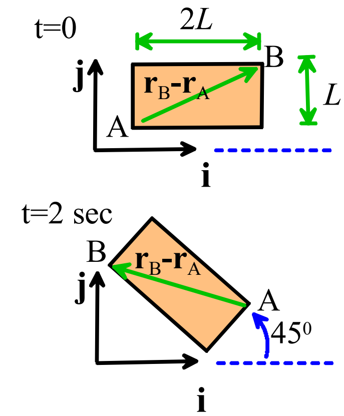
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 7: Rigid Body Kinematics, Inertial properties of rigid bodies
Due Friday April 19, 2019

1. The rigid body shown in the figure is at rest at time t=0, and rotates counterclockwise with constant angular acceleration. After 2 sec it has the orientation shown. Find



1.1 The angular acceleration vector

The body has rotated through 135 degrees $=3\pi/4$ radians in 2 secs. We can use the constant acceleration formula to find the angular acceleration

$$\theta=\frac{1}{2}\alpha t^2 \Rightarrow \alpha = 2\theta/t^2$$

$$\boldsymbol{\alpha}=\frac{3\pi}{8}\mathbf{k} \quad rad/s^2$$

[2 POINTS]

1.2 The angular velocity vector as a function of time

Since the angular acceleration is constant, the angular velocity is $\boldsymbol{\omega}=\boldsymbol{\alpha}t=\dfrac{3\pi}{8}t\mathbf{k}$

[1 POINT]

1.3 The spin tensor **W** (as a 2x2 matrix, also a function of time)

Using the formula

$$\mathbf{W}=\begin{bmatrix}0 & -\omega_z \\ \omega_z & 0\end{bmatrix}=\frac{3\pi t}{8}\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$$

[1 POINT]

1.4 The rotation tensor (a 2x2 matrix for a 2D problem) **R** that rotates the rectangle from its initial to its position at 2 sec.

The 2D rotation matrix is

$$\mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}-1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

[1 POINT]

1.5 Hence, find a formula for the vector $\mathbf{r}_B - \mathbf{r}_A$ in (**i,j**) components at time *t=2*sec

$$\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A) = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2L \\ L \end{bmatrix} = \begin{bmatrix} -3L/\sqrt{2} \\ L/\sqrt{2} \end{bmatrix}$$

[1 POINT]

2. A rigid body is subjected to a sequence of two rotations.

$$\mathbf{R}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \mathbf{R}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.1 Interpret each rotation matrix in a sentence (eg a 30 degree rotation about the **i** axis)

We can either use the formula

$$1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz}$$

$$\mathbf{n} = \frac{1}{2\sin\theta}\left[\left(R_{zy} - R_{yz}\right)\mathbf{i} + \left(R_{xz} - R_{zx}\right)\mathbf{j} + \left(R_{yx} - R_{xy}\right)\mathbf{k}\right]$$

Which shows that $\cos\theta = 0$ for both rotations (i.e. they are both 90 degree rotations); the axes then come out to be

$\mathbf{n}^{(1)} = -\mathbf{i} \qquad \mathbf{n}^{(2)} = -\mathbf{k}$

So the matrices either represent two 90 degree rotations about –**i** and **–k,** or -90 rotations (i.e. clockwise) about **i** and **k.**

[2 POINTS]

2.2 Find the rotation matrix that produces the sequence of rotations $\mathbf{R} = \mathbf{R}^{(2)}\mathbf{R}^{(1)}$

The matrix multiplication is easy -

$$\mathbf{R}^{(2)}\mathbf{R}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

[1 POINT]

2.3 Find the axis **n** and rotation angle θ that will complete the rotation $\mathbf{R}$ directly

The formula gives $1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz} = 0 \Rightarrow \cos\theta = -1/2 \Rightarrow \theta = 120^o$

$$\mathbf{n} = [(R_{zy} - R_{yz})\mathbf{i} + (R_{xz} - R_{zx})\mathbf{j} + (R_{yx} - R_{xy})\mathbf{k}]/2\sin\theta = (-\mathbf{i} + \mathbf{j} - \mathbf{k})/\sqrt{3}$$

You can of course flip the signs of both the axis and the angle as well if you prefer.

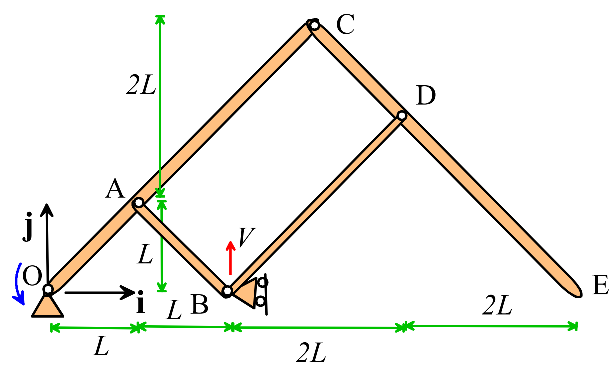
[2 POINTS]

Optional: You can check your answers by downloading a matlab script from the homework page of the course website that will animate a rigid rotation through an angle θ about an axis parallel to a unit vector **n**. You can use the code by navigating to the directory storing the file in the Matlab command window, and then typing

Animate_rotation(*angle,[* n_x, n_y, n_z *])*

Where *angle* is your solution for the rotation angle θ (in radians), and n_x, n_y, n_z are the components of your solution for the unit vector **n**.

3. The figure shows a four-bar chain mechanism. Joint B moves vertically with constant speed *V*. Calculate the angular velocities and angular accelerations of members OC and CE, and find the velocity and acceleration of E.



We can use the standard procedure – start at O and work around.

$$\mathbf{v}_A - \mathbf{v}_O = \omega_{OC}\mathbf{k}\times(L\mathbf{i}+L\mathbf{j}) = \omega_{OC}L(\mathbf{j}-\mathbf{i})$$
$$\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) = \omega_{AB}L(\mathbf{j}+\mathbf{i})$$
$$\Rightarrow \mathbf{v}_B - \mathbf{0} = (\omega_{OC}+\omega_{AB})L\mathbf{j} + (\omega_{AB}-\omega_{OC})L\mathbf{i} = V\mathbf{j}$$

Hence using the **i,j** components give two equations

$$\omega_{OC}+\omega_{AB} = V/L \qquad \omega_{AB}-\omega_{OC} = 0 \ \Rightarrow \omega_{AB} = \omega_{OC} = V/2L$$

[2 POINTS]

We can now do C, D, B

$$\mathbf{v}_C - \mathbf{v}_O = \omega_{OC}\mathbf{k}\times 3(L\mathbf{i}+L\mathbf{j}) = \omega_{OC}3L(\mathbf{j}-\mathbf{i}) = (3V/2)(\mathbf{j}-\mathbf{i})$$
$$\mathbf{v}_D - \mathbf{v}_C = \omega_{CE}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) = \omega_{CE}L(\mathbf{j}+\mathbf{i})$$
$$\mathbf{v}_B - \mathbf{v}_D = \omega_{BD}\mathbf{k}\times(-2L\mathbf{i}-2L\mathbf{j}) = 2L\omega_{BD}(\mathbf{j}-\mathbf{i})$$
$$\Rightarrow \mathbf{v}_B - \mathbf{0} = \left\{(3V/2L)+\omega_{CE}+2\omega_{BD})\right\}L\mathbf{j} + \left\{(-3V/2L)+\omega_{CE}-2\omega_{BD})\right\}L\mathbf{i} = V\mathbf{j}$$

Hence using the **i,j** components give two equations

$$\omega_{CE}+2\omega_{BD} = -V/2L \qquad \omega_{CE}-2\omega_{BD} = 3V/2L \ \Rightarrow \omega_{CE} = V/2L$$

[2 POINTS]

And finally

$$\mathbf{v}_E - \mathbf{v}_C = \omega_{CE}\mathbf{k}\times(3L\mathbf{i}-3L\mathbf{j}) = 3L\omega_{CE}(\mathbf{j}+\mathbf{i})$$
$$\mathbf{v}_E = 3L\omega_{CE}(\mathbf{j}+\mathbf{i}) + (3V/2)(\mathbf{j}-\mathbf{i}) = 3V\mathbf{j}$$

(this is obvious from the geometry)

[2 POINTS]

We can repeat the process for accelerations

$$\mathbf{a}_A - \mathbf{a}_O = \alpha_{OC}\mathbf{k}\times(L\mathbf{i}+L\mathbf{j}) - \omega_{OC}^2(L\mathbf{i}+L\mathbf{j}) = \alpha_{OC}L(\mathbf{j}-\mathbf{i}) - \omega_{OC}^2(L\mathbf{i}+L\mathbf{j})$$

$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{AB}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) - \omega_{AB}^2(L\mathbf{i}-L\mathbf{j}) = \alpha_{AB}L(\mathbf{j}+\mathbf{i}) - \omega_{AB}^2(L\mathbf{i}-L\mathbf{j})$$

$$\Rightarrow \mathbf{a}_B - \mathbf{0} = (\alpha_{OC}+\alpha_{AB}-\omega_{OC}^2+\omega_{AB}^2)L\mathbf{j} + (\alpha_{AB}-\alpha_{OC}-\omega_{OC}^2-\omega_{AB}^2)L\mathbf{i} = \mathbf{0}$$

Hence using the **i,j** components give two equations

$$(\alpha_{OC}+\alpha_{AB})L - \omega_{OC}^2L^2 + \omega_{AB}^2L^2 = 0 \qquad (\alpha_{AB}-\alpha_{OC})L - \omega_{OC}^2L - \omega_{AB}^2L = 0$$

$$\Rightarrow \alpha_{AB} = \omega_{OC}^2 = V^2/4L^2 \qquad \alpha_{OC} = -V^2/4L^2$$

[2 POINTS]

We can now do C, D, B

$$\mathbf{a}_C - \mathbf{a}_O = \alpha_{OC}\mathbf{k}\times 3(L\mathbf{i}+L\mathbf{j}) - \omega_{OC}^2 3(L\mathbf{i}+L\mathbf{j}) = \alpha_{OC}3L(\mathbf{j}-\mathbf{i}) - \omega_{OC}^2 3(L\mathbf{i}+L\mathbf{j})$$

$$\mathbf{a}_D - \mathbf{a}_C = \alpha_{CE}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) - \omega_{CE}^2(L\mathbf{i}-L\mathbf{j}) = \alpha_{CE}L(\mathbf{j}+\mathbf{i}) - \omega_{CE}^2(L\mathbf{i}-L\mathbf{j})$$

$$\mathbf{a}_B - \mathbf{a}_D = \alpha_{BD}\mathbf{k}\times(-2L\mathbf{i}-2L\mathbf{j}) - \omega_{BD}^2(-2L\mathbf{i}-2L\mathbf{j}) = 2L\alpha_{BD}(\mathbf{j}-\mathbf{i}) - \omega_{BD}^2(-2L\mathbf{i}-2L\mathbf{j})$$

$$\Rightarrow \mathbf{a}_B - \mathbf{0} = \left\{+\alpha_{CE}+2\alpha_{BD}-\omega_{OC}^2 3+\omega_{CE}^2+2\omega_{BD}^2)\right\}L\mathbf{j} + \left\{+\alpha_{CE}-2\alpha_{BD}-3\omega_{OC}^2-\omega_{CE}^2+2\omega_{BD}^2)\right\}L\mathbf{i} = \mathbf{0}$$

Hence using the **i,j** components give two equations

$$\left\{+\alpha_{CE}+2\alpha_{BD}-\omega_{OC}^2 3+\omega_{CE}^2+2\omega_{BD}^2)\right\} = 0 \qquad \left\{+\alpha_{CE}-2\alpha_{BD}-3\omega_{OC}^2-\omega_{CE}^2+2\omega_{BD}^2)\right\} = 0$$

$$\Rightarrow 2\alpha_{CE} = 6\omega_{OC}^2 - 4\omega_{BD}^2$$

$$\Rightarrow \alpha_{CE} = 3\omega_{OC}^2 - 2\omega_{BD}^2 = 3(V/2L)^2 - 2(-V/2L)^2 = (V/2L)^2$$

[2 POINTS]

And finally

$$\mathbf{a}_E - \mathbf{a}_C = \alpha_{CE}\mathbf{k}\times(3L\mathbf{i}-3L\mathbf{j}) - \omega_{CE}^2(3L\mathbf{i}-3L\mathbf{j}) = 3L\alpha_{CE}(\mathbf{j}+\mathbf{i}) - \omega_{CE}^2(3L\mathbf{i}-3L\mathbf{j})$$

$$\mathbf{a}_E = \alpha_{OC}3L(\mathbf{j}-\mathbf{i}) - \omega_{OC}^2 3(L\mathbf{i}+L\mathbf{j}) + 3L\alpha_{CE}(\mathbf{j}+\mathbf{i}) - \omega_{CE}^2(3L\mathbf{i}-3L\mathbf{j})$$

$$= 3L\left\{(\alpha_{CE}-\alpha_{OC}-\omega_{OC}^2-\omega_{CE}^2)\mathbf{i} + (\alpha_{OC}+\alpha_{CE}-\omega_{OC}^2+\omega_{CE}^2)\mathbf{j}\right\}$$

$$= 3L\left\{((V/2L)^2 + V^2/4L - (V/2L)^2 - (V/2L)^2)\mathbf{i} + (-V^2/4L + (V/2L)^2 - (V/2L)^2 + (V/2L)^2)\mathbf{j}\right\}$$

$$= \mathbf{0}$$

[2 POINTS]

4. Pratt and Whitney started production of the first geared turbofan engine in 2016. The engine allows the fan and compressor stages (which normally run on the same shaft) to spin at different speeds, which improves the efficiency of the engine (but the additional complexity of the engine has led to some reliability issues). Rolls Royce are currently rolling out their 'Ultrafan' engine, which will keep patent lawyers in business for years. The figure shows the P&W gearbox. It operates with the planet carrier held fixed, the fan

connected to the ring gear, and the compressor to the sun. Use the figure to calculate the ratio of the angular speeds of the compressor and fan.

We can count the teeth: the sun has 34 teeth, the planet has 31 teeth. We know that $N_R = N_S + 2N_P \Rightarrow N_R = 96$. With the planet carrier fixed,

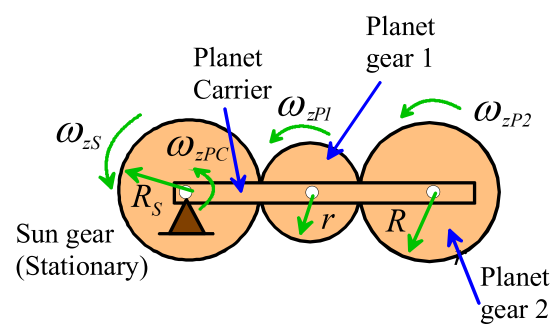
$\omega_S / \omega_R = -N_R / N_S = 96/34 = 2.82$

(i.e. the compressor spins 2.82 times faster than the fan).

(Graders – the inverse is fine; the exact number of teeth is not important (it's a pain to count them))

[3 POINTS]

5. '[Ferguson's paradox]' is a special arrangement of epicyclic gears used by its inventor to make an [orrery]. Ferguson's version made three non-standard co-axial sun gears mesh with the same planet gear, but the basic principle can be illustrated by the simple design shown in the figure (see also [this animation]). The sun gear is stationary, and the planet carrier rotates about O. Find a formula for the angular speed ω_{zP2} of planet gear 2, in terms of the angular speed of the planet carrier ω_{zPC} and the radii R and R_S .



We can use the same trick to analyze this system as we use to analyze an epicyclic gear. We subtract the angular speed of the planet carrier from all the gears (and the planet carrier) – so we are see the motion of the system in a rotating reference frame with the carrier stationary, and the three gears moving with angular speeds

$$\omega_{zS} - \omega_{PC}, \; \omega_{zP1} - \omega_{PC}, \; \omega_{zP2} - \omega_{PC}$$

In the rotating reference frame we see a standard gear system – it follows that

$$\frac{\omega_{zP1} - \omega_{PC}}{\omega_{zS} - \omega_{PC}} = -\frac{R_s}{r} \qquad \frac{\omega_{zP2} - \omega_{PC}}{\omega_{z1} - \omega_{PC}} = -\frac{r}{R} \Rightarrow \frac{\omega_{zP2} - \omega_{PC}}{\omega_{zs} - \omega_{PC}} = \left(-\frac{R_s}{r}\right)\left(-\frac{r}{R}\right) = \frac{R_s}{R}$$

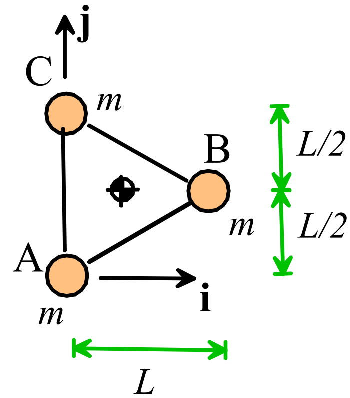
In the Ferguson mechanism $\omega_{zS} = 0$ and so

$$\omega_{zP2} = \omega_{PC} \frac{R - R_S}{R}$$

This shows that the second planet gear can be made to rotate in either direction – if it is larger than the sun, it will rotate in the same direction as the planet carrier; if it is smaller, it will rotate in the opposite direction. It is this reversal of motion that is the 'paradox' (and it's really fairly obvious once you see how the system works….)

[3 POINTS]

6. The figure shows three particles with equal mass m connected by rigid massless links.



6.1 Calculate the position of the center of mass of the assembly

$$\mathbf{r}_G = \frac{1}{M}\sum_i m_i \mathbf{r}_i = \frac{1}{3m}\left(mL\mathbf{j} + m\frac{L}{2}(2\mathbf{i}+\mathbf{j}) \right) = (\frac{L}{3}\mathbf{i} + \frac{L}{2}\mathbf{j})$$

[1 POINT]

6.2 Calculate the 2D mass moment of inertia of the system about the center of mass

$$I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2 \right)$$

where $\mathbf{d}_i = d_{xi}\mathbf{i} + d_{yi}\mathbf{j} = \mathbf{r}_i - \mathbf{r}_G$ is the position vector of the ith particle with respect to the center of mass.

Use the formula $I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2 \right) = m\left[\left(\frac{2L}{3} \right)^2 \right] + 2m\left[\left(\frac{L}{2} \right)^2 + \left(\frac{L}{3} \right)^2 \right] = \frac{7}{6}mL^2$

[1 POINT]

6.3 Suppose that the assembly rotates about its center of mass with angular velocity $\omega\mathbf{k}$ (the center of mass is stationary). What are the speeds of the particles A,B and C?

We can use the circular motion formula. The distances of A and C from the COM are

$$\sqrt{(L/2)^2 + (L/3)^2} = \frac{\sqrt{13}}{6}L$$

while B is a distance 2L/3 from the COM. The speeds are

$$V_A = V_C = \omega\frac{\sqrt{13}}{6}L \qquad V_B = \omega\frac{2L}{3}$$

[2 POINTS]

6.4 Calculate the total kinetic energy of the system (a) using your answer to 6.2; and (b) using your answer to 6.3. (The point of this problem is to demonstrate that the rigid body formula $(1/2)I\omega^2$ is just a quick way of summing the kinetic energies of the 3 masses. For the simple 2D system here it is quite simple to prove the equivalence for any arrangement of masses. For 3D the derivation is more complicated, but the idea is the same.)

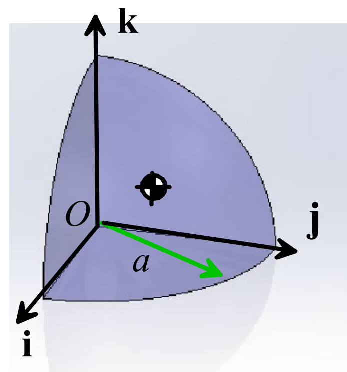
We can use the rigid body formula: $KE = \frac{1}{2}I_G\omega^2 = \frac{7}{12}mL^2\omega^2$

Or we can sum the KEs of the three masses

$$KE = \frac{1}{2}\left(2\frac{13}{36}mL^2 + \frac{4}{9}mL^2\right) = \frac{7}{12}mL^2\omega^2$$

[2 POINTS]

7. The figure shows a 1/8 segment of a sphere with radius a and uniform mass density ρ Using a Matlab 'Live Script', calculate



7.1 The total mass M (you will need to do the relevant integrals using spherical-polar coordinates)

$$M = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho R \sin\theta dR d\theta d\phi = \rho\pi a^3 / 6 \quad \text{(see below for Live Script)}$$

[1 POINT]

7.2 The position vector of the center of mass (with respect to the origin shown in the figure)

$$\mathbf{r}_{COM} = \frac{1}{M}\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho(R\sin\theta\cos\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\theta\mathbf{k})R\sin\theta dR d\theta d\phi = (3a/8)(\mathbf{i}+\mathbf{j}+\mathbf{k})$$ (see below for Live Script)

[2 POINTS]

7.3 The inertia tensor (matrix) about the center of mass, in the basis shown

$$\mathbf{I}_{COM} = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} \rho \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix} R \sin\theta dR d\theta d\phi \quad d_x = R\sin\theta\cos\phi - (3a/8)$$

$$d_y = R\sin\theta\sin\phi - (3a/8) \qquad d_z = R\cos\theta - (3a/8)$$

[3 POINTS]

7.4 Using the parallel axis theorem, calculate the mass moment of inertia about the tip O.

The general formula is

$$\mathbf{I}_O = \mathbf{I}_G + M \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix}$$

[2 POINTS]

Please upload your 'Live script' solution to Canvas.

Graders – only the Live Script upload is required….

A 'Live Script' solution is shown below

```
clear all
syms x y z R r dx dy dz phi theta a M rho mass real
dm = rho*R^2*sin(theta)
M = simplify(int(int(int(dm,R,[0,a]),theta,[0,pi/2]),phi,[0,pi/2]))
r = [R*sin(theta)*cos(phi),R*sin(theta)*sin(phi),R*cos(theta)]
rG = simplify(int(int(int(r*dm,R,[0,a]),theta,[0,pi/2]),phi,[0,pi/2])/M)
dx = r(1)-rG(1); dy = r(2)-rG(2); dz = r(3)-rG(3);
integrand = dm*[dy^2+dz^2,-dx*dy,-dx*dz;...
                -dx*dy,dx^2+dz^2,-dy*dz;...
                -dx*dz,-dy*dz,dx^2+dy^2];
IG = simplify(int(int(int(integrand,R,[0,a]),theta,[0,pi/2]),phi,[0,pi/2]))
IGwithmass = simplify(mass*IG/M)
dx = -rG(1); dy = -rG(2); dz = -rG(3);
Io = simplify(IG + M*[dy^2+dz^2,-dx*dy,-dx*dz;...
                      -dx*dy,dx^2+dz^2,-dx*dz;...
                      -dx*dz,-dy*dz,dx^2+dy^2])
Iowithmass = simplify(mass*Io/M)
```

M =

$$\frac{\pi a^3 \rho}{6}$$

r = $\begin{pmatrix} R\cos(\varphi)\sin(\theta) & R\sin(\varphi)\sin(\theta) & R\cos(\theta) \end{pmatrix}$

rG =

$$\begin{pmatrix} \frac{3a}{8} & \frac{3a}{8} & \frac{3a}{8} \end{pmatrix}$$

IG =

$$\begin{pmatrix} \frac{19\,\pi\,a^5\,\rho}{960} & \sigma_1 & \sigma_1 \\ \sigma_1 & \frac{19\,\pi\,a^5\,\rho}{960} & \sigma_1 \\ \sigma_1 & \sigma_1 & \frac{19\,\pi\,a^5\,\rho}{960} \end{pmatrix}$$

where

$$\sigma_1 = \frac{a^5\,\rho\,\left(45\,\pi - 128\right)}{1920}$$

IGwithmass =

$$\begin{pmatrix} \frac{19\,a^2\,\text{mass}}{160} & \sigma_1 & \sigma_1 \\ \sigma_1 & \frac{19\,a^2\,\text{mass}}{160} & \sigma_1 \\ \sigma_1 & \sigma_1 & \frac{19\,a^2\,\text{mass}}{160} \end{pmatrix}$$

where

$$\sigma_1 = \frac{a^2\,\text{mass}\,\left(45\,\pi - 128\right)}{320\,\pi}$$

Io =

$$\begin{pmatrix} \frac{\pi\,a^5\,\rho}{15} & -\frac{a^5\,\rho}{15} & -\frac{a^5\,\rho}{15} \\ -\frac{a^5\,\rho}{15} & \frac{\pi\,a^5\,\rho}{15} & -\frac{a^5\,\rho}{15} \\ -\frac{a^5\,\rho}{15} & -\frac{a^5\,\rho}{15} & \frac{\pi\,a^5\,\rho}{15} \end{pmatrix}$$

Iowithmass =

$$\begin{pmatrix} \frac{2\,a^2\,\text{mass}}{5} & \sigma_1 & \sigma_1 \\ \sigma_1 & \frac{2\,a^2\,\text{mass}}{5} & \sigma_1 \\ \sigma_1 & \sigma_1 & \frac{2\,a^2\,\text{mass}}{5} \end{pmatrix}$$

where

$$\sigma_1 = -\frac{2\,a^2\,\text{mass}}{5\,\pi}$$