



School of Engineering  
Brown University

## EN40: Dynamics and Vibrations

### Homework 2: Kinematics and Dynamics of Particles Due Friday Feb 7, 2020

**1. Straight Line Motion.** NASA operate a [micro-gravity research facility](#) that can be used to conduct experiments in a (near) weightless environment over a short period of time. It consists of a long vertical shaft, which is pumped down to a near vacuum to minimize air drag. A capsule is dropped from rest at the top of the tube, and while the capsule is falling with gravitational acceleration, its contents experience a weightless environment. NASA specs are:

- The weightless portion of the drop lasts for 5.18s
- The capsule is decelerated at 35g (where  $g=9.81\text{m/s}^2$ ) is the gravitational acceleration



1.1 Calculate the length of the shaft, and the maximum velocity of the capsule during the drop

Using the straight-line motion formula  $v = v_0 + at \Rightarrow v = 9.81 \times 5.18 = 50.82\text{m/s}$

The distance traveled is  $x = x_0 + v_0t + \frac{1}{2}at^2 = \frac{1}{2} \times 9.81 \times 5.18^2 = 131.6\text{m}$  (NASA says 132m...)

[2 POINTS]

1.2 What is the length of the decelerator?

We could solve this by calculating the time to stop ( $v = v_0 + at$ ) and then using the distance-acceleration formula; or more directly use the formula relating distance traveled to velocity and acceleration  $v^2 = v_0^2 + 2a(x - x_0) \Rightarrow 0 = 50.82^2 - 2 \times 35 \times 9.81 \times x \Rightarrow x = 3.76\text{m}$  (the picture on the NASA site looks a bit longer than this but the capsule itself is 4m long...)

[1 POINT]

**2. Calculus review** (apologies for inflicting this on you but hopefully it will be helpful. We suggest doing the problems by hand rather than MATLAB; the problems are meant to help you brush up on calculus that sometimes appears on exams. The calculus in this problem appears again in later parts of the course, so try to remember it!): A particle at position  $x = 0$  and has speed  $V_0$  at time  $t=0$ . For each case below, please find formulas for the speed  $v$  and position  $x$  of the particle as functions of time.

2.1 The acceleration of the particle depends on time  $a(t) = A_0 \sin \omega t$  where  $A_0, \omega$  are constants

$$\begin{aligned}
a &= \frac{dv}{dt} = A_0 \sin \omega t \Rightarrow \int_{V_0}^v dv = \int_0^t A_0 \sin \omega t dt \\
\Rightarrow v - V_0 &= -\frac{A_0}{\omega} (\cos \omega t - 1) \Rightarrow v = V_0 + \frac{A_0}{\omega} (1 - \cos \omega t) \\
\Rightarrow \frac{dx}{dt} &= V_0 + \frac{A_0}{\omega} (1 - \cos \omega t) \Rightarrow \int_0^x dx = \int_0^t \left[ V_0 + \frac{A_0}{\omega} (1 - \cos \omega t) \right] dt \\
\Rightarrow x &= \left( V_0 + \frac{A_0}{\omega} \right) t - \frac{A_0}{\omega^2} \sin \omega t
\end{aligned}$$

[3 POINTS]

2.2 The acceleration of the particle depends on its speed  $a(v) = \sqrt{1 - c^2 v^2}$ , where  $c$  is a constant

$$\begin{aligned}
a &= \frac{dv}{dt} = \sqrt{1 - c^2 v^2} \Rightarrow \int_{V_0}^v \frac{dv}{\sqrt{1 - c^2 v^2}} = \int_0^t dt \\
\Rightarrow \int_{\sin^{-1} c V_0}^{\sin^{-1} cv} \frac{\cos \theta d\theta}{c \sqrt{1 - \sin^2 \theta}} &= \int_0^t dt \Rightarrow v = \frac{1}{c} \sin(ct + \sin^{-1} c V_0) \\
\Rightarrow \frac{dx}{dt} &= \frac{1}{c} \sin(ct + \sin^{-1} c V_0) \Rightarrow \int_0^x dx = \int_0^t \frac{1}{c} \sin(ct + \sin^{-1} c V_0) dt \\
\Rightarrow x &= \frac{1}{c^2} \cos(\sin^{-1} c V_0) - \frac{1}{c^2} \cos(ct + \sin^{-1} c V_0)
\end{aligned}$$

[3 POINTS]

2.3 The acceleration of the particle depends on its position

$$a(x) = \begin{cases} -V_0^2 (1-x) & x < 1 \\ 0 & x > 1 \end{cases}$$

(You can solve this problem by finding  $v$  as a function of  $x$  first, then find  $x$  as a function of  $t$ , and finally use your answer to find  $v$  as a function of  $t$ . There are other ways too, if you know how to solve differential equations).

$$\begin{aligned}
a &= v \frac{dv}{dx} = -V_0^2 (1-x) \Rightarrow \int_{V_0}^v v dv = \int_0^x -V_0^2 (1-x) dx \\
\Rightarrow \frac{1}{2} (v^2 - V_0^2) &= -V_0^2 \left( x - \frac{1}{2} x^2 \right) \Rightarrow v^2 = V_0^2 (x^2 - 2x + 1) = V_0^2 (1-x)^2 \Rightarrow v = V_0 (1-x) \\
\Rightarrow \frac{dx}{dt} &= V_0 (1-x) \Rightarrow \int_0^x \frac{1}{(1-x)} dx = \int_0^t V_0 dt \Rightarrow -\log(1-x) = V_0 t \\
\Rightarrow x &= 1 - \exp(-V_0 t) \quad v = V_0 \exp(-V_0 t)
\end{aligned}$$

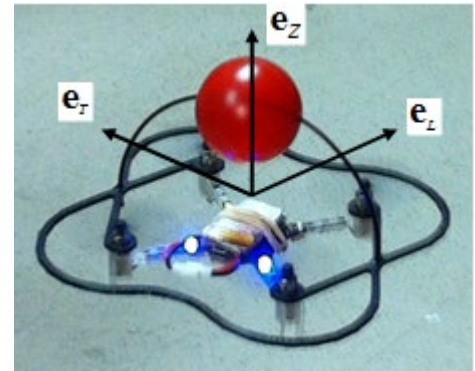
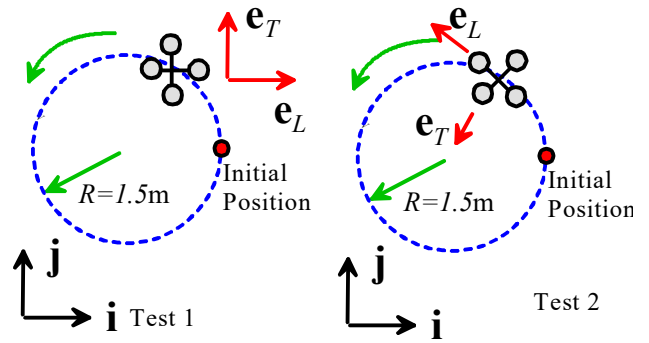
[3 POINTS]

**3. Using MATLAB to process position / acceleration / velocity measurements:** In a class demonstration (see lecture capture to review if needed) we flew a small quadcopter around a circular path, while recording its position, velocity and acceleration.

1. We measured position using a radio system located in a fixed position in the room. The coordinate system is shown in the figure (the origin is just some random position determined by the location of the radio anchors).
2. Velocity, acceleration, and the yaw of the quadcopter were measured using on-board sensors, and are recorded in a coordinate system attached to the quadcopter (to keep things simple you can assume that the quadcopter remains approximately level).

The following flight was programmed into the controller:

- (0) The quadcopter started on its flight stand, with the  $\{e_L, e_T, e_N\}$  quadcopter axes parallel to  $\{i, j, k\}$
- (1) Ascend to the target altitude with constant vertical speed in 5 seconds, while rotating to the correct initial orientation (with constant angular speed)
- (2) Fly two circles with 1.5m radius at constant speed, in 15 seconds (the controller was programmed to fly the path  $\mathbf{r} = \mathbf{r}_0 + R \cos \omega(t-5)\mathbf{i} + R \sin \omega(t-5)\mathbf{j}$   $\omega = 4\pi / 15 \text{ rad} / \text{s}$   $R = 1.5\text{m}$ )
- (3) Descend to a position next to the take-off stand in 5 seconds
- (4) Cut motors



Two tests were run. In the first test, the quadcopter was programmed to keep a fixed orientation, with the  $\{e_L, e_T, e_N\}$  quadcopter axes parallel to  $\{i, j, k\}$  throughout the flight. Data from this test are stored in the file called circle\_fixedorientation.csv. In the second test, the quadcopter turned so that the  $e_L$  direction was parallel to the (desired) direction of motion around the circle. Data from this test are stored in the file called circle\_with\_turn.csv. In both files, the first column of data is the time (s); the second-4<sup>th</sup> columns are  $(x, y, z)$  position; the 5<sup>th</sup>-7<sup>th</sup> columns are velocity  $(v_L, v_T, v_N)$ ; the 8<sup>th</sup>-10<sup>th</sup> columns are acceleration  $(a_L, a_T, a_N)$ , and the last column is the yaw.

3.1 Before trying to process the data, sketch graphs showing:

- (a) The expected variation of  $(v_L, v_T)$  for the two tests during the time interval  $5 < t < 20$  (assuming the path is a perfect circle, which of course it is not...)

For the first test,  $\{e_L, e_T, e_N\}$  and  $\{i, j, k\}$  are the same, so we use the formulas for position and velocity in the  $\{i, j, k\}$  basis

$$\mathbf{r} = \mathbf{r}_0 + R \cos \omega t \mathbf{i} + R \sin \omega t \mathbf{j} \quad \omega = 4\pi / 15 \text{ rad} / \text{s} \quad R = 1.5\text{m}$$

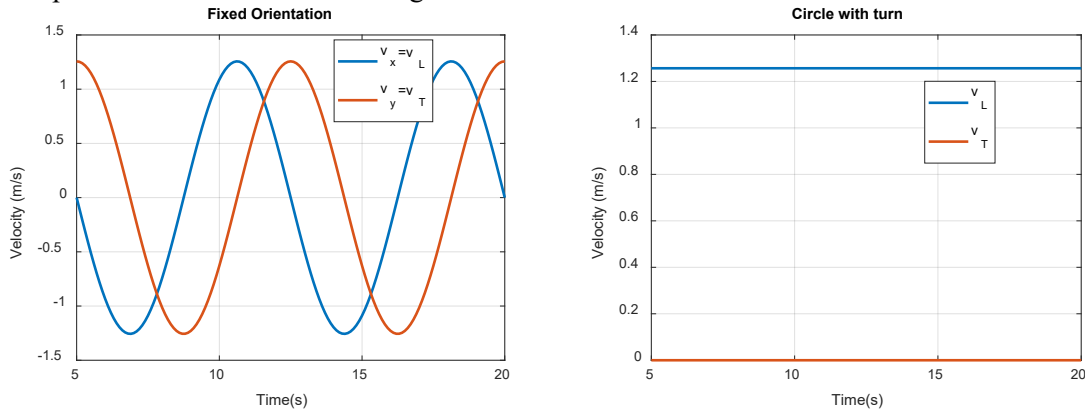
$$\mathbf{v} = \omega R (-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j})$$

So we expect sin/cosine curves, with period of 7.5 sec and amplitude  $6\pi / 15 = 1.26\text{m/s}$

In the second test, we can use the formulas for velocity in the normal-tangential coordinate system

$$\mathbf{v} = V\mathbf{e}_L = R\omega\mathbf{e}_L$$

So we expect a constant vector with magnitude 1.26 m/s



[2 POINTS]

(b) The expected variation of  $(a_L, a_T)$  for the two tests during the time interval  $5 < t < 20$  (again, assuming perfectly circular motion)

For the first test,  $\{\mathbf{e}_L, \mathbf{e}_T, \mathbf{e}_N\}$  and  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  are the same, so we use the formulas for position and velocity in the  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  basis

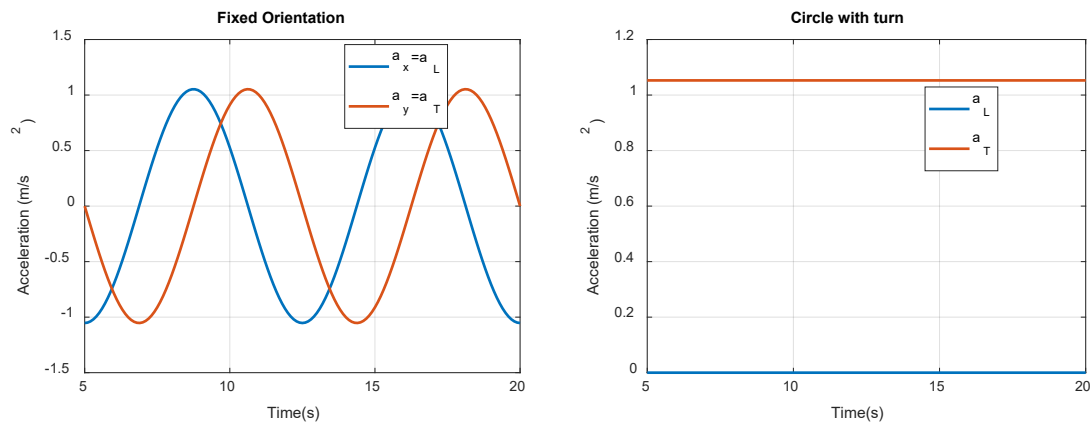
$$\mathbf{a} = -R\omega^2 (\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j}) \quad \omega = 4\pi / 15 \text{ rad / s} \quad R = 1.5\text{m}$$

So we can expect sin/cosine curves with amplitude  $1.05 \text{ m/s}^2$

For the second test we can use the formula in normal-tangential coordinates

$$\mathbf{a} = \frac{dV}{dt} \mathbf{e}_L + \frac{V^2}{R} \mathbf{e}_T = R\omega^2 \mathbf{e}_T \quad \omega = 4\pi / 15 \text{ rad / s} \quad R = 1.5\text{m}$$

So we expect a constant vector with magnitude 1.05 parallel to  $\mathbf{e}_T$



[2 POINTS]

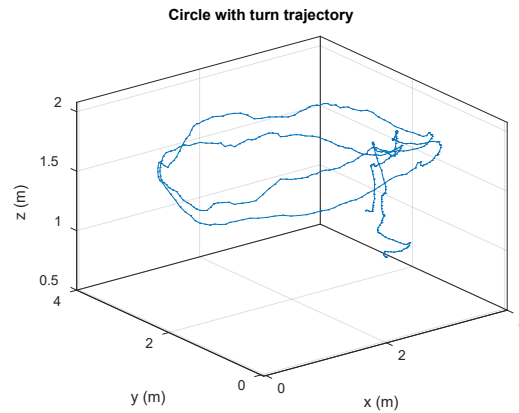
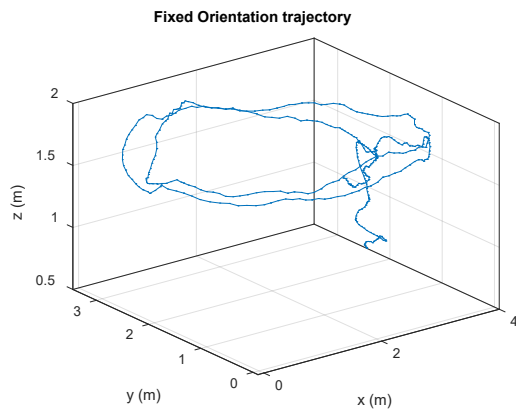
3.2 Write a MATLAB script that accomplishes the following tasks (Your MATLAB code should be uploaded to CANVAS as a submission to this problem): For each data file

(a) Read the data into a matrix using the MATLAB 'csvread' command

[0 POINTS]

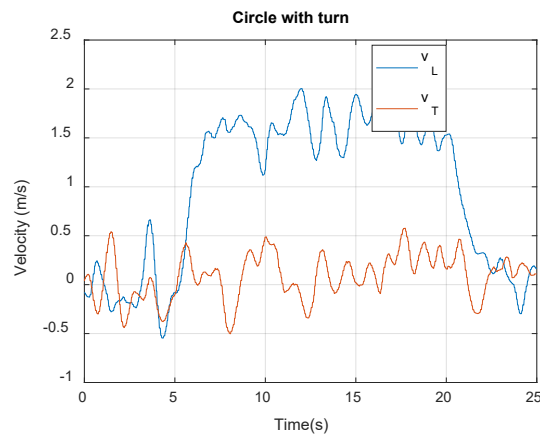
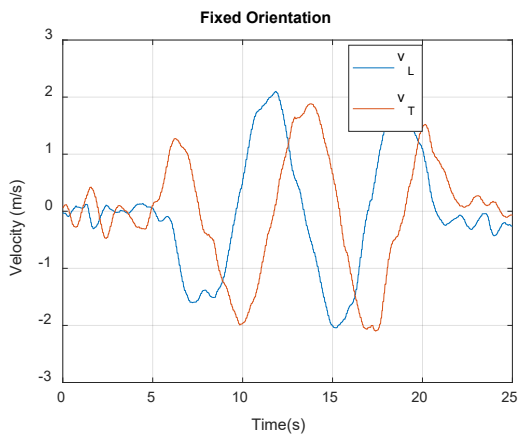
(b) Plot the trajectory (on a 3D plot).

**GRADERS/TAS NOTE: The data below were recorded in January – they are not the data recorded in class. Run the matlab codes provided to see the results with the data from class.**



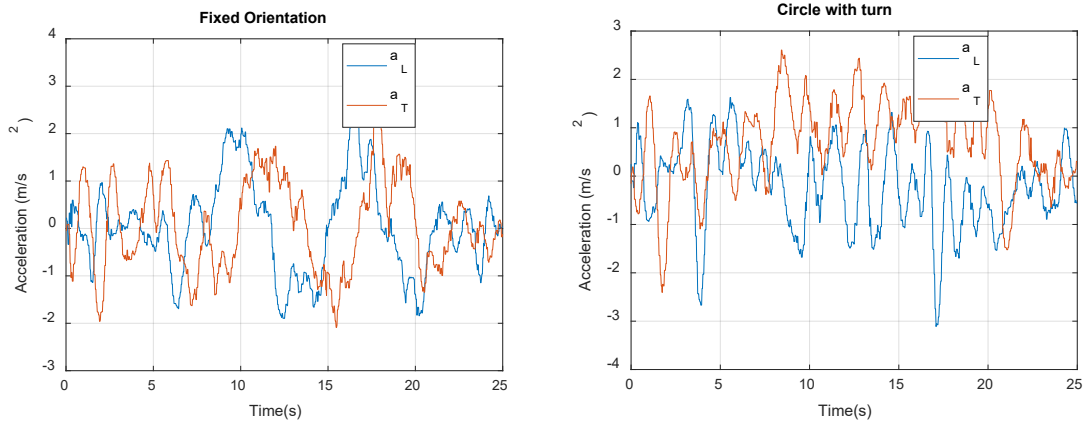
[2 POINTS]

(c) Plot the measured ( $v_L, v_T$ ) as a function of time



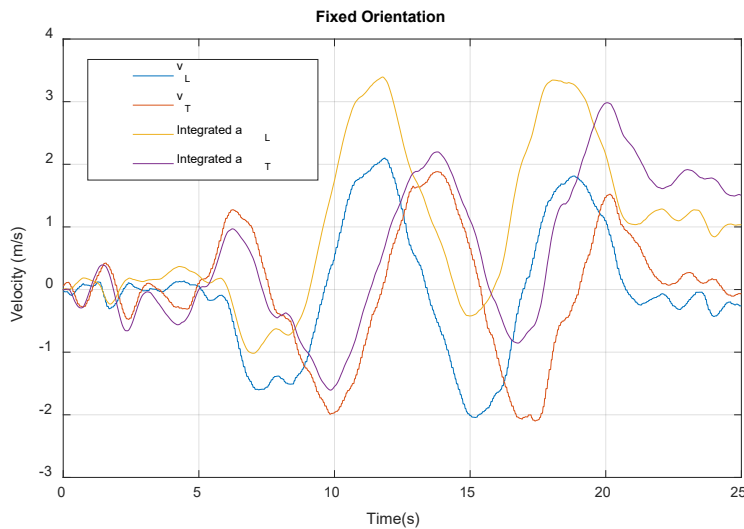
[2 POINTS]

(d) Plot the measured  $(a_L, a_T)$  as a function of time



[2 POINTS]

3.3 Add MATLAB code to check the consistency in the data in the two files. For the data recorded with fixed quadcopter orientation, calculate the velocity by integrating the accelerometer readings  $(a_L, a_T)$  using the 'cumtrapz' function, and plot the results along with the measured  $(v_L, v_T)$ . Explain why this procedure will not yield meaningful results for the second data set.



[2 POINTS]

You can do this for the fixed orientation data because  $(v_L = v_x, v_T = v_y), (a_L = a_x, a_T = a_y)$  and we know that  $v_x = \int a_x, v_y = \int a_y$ . For the data with the quadcopter turning,  $v_L = \int a_L$ , and we could try integrating the longitudinal acceleration – but it's supposed to be zero, so this is not very helpful. For the normal components  $v_T \neq \int a_T$ .

[2 POINTS]

**4. Simple circular motion problem** The rotors on the fields point wind turbines have a diameter of 92m. Suppose that the turbines are spun up from rest with an angular acceleration  $\alpha = 0.1 \text{ rad} / \text{s}^2$



4.1 Find formulas for the angular speed  $\omega = d\theta / dt$  and the angle  $\theta$  turned by the rotor as a function of time.

Just use regular calculus –

$$\alpha = \frac{d\omega}{dt} = 0.1 \Rightarrow \int_0^{\omega} d\omega = \int_0^t 0.1 dt \Rightarrow \omega = 0.1t$$

$$\omega = \frac{d\theta}{dt} = 0.1t \Rightarrow \int_0^{\theta} d\theta = \int_0^t 0.1t dt \Rightarrow \theta = 0.05t^2$$

[2 POINTS]

4.2 How long does it take for the rotor to reach 2 rad/s?

From 4.1  $\omega = 0.1t \Rightarrow 2 = 0.1t \Rightarrow t = 20s$

[1 POINT]

4.3 Calculate the speed of the blade tips on rotor turning at 2 rad/s

The circular motion formula gives  $V = R\omega$  . At 2 rad/sec, the tip speed is 92m/s.

[1 POINT]

4.4 Calculate the magnitude of the acceleration of the blade tips for the blade at the instant when  $\omega = 0$  and the instant when  $\omega = 2 \text{ rad} / \text{s}$

Formulas from class (we can use any version of the circular motion formulas, but the **n-t** version is perhaps most convenient)

$$\mathbf{a} = R\alpha\mathbf{t} + R\omega^2\mathbf{n} \Rightarrow |\mathbf{a}| = R\sqrt{\alpha^2 + \omega^4}$$

$$\text{So when } \omega = 0 \text{ we get } |\mathbf{a}| = R\alpha = 4.6m / s^2$$

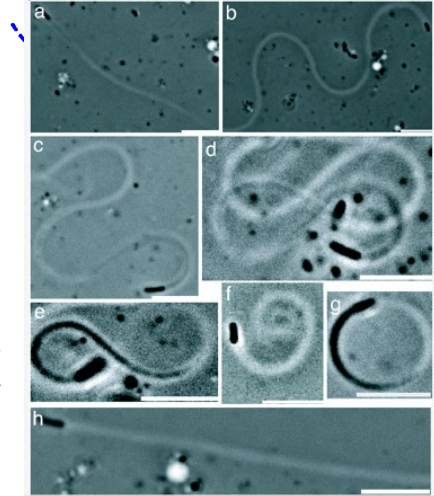
$$\text{When } \omega = 2 \text{ we get } |\mathbf{a}| = (92 / 2)\sqrt{0.1^2 + 2^4} = 184.1m / s^2$$

[3 POINTS]

**5. Normal-Tangential Coordinates.** [This publication](#) shows that a simple formula can predict the trajectories of the Listeria bacterium (some example trajectories are shown in the figure). The authors suggest that the (2D) velocity vector can be expressed as

$$\mathbf{v} = -v_0 \sin\left(\frac{\Omega}{\omega} \sin \omega t\right) \mathbf{i} + v_0 \cos\left(\frac{\Omega}{\omega} \sin \omega t\right) \mathbf{j}$$

with  $v_0, \Omega, \omega$  constants. The research raises many interesting questions, such as why anyone would want to predict the trajectory of a bacterium, but people were probably asking Newton and Kepler why anyone would want to predict the trajectory of a planet...



5.1 Find a formula for the speed of the bacterium, in terms of  $v_0$

$$V = \sqrt{v_x^2 + v_y^2} = v_0$$

[1 POINT]

5.2 Find formulas for the acceleration vector in the  $\{\mathbf{i}, \mathbf{j}\}$  basis

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -v_0 \Omega \cos\left(\frac{\Omega}{\omega} \sin \omega t\right) \cos \omega t \mathbf{i} - v_0 \Omega \sin\left(\frac{\Omega}{\omega} \sin \omega t\right) \cos \omega t \mathbf{j}$$

[2 POINTS]

5.3 Find the normal and tangential components of the acceleration vector

The tangent vector is

$$\mathbf{t} = \frac{\mathbf{v}}{V} = -\sin\left(\frac{\Omega}{\omega} \sin \omega t\right) \mathbf{i} + \cos\left(\frac{\Omega}{\omega} \sin \omega t\right) \mathbf{j}$$

The tangential acceleration is  $a_t = \mathbf{a} \cdot \mathbf{t} = 0$  (we know this because the speed is constant)

The normal acceleration can be found in various different ways – one way is to note that  $\mathbf{a} = a_t \mathbf{t} + a_n \mathbf{n}$  and since  $a_t = 0$  and  $|\mathbf{n}| = 1$  it follows that  $a_n = |\mathbf{a}| = v_0 \Omega \cos \omega t$ . This method only works for this particular problem though. More generally,  $\mathbf{n} = \mathbf{k} \times \mathbf{t} = -\sin\left(\frac{\Omega}{\omega} \sin \omega t\right) \mathbf{j} - \cos\left(\frac{\Omega}{\omega} \sin \omega t\right) \mathbf{i}$ , and then

$$a_n = \mathbf{a} \cdot \mathbf{n}$$

[2 POINTS]



5.4 Find a formula for the radius of curvature of the path as a function of time (get your answer quickly using physics/dynamics, rather than using the long general formulas for curvature).

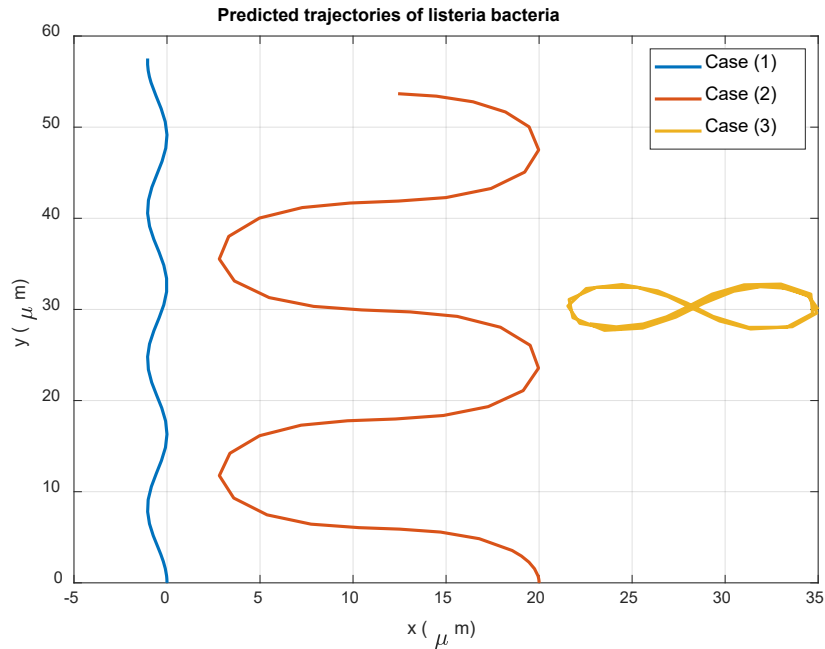
We know that  $\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$ , so the radius of curvature follows as  $R = V^2 / a_n = v_0 / (\Omega \cos \omega t)$

[2 POINTS]

5.5 Write a MATLAB script to calculate the trajectory (include your solution in the same file that contains your solution to problem 2). You can calculate the trajectory by: (1) writing a function to calculate  $d\mathbf{w}/dt = [dx/dt, dy/dt]$  given  $t$ ,  $\mathbf{w}=[x,y]$  and  $v_0, \Omega, \omega$ ; then (2) using ode45 to integrate the velocity to calculate  $x$  and  $y$ . Plot the bacterial trajectories for the following combinations of parameters

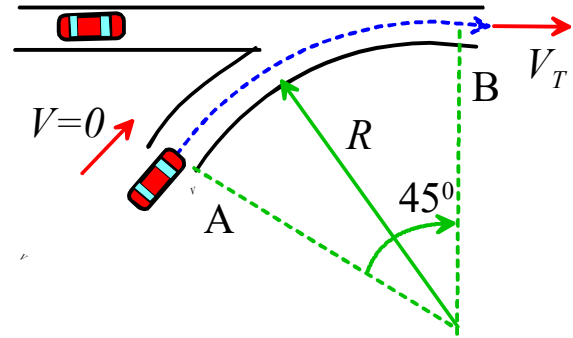
- $v_0 = 0.26 \mu\text{ms}^{-1}$   $\omega = 0.059 \text{rad/s}$   $\Omega = 0.2\omega$   $0 < t < 223.5\text{s}$
- $v_0 = 0.26 \mu\text{ms}^{-1}$   $\omega = 0.035 \text{rad/s}$   $\Omega = 1.5\omega$   $0 < t < 400\text{s}$
- $v_0 = 0.16 \mu\text{ms}^{-1}$   $\omega = 0.028 \text{rad/s}$   $\Omega = 2.39\omega$   $0 < t < 544.5\text{s}$

(these are parameters from Table 1 in the publication). You can put all the trajectories on the same plot, but offset their initial positions so the curves don't overlap. Predictions are shown below.



[4 POINTS]

6. A highway entry ramp is designed as 1/8 arc of a circle with radius  $R$ . Vehicles start at rest at  $A$  and merge with traffic moving with speed  $V_T$  at  $B$ .



The magnitude of the vehicle's acceleration between A and B cannot not exceed  $a_{\max}$ , otherwise the car will skid.

6.1 Find a formula for the maximum possible tangential acceleration of the vehicle between A and B, in terms of  $a_{\max}$ ,  $R$  and the car's speed  $v$ .

We can write down the acceleration of the car in normal-tangential coordinates as  $a_t \mathbf{t} + \frac{v^2}{R} \mathbf{n}$

Taking the magnitude gives  $a_{\max} = \sqrt{a_t^2 + (v^2 / R)^2} \Rightarrow a_t = \sqrt{a_{\max}^2 - v^4 / R^2}$

[1 POINT]

6.2 Hence show that if the vehicle acceleration with its max acceleration, then its speed satisfies

$$Rv \frac{dv}{ds} = \sqrt{a_{\max}^2 R^2 - v^4}$$

where  $s$  is the distance traveled by the vehicle from A.

Use the chain rule  $a_t = \frac{dV}{dt} = \frac{dV}{ds} \frac{ds}{dt} = V \frac{dV}{ds}$

[1 POINT]

6.3 By separating variables and evaluating the integral with respect to  $v$  using MATLAB, show that the car's speed at B is

$$v = \sqrt{Ra_{\max}}$$

Separate variables and integrate as instructed:  $\int_0^v \frac{Rv dv}{\sqrt{a_{\max}^2 R^2 - v^4}} = \int_0^{R\pi/4} ds$

Do the integral on the left with MATLAB (the one on the right is just  $R\pi / 4$ )

```
syms v R a_max real
int(R*v/sqrt(a_max^2*R^2-v^4),v)
```

ans =

$$\frac{R \operatorname{atan}\left(\frac{v^2}{\sqrt{R^2 a_{\max}^2 - v^4}}\right)}{2}$$

So  $\frac{v^2}{\sqrt{R^2 a_{\max}^2 - v^4}} = \tan(\pi/2) \rightarrow \infty \Rightarrow v^4 = R^2 a_{\max}^2 \Rightarrow v = \sqrt{R a_{\max}}$  ; note that  
(the tangential acceleration is therefore zero at B).

[3 POINTS]

6.4 What is the radius of curve required to merge with traffic travelling at 55mph, if  $a_{\max} = 0.2g$  ?

55 mph = 24.59 m/s according to Google, so  $R = v^2 / a_{\max} = 24.59^2 / (0.2 \times 9.81) = 308m$