School of Engineering
Brown University

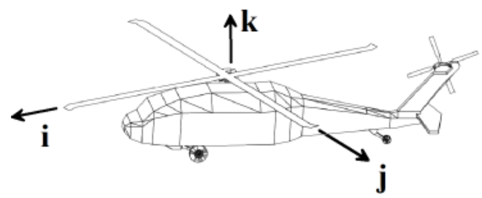
EN40: Dynamics and Vibrations

Homework 8: Rigid Body Dynamics
Due Friday April 26, 2018

1. The Sikorsky UH-60 Black Hawk helicopter has a 4 bladed rotor with the following specifications

- Blade mass 113kg (for one blade)
- Rotor diameter: 16.36m
- Rotor speed 258 rpm

1.1 Calculate the mass moment of inertia matrix of the rotor. Take the basis vectors to be aligned with the blades as shown in the figure.



Idealize the rotor blades as slender bars length L mass m; the mass moment of inertia matrix of one blade with its axis parallel to **i** about its COM is

$$I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & mL^2/12 & 0 \\ 0 & 0 & mL^2/12 \end{bmatrix}$$

shift this to the end of the bar with the parallel axis theorem $I_0 = mL^2/12 + m(L/2)^2 = mL^2/3$

$$I_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & mL^2/12 & 0 \\ 0 & 0 & mL^2/12 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & mL^2/4 & 0 \\ 0 & 0 & mL^2/4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & mL^2/3 & 0 \\ 0 & 0 & mL^2/3 \end{bmatrix}$$

The bars with axis parallel to the **j** axis have inertia

$$I_0 = \begin{bmatrix} mL^2/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & mL^2/3 \end{bmatrix}$$

Adding:

$$[I] = \begin{bmatrix} 2mL^2/3 & 0 & 0 \\ 0 & 2mL^2/3 & 0 \\ 0 & 0 & 4mL^2/3 \end{bmatrix}$$

Substituting numbers

$$[I] = \begin{bmatrix} 5041 & 0 & 0 \\ 0 & 5041 & 0 \\ 0 & 0 & 10082 \end{bmatrix} \text{kg m}^2$$

[4 POINTS]

1.2 What (constant) torque (moment) must be applied to the rotor to spin it up to operating speed in 60sec (neglect drag on the rotors)?

We can use the rigid body equation $\sum \mathbf{M} = \frac{d\mathbf{h}}{dt}$, the angular momentum is

$$\mathbf{h} = [I]\boldsymbol{\omega} = \begin{bmatrix} 5041 & 0 & 0 \\ 0 & 5041 & 0 \\ 0 & 0 & 10082 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} = 10082\omega_z \mathbf{k}$$

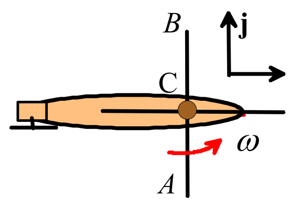
Therefore $M_z \mathbf{k} = 10082 \frac{d\omega_z}{dt} \mathbf{k}$. If the moment is constant the angular acceleration $d\omega_z / dt$ must also be constant. We can calculate it from the change in angular speed and the time (i.e. using the constant acceleration formula)

$$\frac{d\omega_z}{dt} = \frac{258 \times 2\pi}{60} \times \frac{1}{60} = 0.4503 rad / s^2$$

Therefore $M_z = 4.5 \times 10^3 \, Nm$

[2 POINTS]

1.3 If the helicopter is in cruise flight at its maximum airspeed of 294 km/hr, what are the maximum and minimum tip speeds of the rotors through the air?



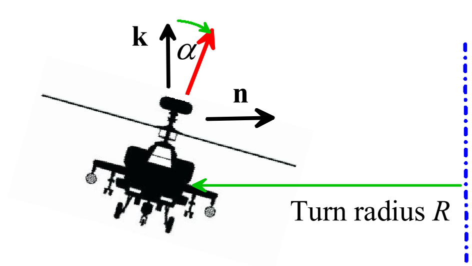
We can use the rigid body kinematics equation: the rotor center moves at the same speed as the helicopter, while the blade tips at A and B have velocities

$$\mathbf{v}_A = \mathbf{v}_C + \omega\mathbf{k} \times (-L\mathbf{j}) \Rightarrow \mathbf{v}_A = (V + \omega L)\mathbf{i}$$
$$\mathbf{v}_B = \mathbf{v}_C + \omega\mathbf{k} \times (L\mathbf{j}) \Rightarrow \mathbf{v}_B = (V - \omega L)\mathbf{i}$$

Substituting numbers gives $\mathbf{v}_A = 302\mathbf{i} \ \ m/s \quad \mathbf{v}_B = -139\mathbf{i} m/s$. Sound speed in air is about 340 m/s so the blade tip at A approaches sound speed – this is one of the reasons the speed of the vehicle is limited.

[2 POINTS]

1.4 If the helicopter is in a standard rate turn at 60 knots (30.86m/s), (with the direction of turn in the same direction as the direction of rotation of the rotor), what is the angular velocity vector of the rotor, in the $\{\mathbf{n}, \mathbf{t}, \mathbf{k}\}$ basis shown in the figure? (Recall that aircraft in a standard rate turn was analyzed in Lecture 3. Neglect drag and the **t** component of lift for simplicity.)



The helicopter body rotates about the **k** axis at a rate of 2π radians in 2 mins. This is an angular velocity

$$\boldsymbol{\omega}_{body} = \frac{\pi}{60}\mathbf{k}$$

The rotor spins with an angular velocity perpendicular to the plane of the rotor. Relative to the aircraft, the angular velocity is $\boldsymbol{\omega}_{rel} = \Omega\cos\alpha\mathbf{k} + \Omega\sin\alpha\mathbf{n}$ where Ω is the angular speed of the rotor (258rpm in rad/s). The bank angle was calculated in L3 as

$$\alpha = \tan^{-1}\left(\frac{V\omega_{body}}{g}\right)$$

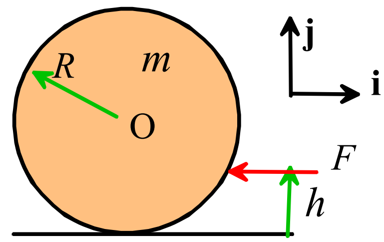
The total angular velocity is the sum of these two, i.e.

$$\boldsymbol{\omega}_{rotor} = (\Omega\cos\alpha + \frac{\pi}{60})\mathbf{k} + \Omega\sin\alpha\mathbf{n}$$

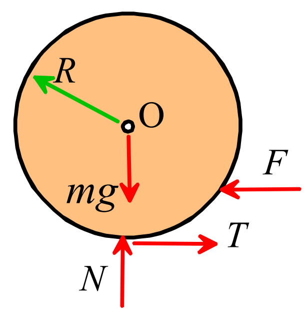
Substituting numbers gives $\boldsymbol{\omega}_{rotor} = 26.711\mathbf{k} + 4.391\mathbf{n}$ rad/s

[4 POINTS]

2. The figure shows a spherical billiard ball with mass m and radius R at rest on a pool table. It is subjected to a horizontal force F by a cue at a height h above the table.



2.1 Draw a free body diagram showing all the forces acting on the ball (include gravity and assume no slip at the contact)



[2 POINTS]

2.2 Write down the equations of motion (Newton's law, and the angular momentum equation)

$$\mathbf{F} = m\mathbf{a} \Rightarrow (T - F)\mathbf{i} + (N - mg)\mathbf{j} = ma_{Gx}\mathbf{i}$$

$$\sum \mathbf{r}\times\mathbf{F} + \mathbf{Q} = \frac{d\mathbf{h}}{dt} \Rightarrow hF\mathbf{k} = R\mathbf{j}\times ma_{Gx}\mathbf{i} + I_{Gzz}\alpha_z\mathbf{k}$$

$$\Rightarrow hF\mathbf{k} = -Rma_{Gx}\mathbf{k} + \frac{2}{5}mR^2\alpha_z\mathbf{k}$$

[2 POINTS]

2.3 Write down the kinematics equation relating the acceleration of the center at O and the angular acceleration of the sphere

$$a_{Gx} = -R\alpha_z$$

[1 POINT]

2.4 Hence, calculate the acceleration of the sphere

Using 2.3, 2.4, note that $hF = -Rma_{Gx} - \frac{2}{5}mRa_{Gx} \Rightarrow a_{Gx} = -\frac{5h}{7R}\frac{F}{m}$

[2 POINTS]

2.5 Calculate the reaction forces at the contact. If the coefficient of friction at the contact is μ , find a formula for the critical value of *F* that will cause slip at the contact, in terms of *m,g,h,R*.

$(T-F)\mathbf{i} + (N-mg)\mathbf{j} = ma_{Gx}\mathbf{i}$ shows that

$$N = mg \qquad T = -m\frac{5h}{7R}\frac{F}{m} + F = F(1-\frac{5h}{7R})$$

At the onset of slip $|T| = \mu N \Rightarrow F\left(1-\frac{5h}{7R}\right) = \mu mg \Rightarrow F = \mu mg \Big/ \left|\left(1-\frac{5h}{7R}\right)\right|$

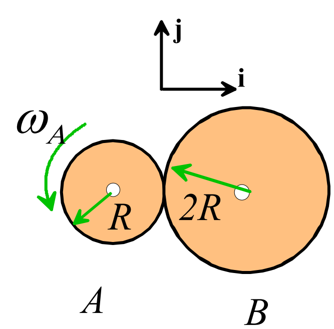
[2 POINTS]

2.6 Where should the ball be struck to guarantee no slip for any value of *F*?

We can choose *h* to make *T=0*, which requires $h = 7R/5$

[1 POINT]

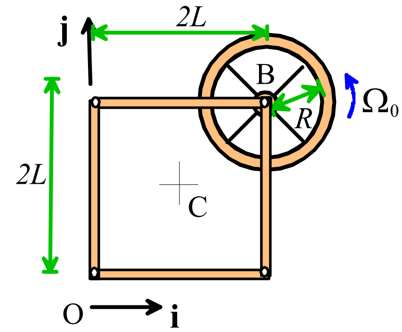
3. The two gears A and B in the figure have radii *R* and *2R*, and mass moments of inertia $mR^2/2$ and $2mR^2$, respectively. Their centers are stationary. If gear A rotates counterclockwise at angular speed ω_A , find a formula for the total angular momentum of the system (including both gears)



The total angular momentum is $(I_{Azz}\boldsymbol{\omega}_A + I_{Bzz}\boldsymbol{\omega}_B)$; we know $\omega_B = -\omega_A R_A / R_B = -\omega_A/2$ and so the total angular momentum is $\left((mR^2/2)\omega_A - 2mR^2\omega_A/2\right)\mathbf{k} = -mR^2\omega_A/2$

[2 POINTS]

4. Reaction/Momentum wheels are used to control the attitude of spacecraft



The figure shows a 2D idealization of a cube-sat (idealized as a frame with four members with length L and combined mass M, with a ring-shaped momentum wheel with mas m and radius R at one corner.

At time $t=0$ the frame is at rest (no translation or rotation), and the wheel spins counterclockwise with angular speed Ω_0 . The motor is then spun up rapidly to increase the speed of the motor to a speed Ω_1 The goal of this problem is to calculate the resulting angular speed of the frame.

4.1 Find formulas for the out-of-plane components of the mass moments of inertia (I_{zz})of the frame (about its center of mass at C) and the wheel (about the center of the wheel), in terms of m, M, R and L.

The wheel is $I_{zzr} = mR^2$.

For the frame, the mass moment of inertia of one bar about its COM is $\frac{4}{12}\frac{M}{4}L^2$. The parallel axis theorem gives the COM of one bar about C as $\frac{4}{12}\frac{M}{4}L^2 + \frac{M}{4}L^2 = \frac{1}{3}ML^2$. The total (4 bars) is thus

$$I_f = \frac{4}{3}ML^2$$

[2 POINTS]

4.2 Find a formula for the total angular momentum of the system about the corner at O at time $t=0$, in terms of Ω_0 and other relevant variables.

The angular momentum is just that of the ring, i.e. $\mathbf{h} = mR^2\Omega_0\mathbf{k}$

[1 POINT]

4.3 What is the total linear momentum of the system at time $t=0$?

Since the system is stationary the linear momentum is zero.

[1 POINT]

4.4 Suppose that after the motor is spun up the center of the frame (at C) has a velocity $\mathbf{v}_C = v_x\mathbf{i} + v_y\mathbf{j}$ and the frame rotates with an angular velocity $\omega_f\mathbf{k}$. Find a formula for the velocity $\mathbf{v}_B$ of the corner at B, in terms of $\omega_f, \mathbf{v}_C, L$

We can use the rigid body kinematics formula

$$\mathbf{v}_B - \mathbf{v}_C = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_C)$$
$$\Rightarrow \mathbf{v}_B = \mathbf{v}_C + \omega_f \mathbf{k} \times L(\mathbf{i} + \mathbf{j}) = \mathbf{v}_C + \omega_f L(-\mathbf{i} + \mathbf{j})$$

[2 POINTS]

4.5 Why is the total linear momentum of the system conserved? Use linear momentum conservation to find formulas for $\mathbf{v}_C$ and $\mathbf{v}_B$ in terms of ω_f, M, m, L

Linear momentum of the system is conserved because no external forces act on the combined frame and ring, and therefore no external impulse is exerted on the system while the motor is spun up. Impulse = change in momentum => change in momentum is zero.

Momentum conservation gives

$$M\mathbf{v}_C + m\left(\mathbf{v}_C + \omega_f L(-\mathbf{i} + \mathbf{j})\right) = \mathbf{0}$$
$$\Rightarrow \mathbf{v}_C = -\frac{m}{M+m}\omega_f L(-\mathbf{i} + \mathbf{j})$$

$$\Rightarrow \mathbf{v}_B = \mathbf{v}_C + \omega_f L(-\mathbf{i} + \mathbf{j}) = \frac{M}{M+m}\omega_f L(-\mathbf{i} + \mathbf{j})$$

(you can also get these results by noting that the COM of the overall system must be stationary)

[2 POINTS]

4.6 Hence, write down the total angular momentum of the frame and the motor about O after the change in motor speed. Note that the motor speed Ω_1 specifies the rate of rotation of the motor, not the angular speed of the ring.

The angular momentum of the frame is

$$\mathbf{r}_C \times M\mathbf{v}_C + I_f \omega_f \mathbf{k} = L(\mathbf{i} + \mathbf{j}) \times \frac{-mM}{M+m}\omega_f L(-\mathbf{i} + \mathbf{j}) + \frac{4}{3}ML^2\omega_f \mathbf{k}$$
$$= \frac{-2mML^2}{M+m}\omega_f \mathbf{k} + \frac{4}{3}ML^2\omega_f \mathbf{k} = \frac{2M(2M-3m)L^2}{3(M+m)}\omega_f \mathbf{k}$$

The angular momentum of the ring is $\mathbf{r}_B \times m\mathbf{v}_B + I_r \omega_r \mathbf{k}$ where $\omega_r = \omega_f + \Omega_1$

$$\mathbf{r}_B \times m\mathbf{v}_B + I_r \omega_r \mathbf{k} = 2L(\mathbf{i} + \mathbf{j}) \times \frac{mM}{M+m}\omega_f L(-\mathbf{i} + \mathbf{j}) + mR^2(\omega_f + \Omega_1)\mathbf{k}$$
$$= \frac{4mML^2}{M+m}\omega_f \mathbf{k} + mR^2(\omega_f + \Omega_1)\mathbf{k}$$

The total angular momentum is the sum of these two, which gives

$$\mathbf{h} = \frac{2M(2M-3m)L^2}{3(M+m)}\omega_f \mathbf{k} + \frac{4mML^2}{M+m}\omega_f \mathbf{k} + mR^2(\omega_f + \Omega_1)\mathbf{k}$$
$$= \frac{2M(2M+3m)L^2}{3(M+m)}\omega_f \mathbf{k} + mR^2(\omega_f + \Omega_1)\mathbf{k}$$

[4 POINTS]

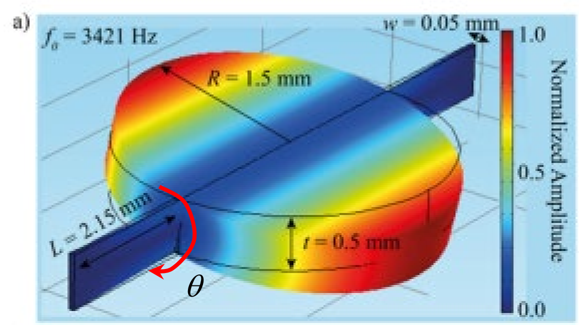
4.7 Finally, find a formula for the angular velocity of the frame after the change in motor speed in terms of $\Omega_1 mM, m, L, R$

Since the angular momentum is conserved we have that

$$mR^2\Omega_0\mathbf{k} = \frac{2M(2M+3m)L^2}{3(M+m)}\omega_f \mathbf{k} + mR^2(\omega_f + \Omega_1)\mathbf{k}$$
$$\Rightarrow \omega_f = \frac{3m(M+m)R^2(\Omega_0 - \Omega_1)}{2M(2M+3m)L^2 + 3mR^2(M+m)}\mathbf{k}$$

[2 POINTS]

5. The figure (from this publication) shows a MEMS mirror (used in laser scanning applications). It consists of a disk with radius R and thickness t, made from Silicon with mass density ρ with dimensions shown in the figure. The disk is mounted on two slender beams that operate as torsional springs with stiffness (Nm/radian) κ (for one beam). For laser scanning the mirror is driven at resonance.



5.1 Write down the total kinetic and potential energy of the system, in terms of the angle of rotation of the mirror θ and its time derivative $d\theta / dt$

The mass moment of inertia of the disk about the axis shown is $\frac{1}{12}m(3R^2+t^2)$ $\quad m = \pi\rho R^2 t$ (this can be approximated as $mR^2/4$ without affecting the numerical answer below). The total KE is therefore $KE = \frac{\pi}{24}\rho R^2 t(3R^2+t^2)\left(\frac{d\theta}{dt}\right)^2$

The two springs are in parallel and therefore have effective torsional stiffens 2κ . The potential energy is therefore $\kappa\theta^2$

[2 POINTS]

5.2 Use the energy method to derive an equation of motion for θ

This is a conservative system, so the sum of KE and PE is constant. Take the time derivative of (KE+PE) to get

$$\frac{d}{dt}(KE+PE) = \frac{\pi}{12}\rho R^2 t(3R^2+t^2)\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + 2\kappa\theta\frac{d\theta}{dt} = 0$$
$$\Rightarrow \frac{\pi}{24\kappa}\rho R^2 t(3R^2+t^2)\frac{d^2\theta}{dt^2} + \theta = 0$$

[2 POINTS]

5.3 Find a formula for the natural frequency of vibration

We have a standard case I equation so

$$\omega_n = \sqrt{\frac{24\kappa}{\pi\rho R^2 t(3R^2+t^2)}}$$

[1 POINT]

5.4 The authors report a resonant frequency of 3421Hz. Calculate the torsional stiffness κ. (Si has a mass density of 2329kg/m^3)

The torsional stiffness follows as

$$2\pi f_n = \sqrt{\frac{24\kappa}{\pi\rho R^2 t(3R^2+t^2)}} \Rightarrow \kappa = \frac{\pi\rho R^2 t(3R^2+t^2)}{24}\left(2\pi f_n\right)^2 = 0.0011 Nm/rad$$

[1 POINT]