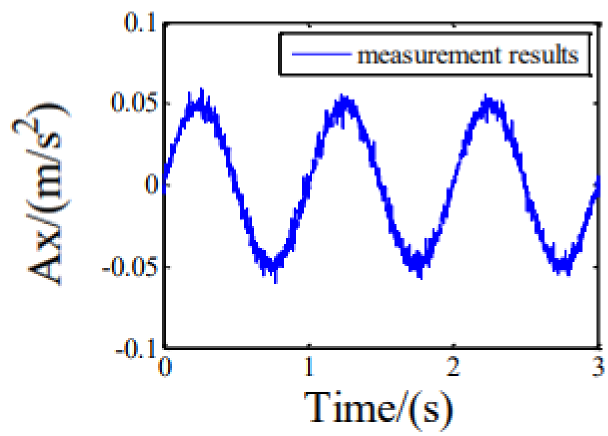
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday March 22, 2019

1. The figure (from this publication) shows a vibration measurement from a test on an accelerometer. Use the figure to estimate



1.1 The amplitude of the acceleration

From the graph, A=0.05 m/s^2

[1 POINT]

1.2 The period of the vibration

There are 3 cycles in 3 sec, so the period is 1s.

[1 POINT]

1.3 The frequency (in Hertz) and angular frequency (in rad/s)

The frequency is $1/T$=1Hz, or 2π rad/s

[1 POINT]

1.4 The amplitude of the velocity

The simple harmonic motion formulas give $V = A/\omega = 0.05/2\pi = 0.008 m/s$

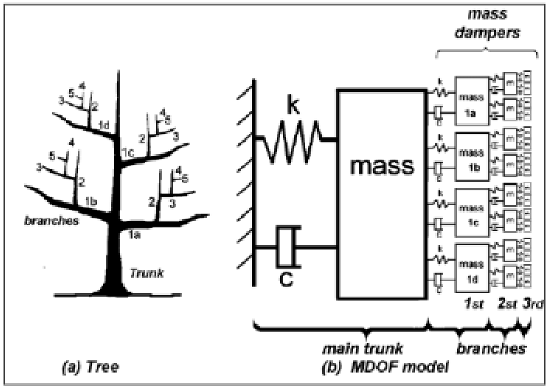
[1 POINT]

1.5 The amplitude of the displacement

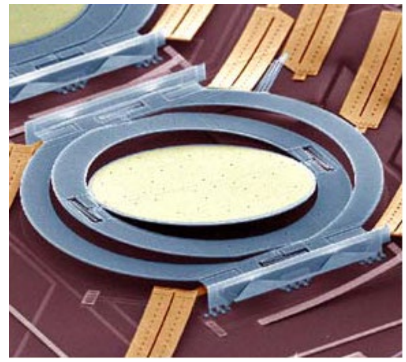
The simple harmonic motion formulas give
$X = V/\omega = 0.008/2\pi = 0.008 m/s = 0.0013m$ $(1.3mm)$

[1 POINT]

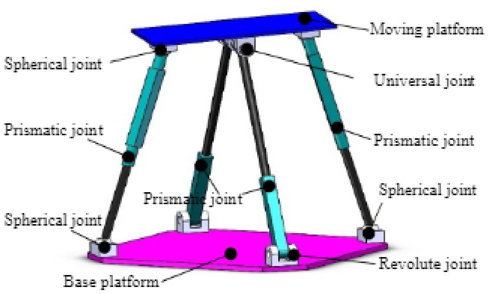
2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures (you may need to consult the publications to understand the system)



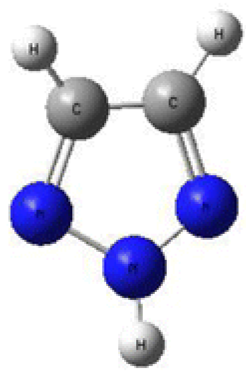
(a) Theoretical model of vibrations in a tree
(use the figure on the right)



(b) Mems Mirror



(c) Model of an articulated platform



(d) Thiazole molecule (the publication shows some of the vibration modes)

(a) Each mass moves only horizontally, so # DOF is equal to # masses, i.e. 29. There are no rigid body modes (the system is attached to the ground) so 29 vibration modes
(b) Each gimbal can rotate about its axis, and no other motion is possible. So 2 DOF. Or 2 rigid bodies (the mirror and the ring) and 2 joints with 3 constraints each, and 2 joints with 2 constraints each. No rigid body modes so 2 vibration modes

(c) Each leg is two rigid bodies; the platform is also a rigid body. So there are 9 rigid bodies.
There are 4 spherical joints (3 constraints each), two revolute joints (5 constraints), two universal joints (4 constraints) and 4 prismatic joints (5 constraints each)
The formula gives #DOF = 6r – c = 54 - 3x4 – 2x5 – 2x4 – 4x5 = 4
No rigid body modes so 4 vibration modes
(or you can just click on the link, which will tell you it is 4 DOF!)

(d) 8 atoms (particles), 3 DOF each = 24 DOF. There are 6 rigid body modes so 18 vibration modes.

[8 POINTS]

3. Solve the following differential equations (please solve them by hand, using the tabulated solutions to differential equation – you can check the answers with matlab if you like)

3.1 $\frac{d^2y}{dt^2} + 9y = 9 \qquad y = 1 \qquad \frac{dy}{dt} = -3 \qquad t = 0$

3.2 $\frac{d^2y}{dt^2} - 9y = -9 \qquad y = 1 \qquad \frac{dy}{dt} = -1 \qquad t = 0$

3.3 $\frac{d^2y}{dt^2} + 100\frac{dy}{dt} + 25y = 50\sin(t) \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

3.1

Rearrange in standard form

$\frac{1}{9}\frac{d^2y}{dt^2} + y = 1$

This is a Case I equation - compare with the standard form to see that $\omega_n = 3 \qquad C = 1$

The solution is

$$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

We are given $x_0 = 1 \qquad v_0 = -3$ so

$$y(t) = 1 - \sin 3t$$

[3 POINTS]

3.2

Rearrange in standard form

$$\frac{1}{9}\frac{d^2y}{dt^2} - y = -1$$

This is a Case II equation - compare with the standard form to see that $\alpha = 3 \qquad C = 1$

. The solution is

$$x(t) = C + \frac{1}{2}\left((x_0 - C) + \frac{v_0}{\alpha}\right)\exp(\alpha t) + \frac{1}{2}\left((x_0 - C) - \frac{v_0}{\alpha}\right)\exp(-\alpha t)$$

We are given $x_0 = 1 \qquad v_0 = -1$ so

$$y(t) = 1 - \frac{1}{6}\exp(\alpha t) + \frac{1}{6}\exp(-\alpha t)$$

[3 POINTS]

3.3 $\dfrac{d^2 y}{dt^2} + 100\dfrac{dy}{dt} + 25y = 50\sin(t) \qquad y = 0 \qquad \dfrac{dy}{dt} = 0 \qquad t = 0$

We can rearrange this as a Case 4 equation

$$\frac{1}{5^2}\frac{d^2 y}{dt^2} + \frac{2\times 10}{5}\frac{dy}{dt} + y = 2\sin t$$

$$\frac{1}{\omega_n^2}\frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = C + K\sin t$$

It appears that $\omega = 1,\ K = 2,\ \omega_n = 5,\ \zeta = 10\ \ C = 0$.

The steady-state solution follows as

$$x_p(t) = X_0 \sin\left(\omega t + \phi\right)$$

$$X_0 = \frac{KF_0}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi = \tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2} \quad (-\pi < \phi < 0)$$

$$\Rightarrow X_0 = \frac{2}{\left\{\left(1-1^2/25\right)^2 + \left(2\times 10\times 1/5\right)^2\right\}^{1/2}} = 0.486 \qquad \phi = \tan^{-1}\frac{-2\times 10\times 1/5}{\left(1-1^2/25\right)} = -1.335$$

The homogeneous solution is

$$x_h(t) = \exp(-\varsigma\omega_n t)\left\{\frac{v_0^h + (\varsigma\omega_n + \omega_d)x_0^h}{2\omega_d}\exp(\omega_d t) - \frac{v_0^h + (\varsigma\omega_n - \omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$$

with

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0\sin\phi = -0.8\sin(-0.6947) = 0.4728$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0\omega\cos\phi = -0.8\times 1\times\cos(-0.6947) = -0.1135$$

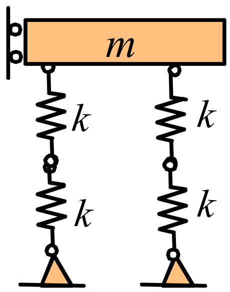
The total solution is therefore

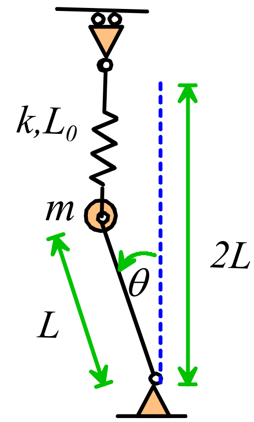
$$y(t) = 0.4862\sin(t - 1.3353) + \exp(-50t)\left\{0.4728\exp(49.75t) - 5.05\times 10^{-5}\times\exp(-49.75t)\right\}$$

We can check that this is correct by substituting it into the differential equation, and by substituting *t=0* into *y* and *dy/dt* and checking that initial conditions are satisfied.

[3 POINTS]

4. Find formulas for the natural frequency of vibration for the systems shown in the figure





For the first system, we can replace the springs with an equivalent single spring. The two springs connected end to end are in series, so

$$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{eff} = k/2$$

Since there are two of these connected in parallel we double this, so the effective stiffness of the entire assembly is just *k*. The formula for natural frequency gives $\omega = \sqrt{k/m}$

[2 POINTS]

We can get an EOM for the second system using the energy method. The mass is in circular motion about the pivot, so its speed (from the circular motion formula) is

$$v = L\left(\frac{d\theta}{dt}\right)$$

and therefore the KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$

The PE includes gravity and the energy of the spring. Geometry shows that the spring length is $2L - L\cos\theta$ so

$$U = mgL\cos\theta + \frac{1}{2}k\left(2L - L\cos\theta - L_0\right)^2$$

$$T + U = const \Rightarrow \frac{d}{dt}(T+U) = 0$$

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) - mgL\sin\theta\frac{d\theta}{dt} + k(2L - L\cos\theta - L_0)L\sin\theta\frac{d\theta}{dt} = 0$$

$$mL^2\left(\frac{d^2\theta}{dt^2}\right) + k(2L - L_0)L\sin\theta - \frac{1}{2}kL^2\sin 2\theta - mgL\sin\theta = 0$$

(here we used the formula $2\sin\theta\cos\theta = \sin 2\theta$ to make finding the small angle approximation easier but its fine to leave this term as just $2\sin\theta\cos\theta$)

To linearize the equation just set $\sin\theta \approx \theta, \quad \sin 2\theta \approx 2\theta$, which gives

$$mL^2\left(\frac{d^2\theta}{dt^2}\right) + k(L-L_0)\theta - mgL\theta = 0$$

$$\Rightarrow \frac{mL}{k(L-L_0)-mg}\left(\frac{d^2\theta}{dt^2}\right) + \theta = 0$$

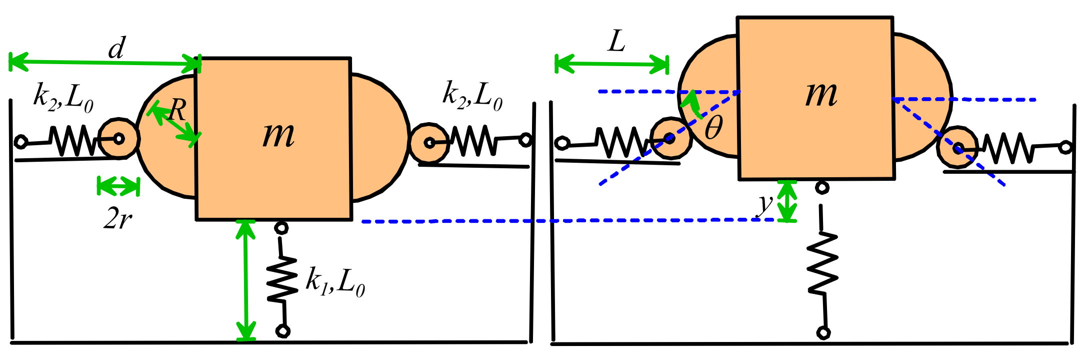
and compare to the standard case I EOM to see that

$$\omega_n = \sqrt{\frac{k(L-L_0)-mg}{mL}}$$

Note that the formula for ω_n gives an imaginary frequency if $k(L-L_0)-mg<0$. This means the system is unstable – it will collapse (or slump to one side) instead of vibrating about $\theta = 0$.

[3 POINTS]

5. This publication shows an alternative design for the 'minus-k' vibration isolation system discussed in class. Like the minus-k design, the platform of mass *m* is supported by a vertical spring, and loaded from the side by two horizontal springs. The horizontal springs are connected to the mass through a system of rollers, instead of being directly attached.



5.1 Using geometry, find a formula for the angle θ in terms of the vertical deflection *y* of the mass. Hence, find a formula for the stretched length *L* of the horizontal springs, in terms of *y*.

Trig gives $\theta = \sin^{-1}(y/(R+r))$

Note that $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-y^2/(R+r)^2}$

Geometry then shows

$$L+(r+R)\cos\theta = d$$
$$L = d-\sqrt{(R+r)^2-y^2}$$

[2 POINTS]

5.2 Write down the total kinetic and potential energy of the system, in terms of y (and its time derivatives), d, k_1, k_2, L_0, r, R, m . **Neglect gravity**.

$$U = k_2(L-L_0)^2+\frac{1}{2}k_1y^2$$
$$= k_2\left(d-\sqrt{(R+r)^2-y^2}-L_0\right)^2+\frac{1}{2}k_1y^2$$
$$T=\frac{1}{2}m\left(\frac{dy}{dt}\right)^2$$

[2 POINTS]

5.3 Hence, find the equation of motion for y

The system is conservative, so

$$T+U = const \Rightarrow \frac{d}{dt}(T+U)=0$$
$$m\left(\frac{dy}{dt}\right)\left(\frac{d^2y}{dt^2}\right)+k_2\left(d-L_0-\sqrt{\left(R+r\right)^2-y^2}\right)\frac{2y}{\sqrt{\left(R+r\right)^2-y^2}}\frac{dy}{dt}+k_1y\frac{dy}{dt}=0$$

We can simplify this to

$$m\frac{d^2y}{dt^2}+2k_2\left(\frac{(d-L_0)y}{\sqrt{\left(R+r\right)^2-y^2}}\right)+(k_1-2k_2)y=0$$

[2 POINTS]

5.4 Linearize the equation for small y, and hence find the natural frequency of the system.

The only nonlinear term in the equation of motion is

$$k_2\left(\frac{(d-L_0)y}{\sqrt{\left(R+r\right)^2-y^2}}\right)\approx k_2\frac{(d-L_0)}{R+r}y$$

Hence

$$m\frac{d^2y}{dt^2}+\left[k_1-2k_2\left(1-\frac{d-L_0}{R+r}\right)\right]y=0$$

Rearrange in standard form

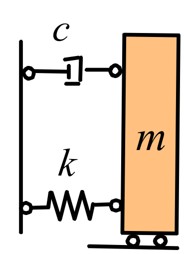
$$\frac{m}{\left[k_1-2k_2\left(1-\frac{d-L_0}{R+r}\right)\right]}\frac{d^2y}{dt^2}+y=0$$

So the natural frequency is

$$\sqrt{\frac{\left[k_1-2k_2\left(1-\frac{d-L_0}{R+r}\right)\right]}{m}}$$

[2 POINTS]

6. The spring-mass-dashpot system shown in the figure has $k=10^3 N/m$, $c=100Ns/m$, $m=100$grams.



6.1 Calculate the natural frequency and damping factor for the system (just use the standard formulas)

The formulas are $\omega_n=\sqrt{\frac{k}{m}}=100rad/s \qquad \zeta=\frac{c}{2\sqrt{km}}=5$

[2 POINTS]

6.2 If the system is released from rest with a 1cm deflection from its equilibrium position, how long will it take to reach 0.1cm deflection? (there is no need to derive and solve the EOM; just use the tabulated standard solutions)

Motion of the system is governed by the Case III EOM. Since the solution is overdamped

$$x(t)=C+\exp(-\varsigma\omega_n t)\left\{\frac{v_0+(\varsigma\omega_n+\omega_d)(x_0-C)}{2\omega_d}\exp(\omega_d t)-\frac{v_0+(\varsigma\omega_n-\omega_d)(x_0-C)}{2\omega_d}\exp(-\omega_d t)\right\}$$

where $\omega_d=\omega_n\sqrt{\varsigma^2-1}$

For the system of interest $C=0, v_0=0, x_0=1cm,\ \omega_n=100\ \ \omega_d=100\sqrt{24}$. Therefore

$$x(t) = \exp(-500t)\left\{\frac{(500+100\sqrt{24})}{200\sqrt{24}}\exp(100\sqrt{24}t) - \frac{(500-100\sqrt{24})}{200\sqrt{24}}\exp(-100\sqrt{24}t)\right\} cm$$
$$= \left\{\frac{(500+100\sqrt{24})}{200\sqrt{24}}\exp([100\sqrt{24}-500]t) - \frac{(500-100\sqrt{24})}{200\sqrt{24}}\exp(-[100\sqrt{24}+500]t)\right\}$$

The second term dies out very quickly compared to the first, so we can find the time to reach 0.1cm deflection as

$$0.1 \approx \frac{(500+100\sqrt{24})}{200\sqrt{24}}\exp([100\sqrt{24}-500]t) \Rightarrow t = \frac{1}{100\sqrt{24}-500}\log\frac{20\sqrt{24}}{(500+100\sqrt{24})} = 0.23s$$

[2 POINTS]

6.3 How much mass needs to be added to the system (with no change in *c* or *k*) to make it critically damped? What is the new natural frequency?

For critical damping $\zeta = 1 \Rightarrow m = \frac{c^2}{4k} = 2.5kg$. This means 2.4kg must be added to the system.

The natural frequency is now

$$\omega_n = \sqrt{\frac{k}{m}} = 20 rad/s$$

6.4 How long will the modified system (in 6.3) take to reach 0.1 cm deflection? (there is no need to derive and solve the EOM; just use the tabulated standard solutions)

We now have to use the critically damped solution

$$x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n(x_0 - C)\right]t\right\}\exp(-\omega_n t)$$

With the numbers given

$$x(t) = \left\{1 + 20t\right\}\exp(-20t)cm$$

To find the time to reach 0.1cm we need to solve $x=0.1$ for t

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vpasolve(0.1==(1+20*t)*exp(-20*t),[0,1])
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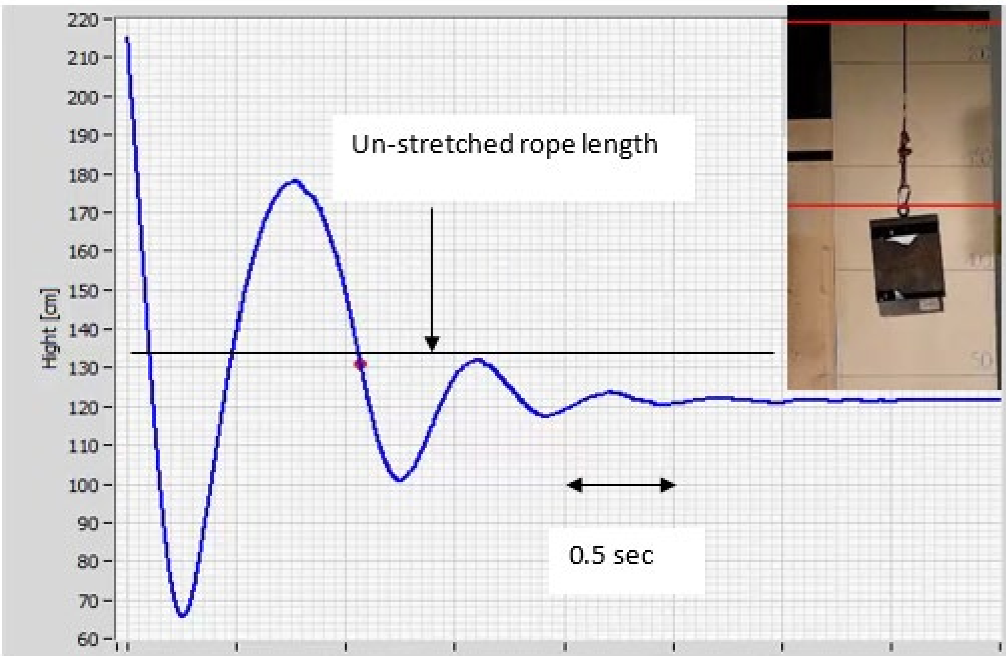
giving $t=0.194s$.

[2 POINTS]

7. This website describes a series of tests to measure the dynamic properties of a climbing rope. The figure shows the results of an experiment with the following parameters:

- Rope length 294.7cm,
- Mass 80 kg
- Fall factor 0.96

Before you do this problem watch a movie of the test here. Note that the rope goes slack during part of the experiment – the standard spring-mass vibration formulas don't apply during this phase of the experiment, so the data has to be interpreted carefully.



7.1 Suppose that the rope and weight can be idealized as a spring-mass-damper system. Use the data in the figure to estimate

(a) The period of oscillation (use only peaks after the rope remains stretched)
(b) The log decrement
(c) The values of ζ and ω_n

(a) Using the time between the 2nd and 3rd negative peaks gives T=0.67s

(b) When calculating the log decrement we have to be careful to measure the peaks from the equilibrium height (122cm). The 2nd negative peak is 11cm below the equilibrium position, the 3rd is 0.45cm below the peak. Since there is one cycle between the two peaks we are using, the formula (with $n=1$) gives

$$\delta = \log \frac{20}{4} = 1.6094$$

(c) Using the formulas from class

$$\zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}} = 0.25 \qquad \omega_n = \frac{\sqrt{\delta^2 + 4\pi^2}}{T} = 9.38 rad/s$$

[3 POINTS]

7.2 Hence, determine values for

(a) the stiffness of the rope

(b) The dashpot coefficient c that best approximates the damping in the system.

We can get the stiffness from the natural frequency and the mass used in the test $k = m\omega_n^2 = 7050 N/m$

We can check this from the static deflection, which gives $k = mg/\Delta = 7100 N/m$. The two agree to within 5% error.

The dashpot coefficient can be calculated from the formula $c = 2\zeta\sqrt{km} = 2\zeta m\omega_n = 380 Ns/m$

[2 POINTS]

7.2 If the length of the rope is doubled, what will be its new stiffness and dashpot coefficient?

Doubling the rope length is like connecting two ropes in series. Using the formula

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2} \qquad \frac{1}{c_{eff}} = \frac{1}{c_1} + \frac{1}{c_2}$$

The stiffness and dashpot coefficient will both be halved

[2 POINTS]

7.3 What length of rope would give a critically damped response in the drop test (with an 80kg mass)?

Both the stiffness and dashpot are inversely proportional to the rope length (from the previous problem). Using the formula for the damping coefficient

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{c_0 l_0 / l}{2\sqrt{mk_0 l_0 / l}} = 1$$

$$\Rightarrow \frac{c_0}{2\sqrt{mk_0}}\sqrt{l_0} = \sqrt{l}$$

$$\Rightarrow l = \zeta_0^2 l_0 = 294.7 \times 0.25^2 = 18cm$$

[2 POINTS]