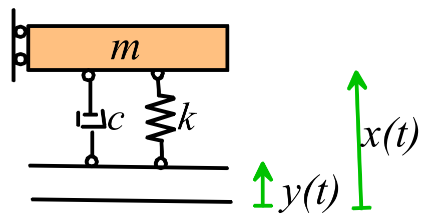
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 5, 2018

1. The vibration isolation system shown in the figure has

- m=20kg,
- $k = 19.8 kN/m$
- $c = 1.259 kNs/m$



The base vibrates harmonically with an amplitude of 1mm and angular frequency ω .

1.1 What is the value of ω that will cause the platform (the mass m) to vibrate with the greatest amplitude? What is the corresponding vibration amplitude?

We find that for this system the natural frequency is $\omega_n = \sqrt{k/m} = 31.4 rad/s \ \ (5Hz)$ and $\zeta = 1/(2\sqrt{km}) = 1$. This is too big for the approximate formulas $M \approx 1/(2\zeta)$ to be valid, so we need to rigorously maximize the vibration amplitude. Note that the vibration amplitude is $X_0 = Y_0 M(\omega/\omega_n, \zeta)$, where the magnification for the critically damped base excited system is

$$M = \frac{\sqrt{1+4r^2}}{\sqrt{(1-r^2)^2 + 4r^2}} \qquad r = \omega/\omega_n$$

The maximum amplitude occurs when M is a maximum. We can maximize M in a live script

```
syms r M
M = sqrt(1+4*r^2)/sqrt( (1-r^2)^2 + 4*r^2)
rmax = solve(diff(f,r)==0,r)
subs(M,r,rmax(3))
```

M =

$$\frac{\sqrt{4\,r^2+1}}{\sqrt{(r^2-1)^2+4\,r^2}}$$

rmax =

$$\begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

ans =

$$\frac{2\,\sqrt{3}}{3}$$

This shows that the biggest vibrations will occur when $\omega = \omega_n/\sqrt{2} = 22.2 rad/s$

The vibration amplitude is $X_0 = Y_0 M_{\max} = 2Y_0/\sqrt{3} = 1.15 mm$

[2 POINTS]

1.2 What is the lowest value of ω for which the vibration isolator is effective (i.e. the amplitude of the platform is less than the amplitude of the base)?

For vibration isolation we need

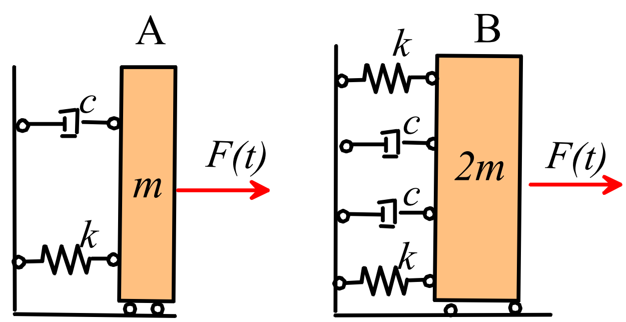
$$X_0 / Y_0 < 1 \Rightarrow M < 1$$

$$M = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

$$\Rightarrow (1-\omega^2/\omega_n^2)^2 > 1 \Rightarrow \omega/\omega_n > \sqrt{2} \Rightarrow \omega > 44.5 rad/s$$

[2 POINTS]

2. Systems A and B in the figure shown are subjected to the same harmonic force *F(t)*. The steady state amplitude of vibration of system A is measured to be 1mm. What is the amplitude of vibration of system B?

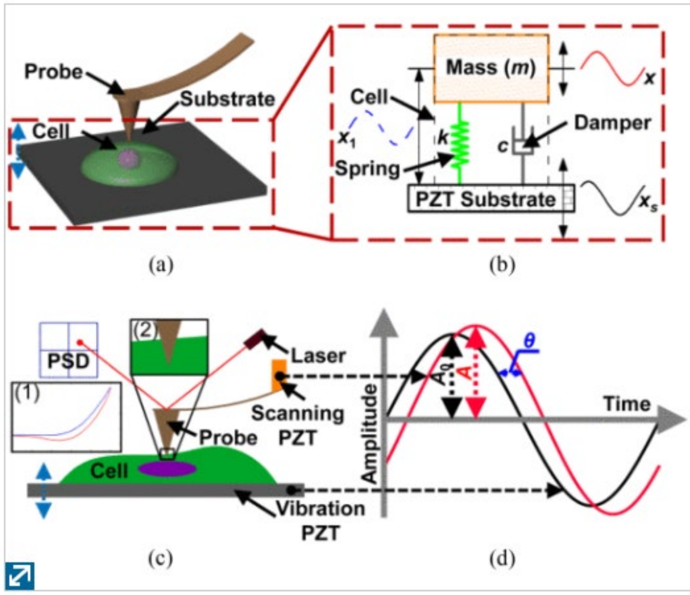


Note that the values of ζ and ω_n are equal for both systems. The vibration amplitude is given by the formula

$X_0 = KF_0 M(\omega/\omega_n, \zeta) \qquad K = 1/k_{eff}$

Where k_{eff} is the effective stiffness. The two springs in B are in parallel, so the effective stiffness of B is twice that of A. This implies that the amplitude of B will be half that of A, i.e. 0.5mm.

[2 POINTS]



3. This IEEE transactions on biomedical engineering paper describes a vibration experiment designed to measure the mass and visco-elastic properties of a cell. The cell is placed on a piezoelectric vibrating platform, and an atomic force microscope is used to measure the displacement of the top of the cell. The authors idealize the cell as a spring-mass-damper system, and extract values for the stiffness, dashpot coefficient, and mass for live and dead cells from their experimental data.

Using a static test, they measure a stiffness of 0.1 N/m for a live cell, and 0.2N/m for a dead one.

Their data for the amplitude ratio (the vibration amplitude of the AFM tip divided by the vibration amplitude of the substrate) and phase lag (related to the time lag θ between the zero crossing of the substrate and the zero crossing of the AFM tip and the period T as $\phi = -2\pi\theta / T$) are shown in the tables below.

Live Cells		
Frequency (kHz)	Amplitude Ratio	Phase Lead (rad)
0.05	1.002	-0.001
0.1	1.003	-0.005
0.15	1.008	-0.01
0.2	1.011	-0.025
0.25	1.018	-0.04
0.3	1.0225	-0.06
0.35	1.025	-0.08
0.4	1.035	-0.09
0.45	1.039	-0.1
0.5	1.04	-0.12

Dead Cells		
Frequency (kHz)	Amplitude Ratio	Phase lead (rad)
0.05	1.0005	-0.0005
0.1	1.0025	-0.0025
0.15	1.004	-0.005
0.2	1.006	-0.015
0.25	1.01	-0.025
0.3	1.0125	-0.03
0.35	1.019	-0.045
0.4	1.021	-0.05
0.45	1.027	-0.051
0.5	1.025	-0.08

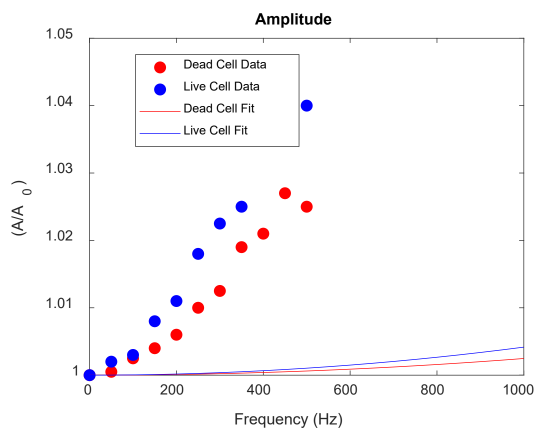
3.1 The paper reports the following values for the cell mass *m* and dashpot coefficient *c* from their data.

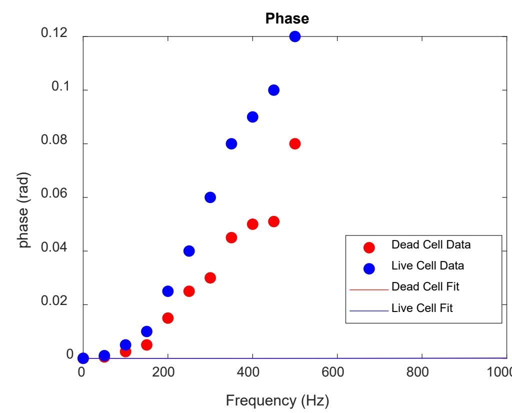
- Live cell: $m = 10.5\text{x}10^{-12}$kg c=0.393x10^{-6}Ns/m
- Dead cell: $m = 12.5\text{x}10^{-12}$kg c=0.7x10^{-6}Ns/m

The paper does not compare the predictions of the model (equations 8 and 9 in the paper) with the experimental data, however, so we will try this.

Calculate the values of damping coefficient ζ, ω_n and hence use the standard solutions for a base-excited spring-mass system to plot (on the same graph) the predicted amplitude ratio (this is *M* in the engn40 notation) and the experimental data. Do a similar second plot for the phase. Don't forget to convert frequency from Hz to rad/s. Don't worry if the theory and experiment don't agree – I couldn't get it to work either.

There is no need to submit MATLAB code for this problem, the graphs are sufficient.





[4 POINTS]

3.2 We can attempt to get better estimates for *m* and *c*. The estimates from the paper for *k* and *m* suggest that the excitation frequency is much less than the natural frequency. If this is the case, we can derive simplified formulas for the amplitude ratio and phase that make it easier to fit numbers to the data. Show that the formulas for M and ϕ for a base-excited spring-mass system can be expressed in terms of *c, m*, and *k* as

$$M = \frac{\sqrt{1+(c\omega/k)^2}}{\left[(1-\omega^2 m/k)^2+(c\omega/k)^2\right]^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{-(c\omega/k)(\omega^2 m/k)}{1-\omega^2 m/k+(c\omega/k)^2}\right)$$

Hence, show that for $\omega << \sqrt{k/m}$ the amplitude ratio can be approximated by (use MATLAB to take the Taylor series, or do it by hand)

$$\frac{A}{A_0} \approx 1+\frac{m}{k}\omega^2$$

while the phase can be approximated by

$$\phi \approx -\frac{cm}{k^2}\omega^3$$

The amplitude ratio is just the magnification

$$M = \frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\left[(1-\omega^2/\omega_n^2)^2+(2\zeta\omega/\omega_n)^2\right]^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{-2\zeta\omega^3/\omega_n^3}{1-(1-4\zeta^2)\omega^2/\omega_n^2}\right)$$

We know that $\omega_n = \sqrt{k/m} \qquad \zeta = \dfrac{c}{2\sqrt{km}}$ so $2\zeta\omega/\omega_n = 2\dfrac{c}{2\sqrt{km}}\dfrac{\omega}{\sqrt{k/m}} = c/k$

$$M = \frac{\sqrt{1+(c\omega/k)^2}}{\left[(1-\omega^2 m/k)^2+(c\omega/k)^2\right]^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{-(c\omega/k)(\omega^2 m/k)}{1-\omega^2 m/k+(c\omega/k)^2}\right)$$

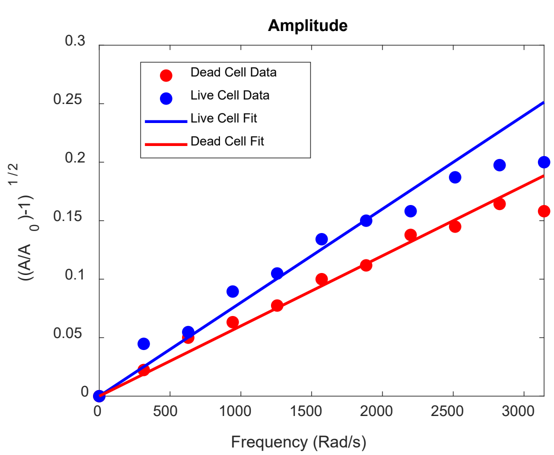
Here's a MATLAB 'Live Script' for the Taylor series but you can write down the answer by inspection….

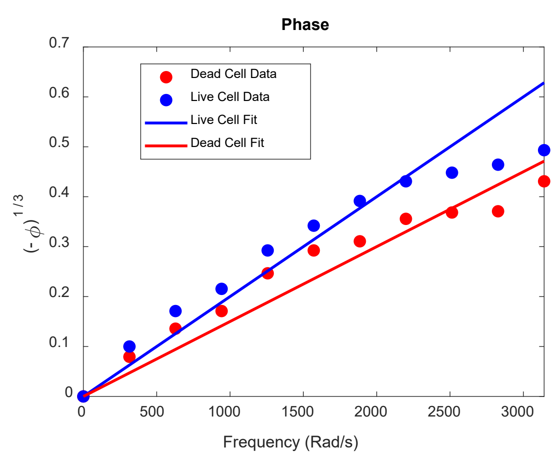
```
clear all
syms c m w k
M = sqrt(1+(c*w/k)^2)/sqrt( (1-w^2*m/k)^2 + (c*w/k)^2)
series(M,w,'Order',4)
p = atan(-(c*w/k)*(w^2*m/k)/(1-w^2*m/k + (c*w/k)^2))
series(p,w,'Order',2)
```

[4 POINTS]

3.3 Use 3.2 and the experimental data to estimate values for *m* and *c* for live and dead cell (Find a way to plot the data to get a straight line relationship between the *y* axis and the frequency, so you can estimate *m* from the slope of the amplitude plot, and *c* from the slope of the phase plot. you will find the data does not give very good straight lines, so you will only be able to get approximate values)

In theory, a graph of $\sqrt{(A/A_0)-1} - v - \omega$ should yield a straight line with slope $\sqrt{m/k}$, while a graph of $(-\phi)^{1/3} - v - \omega$ should yield a straight line with slope $(cm/k^2)^{1/3}$. The relevant plots, with approx. straight line fits, are shown below. The data for the amplitude ratio fits the theory much better than the data for phase.





The slopes used for these plots are:

Live cell amplitude ratio: $6x10^{-5}s^{-1}$
Dead cell amplitude ratio $8x10^{-5}s^{-1}$

Live cell phase: $1.5x10^{-4}s^{-1}$
Dead cell phase: $2x10^{-4}s^{-1}$

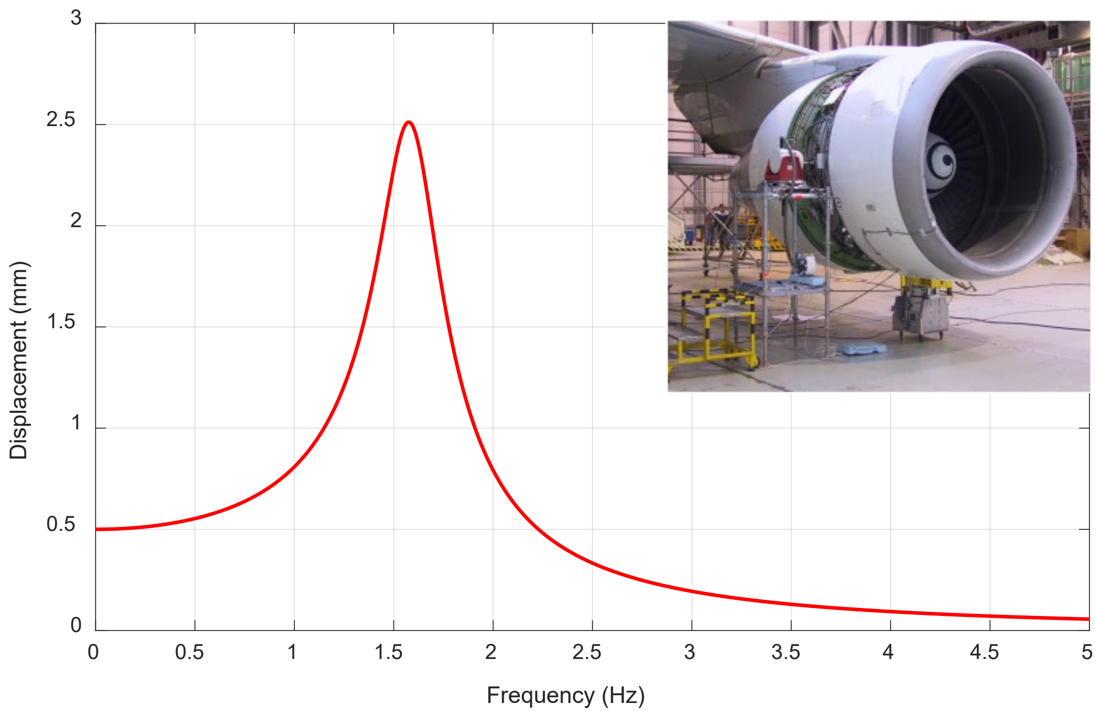
And using the formulas

$$\sqrt{\frac{A}{A_0}-1} \approx \sqrt{\frac{m}{k}}\omega \qquad (-\phi)^{1/3} \approx \left(\frac{cm}{k^2}\right)^{1/3} \omega$$

we can calculate the mass and effective dashpot coefficient for live and dead cells

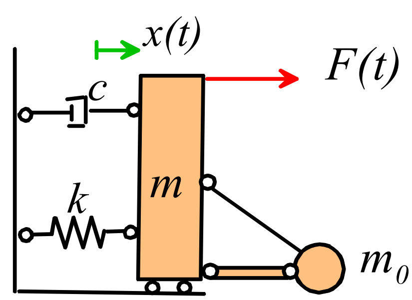
Live cell: $m = 3.6x10^{-10}$kg c=$9.4x10^{-5}$Ns/m
Dead cell: $m = 1.3x10^{-9}$kg c=$2.5x10^{-4}$Ns/m

[6 POINTS]



4 As part of the airworthiness certification process, the rotating parts of a jet engine are prevented from turning, and the engine is subjected to an external horizontal harmonic force $F(t) = F_0 \sin \omega t$ with amplitude $F_0 =$ 250N. The amplitude X_0 of the steady-state horizontal vibration $x(t) = X_0 \sin(\omega t + \phi)$ of the engine is measured.

The measured displacement amplitude X_0 is shown in the figure as a function of frequency (in cycles/sec).



Rotor (prevented from rotating)

4.1 Assuming that the engine and its mounting are idealized as a spring-mass-damper system (with light damping), use the graph provided to estimate values for the following quantities

(a) The natural frequency of vibration of the engine (give both the frequency in cycles per second and the angular frequency)

From the graph, $f_n = 1.6Hz$; $\omega_n = 2\pi f_n = 10 rad/s$

[2 POINTS]

(b) The damping factor ζ . (Use the peak. Note that the graph shows the displacement amplitude, not magnification *M*)

We can get ζ from the peak. The magnification is 2.5/0.5=5, and we know $M \approx 1/2\zeta \Rightarrow \zeta = 0.1$

[2 POINTS]

(c) The spring stiffness (use the displacement at very low frequency)

We can get the stiffness from the zero frequency deflection (static).
$k = F / X_0 \Rightarrow k = 250 / 0.5 \times 10^{-3} = 500 kN / m$

[1 POINT]

(d) The total mass $m + m_0$

The mass follows from the natural frequency $\sqrt{k / (m + m_0)} = \omega_n \Rightarrow m + m_0 = k / \omega_n^2 = 5000 kg$

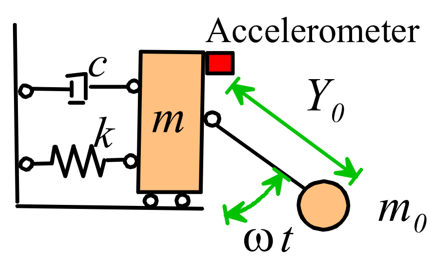
[1 POINT]

(e) The dashpot coefficient c

Using the formula: $\zeta = c / 2\sqrt{k(m + m_0)} \Rightarrow c = 10 kNs / m$

[1 POINT]

4.2 During operation, the engine spins at 9550 rpm. An accelerometer mounted on the outside of the engine measures a harmonic acceleration with amplitude 10 m/s^2. What is the amplitude of the displacement?



The vibration is harmonic, therefore
$X_0 = A_0 / \omega^2 = 10 / (9550 \times 2\pi / 60)^2 = 0.01 mm$

[1 POINT]

4.3 What is the engine speed (in rpm) at which the steady-state displacement amplitude will be a maximum?

The maximum vibration amplitude will occur when the engine spins at the resonant frequency; this corresponds to 1.6x60 =96rpm

[1 POINT]

4.4 What is the steady-state displacement amplitude when the engine runs at the speed in 4.3?

During operation the engine spins much faster than the resonant frequency. We know the magnification is approximately 1 in this regime, and we know the amplitude is 0.01mm from 4.2. At resonance, the magnification is 5. We expect a vibration amplitude of 0.05mm.

Other approaches are possible too – you could use 4.2 and the formulas $X_0 = K Y_0 M$, $K = \dfrac{m_0}{m + m_0}$, use the numbers in 4.2 to calculate the product $Y_0 m_0 = X_0 (m + m_0) / M(\omega = 9550 \times 2\pi / 60)$, and then find the new amplitude at resonance using $X_0 = \dfrac{m_0 Y_0}{m + m_0} M(\omega = 10)$. At resonance the $M \approx 1/(2\zeta)$ approximation works too.

[1 POINT]