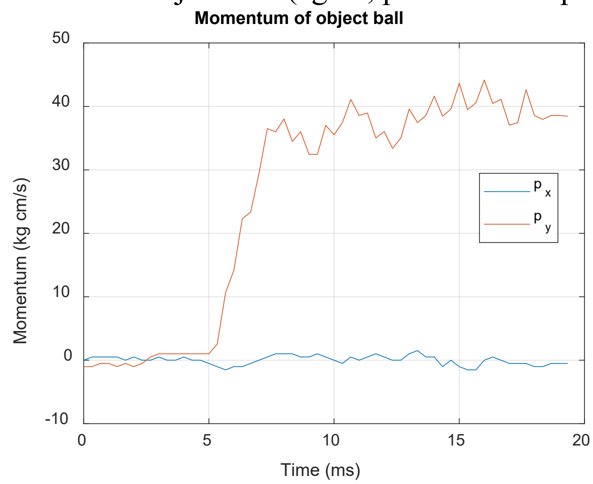
(1) The linear momentum of the cue ball (the one with the black stripe) (plot a separate line, on the same plot, for the *x* and *y* components of momentum).



The momentum of the cue ball is approximately constant in the *x* direction (there is a slow decrease because of friction), and decreases in the *y* direction. This is consistent with the predictions of the preceding problem.

[2 POINTS]

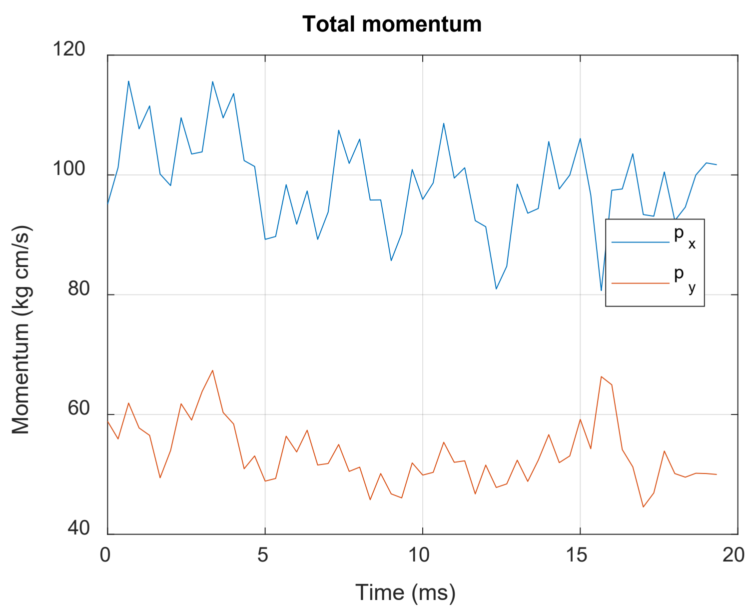
(2) The linear momentum of the object ball (again, plot both components)



The momentum of the object ball is approximately zero in the *x* direction and increases in the *y* direction. This is consistent with the predictions of the preceding problem.

[2 POINTS]

(3) The total linear momentum.



The total linear momentum is approximately constant (there is a slow decrease in both components because of friction).

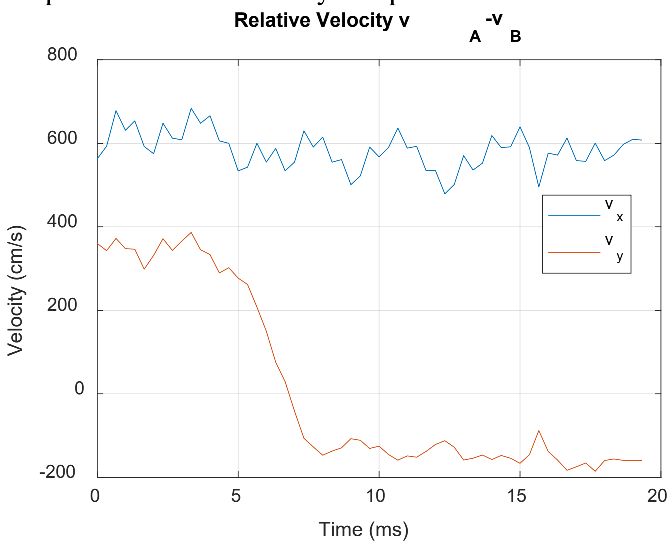
[2 POINTS]

7.2 What is the restitution coefficient for the collision?

Since the collision takes place along the **j** direction we can use the 1-D formulas to calculate the restitution coefficient

$$e = -\frac{v_y^{A1} - v_y^{B1}}{v_y^{A0} - v_y^{B0}}$$

We can use MATLAB to plot the relative velocity components



From the graphs $v_y^{A0} - v_y^{B0} = 350 \qquad v_y^{A1} - v_y^{B1} = -150 \Rightarrow e \approx 150/350 = 0.43$

[2 POINTS]