School of Engineering
Brown University

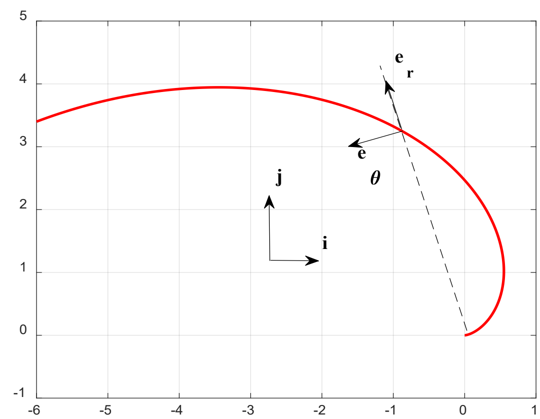
EN40: Dynamics and Vibrations

Homework 3: Kinematics and Dynamics of Particles
Due Friday Feb 14, 2020

Please submit your solutions to the MATLAB coding problems 4, 5 by uploading a single file, with a .m extension, to Canvas.

1. Polar Coordinates: The trajectory of a particle is specified in polar coordinates as a function of time as

$$r = t^2 \qquad \theta = t$$



1.1 Find the components of velocity and acceleration of the particle at time $t=1$s, using the $\mathbf{e}_r, \mathbf{e}_\theta$ coordinate system

The polar coordinate formula for velocity $\mathbf{v} = (dr/dt)\mathbf{e}_r + r(d\theta/dt)\mathbf{e}_\theta$ gives $\mathbf{v} = 2\mathbf{e}_r + \mathbf{e}_\theta$

[1 POINT]

The polar coordinate formula for acceleration
$\mathbf{a} = \left\{(d^2r/dt^2) - r(d\theta/dt)^2\right\}\mathbf{e}_r + \left\{rd^2\theta/dt^2 + 2(dr/dt)(d\theta/dt)\right\}\mathbf{e}_\theta$ gives $\mathbf{a} = \mathbf{e}_r + 4\mathbf{e}_\theta$

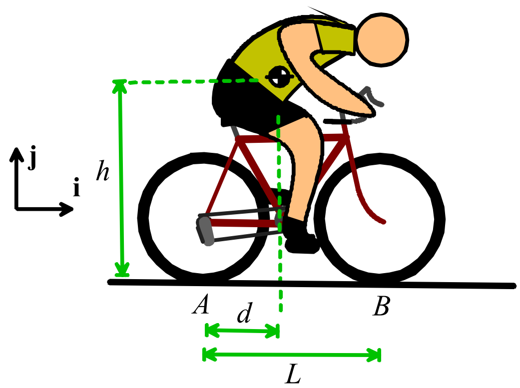
[2 POINTS]

1.2 Hence, find the normal and tangential components of acceleration of the particle at $t=1$s.

$$a_t = \mathbf{t}\cdot\mathbf{a} \qquad \mathbf{t} = \mathbf{v}/|\mathbf{v}| \quad \Rightarrow \quad a_t = 6/\sqrt{5}$$
$$a_n = |\mathbf{n}\cdot\mathbf{a}| \qquad \mathbf{n} = \pm\mathbf{k}\times\mathbf{t} \quad \Rightarrow \quad a_n = 7/\sqrt{5}$$

[2 POINTS]

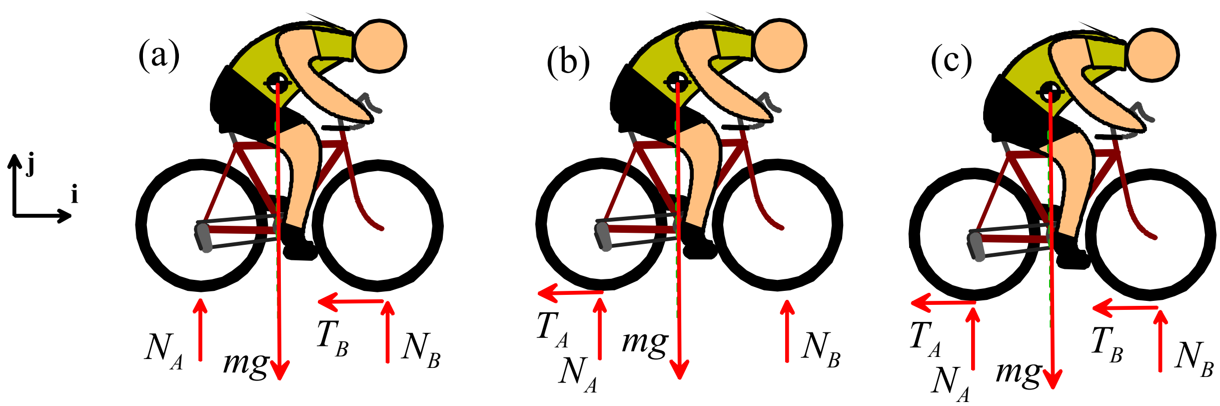
2. Simple Newton's law problem. A 'Fixie' bicycle is a lightweight vehicle with no gears or front brake, intended to be used for track racing. They are ridden on roads (in most places illegally) and have been involved in at least one fatal accident. The goal of this problem is to evaluate assertions made by the prosecution at the subsequent trial that the accident would have been avoided had the bicycle been fitted with a front brake.



For the purposes of this analysis the bicycle and rider can be idealized as a rigid body that translates over the ground without rotation (we won't worry about tipping, which is analyzed in the class notes).

The conditions necessary for a bike braked by only the front wheel to tip is analyzed in Section 3.2 of the class notes. It is shown that the bike will tip if the friction coefficient $\mu \geq (L-d)/h$. The bike will never tip if braked by back wheel only.

2.1 Draw free body diagrams showing the forces acting on the bicycle and rider assuming that (a) only the front wheel is braked, and the rear wheel rolls freely (so no friction force acts at A); (b) only the rear wheel is braked (no friction at B); or (c) both wheels are braked.



[6 POINTS (2 for each figure)]

2.2 Assume that the rider brakes as hard as possible, so the contacts are at the point of slip. Write down Newton's law (translational motion) and the moment balance equation about the COM (rotational motion). Hence, find formulas for the maximum possible deceleration of the bicycle in terms of friction coefficient μ, g, L, d and h for each of case (a), (b), (c) in 2.1.

If the wheels are at the point of slip then $T_A = \mu N_A \qquad T_B = \mu N_B$

Case (a) $$\begin{aligned} -\mu N_B \mathbf{i} + (N_A + N_B - mg)\mathbf{j} &= m a_x \mathbf{i} \\ N_B(L-d) - N_A d - \mu N_B h &= 0 \end{aligned}$$

Hence

$$N_A + N_B = mg$$
$$-N_A + N_B(L/d - 1) - \mu N_B h/d = 0$$

Add these

$$N_B(L/d) - \mu N_B h/d = mg \Rightarrow N_B = \frac{mgd}{L - \mu h}$$

$$\Rightarrow a_x = -\mu N_B/m = -\frac{\mu g d}{L - \mu h}$$

[3 POINTS]

Case (b) $\begin{array}{l} -\mu N_A \mathbf{i} + (N_A + N_B - mg)\mathbf{j} = m a_x \mathbf{i} \\ N_B(L-d) - N_A d - \mu N_A h = 0 \end{array}$

Hence

$$N_A + N_B = mg$$
$$N_A \frac{(d + \mu h)}{(L-d)} - N_B = 0$$

Add these

$$N_A + N_B = mg$$
$$N_A \frac{(d+\mu h)}{(L-d)} + N_A = mg \Rightarrow N_A = \frac{mg(L-d)}{L+\mu h} \Rightarrow a_x = -\mu N_A/m = -\frac{\mu g(L-d)}{L+\mu h}$$

[3 POINTS]

Case (c) $\begin{array}{l} -\mu(N_A + N_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = m a_x \mathbf{i} \\ N_B(L-d) - N_A d - \mu(N_A + N_B)h = 0 \end{array}$

Hence $N_A + N_B = mg$ $\qquad m a_x = -\mu(N_A + N_B) = \mu mg \Rightarrow a_x = -\mu g$

[2 POINTS]

2.3 Assume that $\mu = (L-d)/h$ (so the bike is just at the point of tipping if braked by front wheel only). Show that (for a given initial speed) the stopping distance with only rear wheel braking is a factor $(2L-d)/(L-d)$ greater than braking with both or front wheel only. You can find some relevant dimensions for fixie bikes here

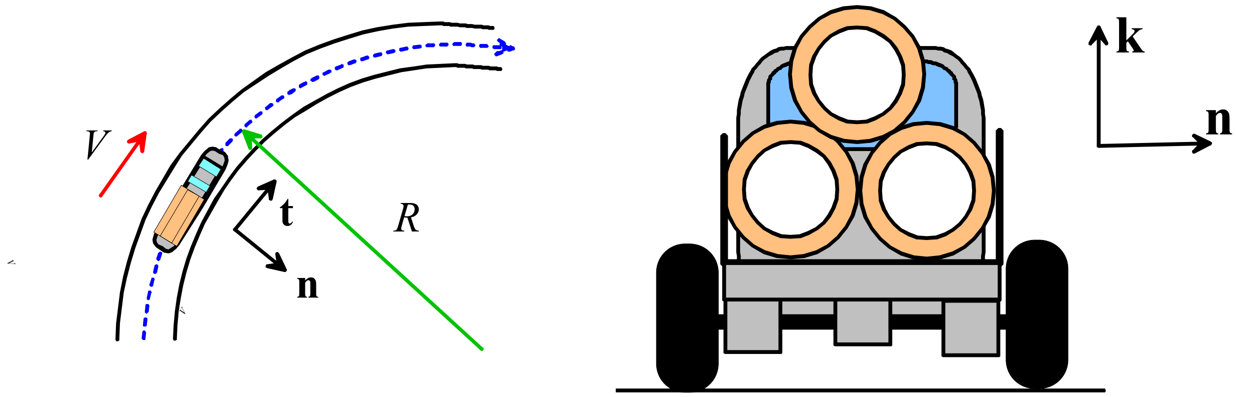
The acceleration with either front wheel or both wheels braked is $a_x = -\mu g$

The acceleration with only rear wheel braking is $a_x = -\frac{\mu g(L-d)}{L+\mu h} = -\frac{\mu g(L-d)}{2L-d}$

The straight line motion formula gives the stopping distance as $s = -v_0^2 / (2a_x)$.

The ratio of stopping distance with only rear wheel braking is therefore $\dfrac{2L-d}{(L-d)}$ greater than braking with both or front wheel only.

[3 POINTS]



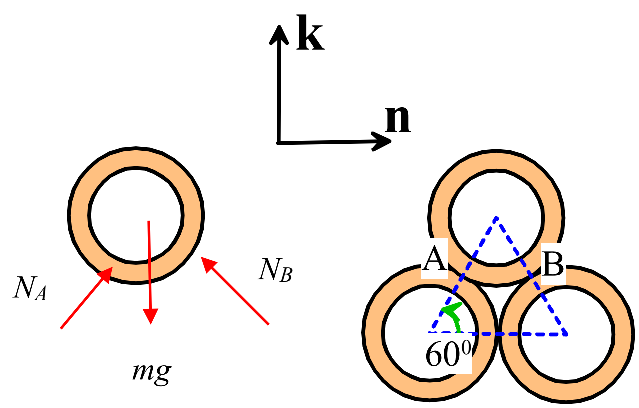
3. A flatbed truck carrying three long cylindrical water-pipes travels around a curved road with radius R at constant speed V. The pipes are arranged on the truckbed as shown in the figure. The goal of this problem is to determine the maximum allowable speed that will ensure that the **topmost** cylinder does not roll off the truck as it negotiates the bend.

3.1 Write down the acceleration of the vehicle in terms of V and R, using the normal-tangential coordinate system shown in the figure.

Use the circular motion formula $\mathbf{a} = \dfrac{V^2}{R}\mathbf{n}$

[1 POINT]

3.2 Draw the forces acting on the topmost cylinder on the diagram below (the figure on the right is provided for information). **Neglect friction** (but do include gravity!)



[2 POINTS]

3.3 Write down Newton's law in the normal-tangential-vertical coordinate system, and hence find formulas for the (normal) reaction forces acting at the two contact points A and B

$$\mathbf{F} = m\mathbf{a} \Rightarrow (N_A - N_B)\cos 60\mathbf{n} + \left[(N_A + N_B)\sin 60 - mg\right]\mathbf{k} = m\frac{V^2}{R}\mathbf{n}$$

The two components of this equation give

$$(N_A - N_B)\frac{1}{2} = m\frac{V^2}{R}$$

$$(N_A + N_B)\frac{\sqrt{3}}{2} = mg$$

$$\Rightarrow \sqrt{3}N_A = \sqrt{3}m\frac{V^2}{R} + mg \Rightarrow N_A = m\frac{V^2}{R} + \frac{mg}{\sqrt{3}}$$

$$\sqrt{3}N_B = mg - \sqrt{3}m\frac{V^2}{R} \Rightarrow N_B = \frac{mg}{\sqrt{3}} - m\frac{V^2}{R}$$

[3 POINTS]

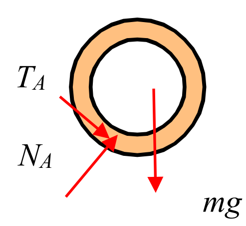
3.4 Hence, find the critical speed at which the topmost cylinder will just start to roll off the truck

The pipe will roll if either reaction force is negative (the negative reaction force is impossible). At the critical speed the reaction force is zero. Clearly the reaction at A will never be negative, so the condition is

$$N_B = \frac{mg}{\sqrt{3}} - m\frac{V^2}{R} = 0 \Rightarrow V = \sqrt{\frac{gR}{\sqrt{3}}}$$

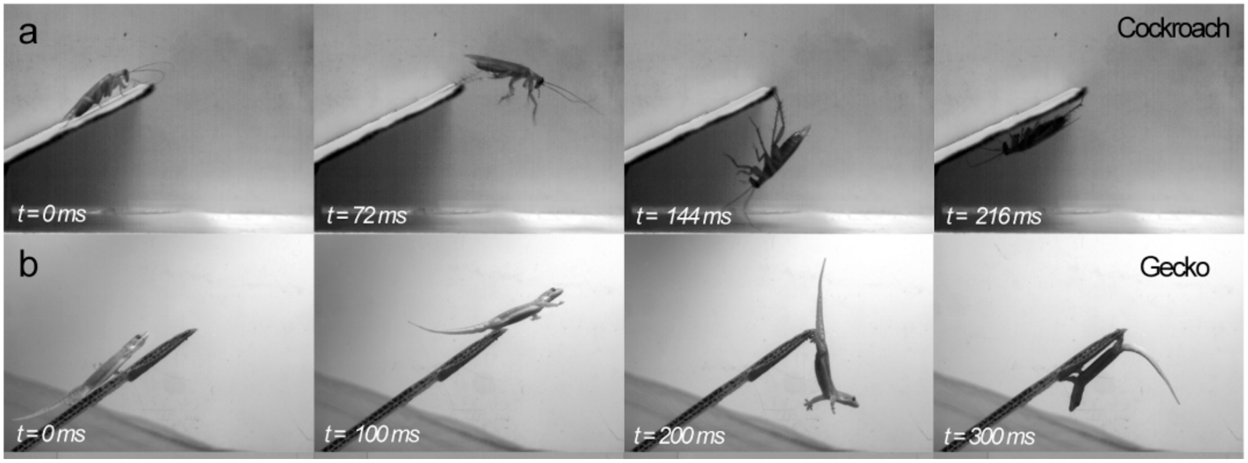
[2 POINTS]

3.5 If friction acts at the contact points, will this change the critical speed? Please explain your reasoning with the aid of relevant calculations (and possibly a new free body diagram).

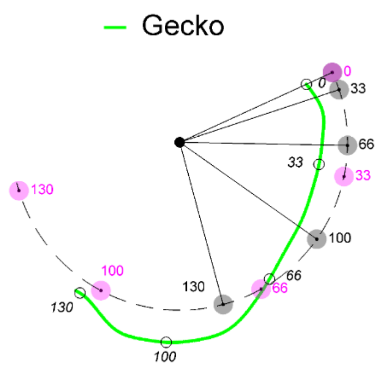


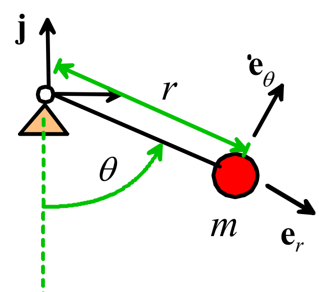
The figure shows a FBD at the critical speed (when the normal force at B is zero), but including a friction force at A. Taking moments about the COM of the cylinder shows that $T_A = 0$. This means that friction makes no difference.

[2 POINTS]



4. This entertaining publication “Rapid Inversion: Running Animals and Robots Swing like a Pendulum under Ledges” studies the behavior of cockroaches and geckos when they reach the end of a ramp (their paper includes some high speed movies). The paper shows that the trajectory of these animals during their flip can be predicted with reasonable accuracy by idealizing the animals as a pendulum with fixed length. The trajectories don’t match the theory very well, however (the pink and gray dots in the picture show model predictions of the position of the animals COM at successive times (in milliseconds) after the start of the swing. with different assumptions for pink and gray; the green curve is the actual trajectory. In this problem, you will test a more sophisticated model in which the pendulum can change its length.





The figure shows the idealization. The length of the pendulum r can vary with time – we will use the simple approximation

$$r(\theta) = L_0 - \Delta L \sin(\theta - \beta)$$

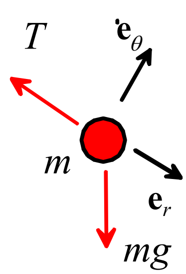
where $L_0, \Delta L, \beta$ are constants.

4.1 Write down the formula for the acceleration vector of the end of the pendulum in terms of r, θ and their time derivatives (use polar coordinates).

Use the polar coordinate formula $\mathbf{a} = \left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\} \mathbf{e}_r + \left\{ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right\} \mathbf{e}_\theta$

[1 POINT]

4.2 Draw a free body diagram showing the force acting on the mass – include gravity, but neglect air resistance.



[2 POINTS]

4.3 Hence show that the angle θ and its time derivative ω are governed by the equations of motion (expressed in a form that MATLAB can solve)

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -\left(g\sin\theta - 2\Delta L\omega^2\cos(\theta-\beta)\right)/r\end{bmatrix}$$

Newton's law gives

$$(mg\cos\theta - T)\mathbf{e}_r - mg\sin\theta\mathbf{e}_\theta = m\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\mathbf{e}_r + m\left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\mathbf{e}_\theta$$

Take the $\mathbf{e}_\theta$ component

$$-mg\sin\theta = +m\left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}$$

Introduce $\omega = \frac{d\theta}{dt}$ and note that

$$\frac{dr}{dt} = -\Delta L\cos(\theta-\beta)\frac{d\theta}{dt} = -\Delta L\omega\cos(\theta-\beta)$$

so

$$-mg\sin\theta = +m\left\{r\frac{d\omega}{dt} - 2\Delta L\omega\cos(\theta-\beta)\omega\right\}$$

$$\Rightarrow \frac{d\omega}{dt} = -\left[g\sin\theta - 2\Delta L\omega^2\cos(\theta-\beta)\right]/r$$

So if we take $[\theta,\omega]$ as the unknowns, then

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -\left(g\sin\theta - 2\Delta L\omega^2\cos(\theta-\beta)\right)/r\end{bmatrix}$$

[3 POINTS]

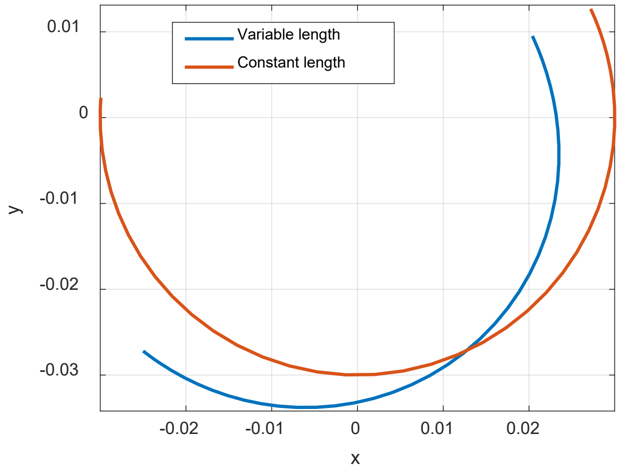
4.4 Write a MATLAB script that will solve the equations of motion. Use the following values for parameters (intended to approximate the Gecko):

- $L_0 = 3cm \quad \Delta L = 0.75cm \quad \beta = 25^0$
- Initial conditions $\theta = 115^0, \quad \omega = -15.95\ rad/s$
- Time interval 0<*t*<130 milliseconds

For comparison, run a second simulation with $L_0 = 3cm \quad \Delta L = 0 \quad \beta = 25^0$ (this repeats the constant length model published in the paper.

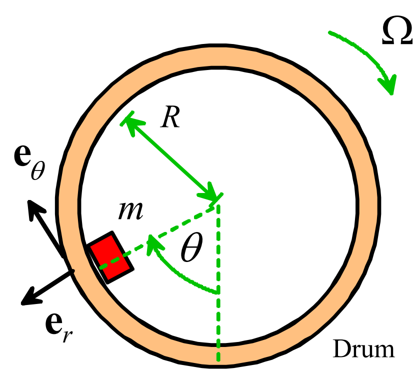
For each simulation plot (on the same graph) the trajectory (i.e. *x,y* coordinates, which you will need to compute after solving the ODE with some trig formulas).

The two trajectories are shown below – the variable length version agrees with experiment much better than the original.



[4 POINTS]

5. The figure shows a hollow drum with radius R (eg a cement mixer) that contains a small particle with mass *m.* The contact between the drum and mass has friction coefficient μ . At time *t=0* the mass is at position $\theta = 0$, and the drum and particle are at rest.. For time *t>0* the drum spins with constant angular speed Ω . Since the mass is stationary at time *t=0*, slip must initially occur at the contact between the mass and the drum. If the speed of the mass reaches the speed of the drum, slip will stop and the mass will move with the same tangential speed as the drum.



The goal of this problem is to analyze the dynamics of the particle inside the drum; and in particular to determine (i) whether the particle will lose contact with the drum wall, (ii) whether the mass will spin 'over the top' and (iii) to calculate how long slip between the particle and drum will continue.

5.1 Assume that the mass remains in contact with the drum. Write down a formula for the acceleration of the mass, in terms of R and time derivatives of θ

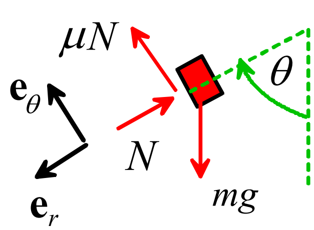
Using the polar coordinate formula

$$\mathbf{a} = -R\left(\frac{d\theta}{dt}\right)^2 \mathbf{e}_r + R\frac{d^2\theta}{dt^2}\mathbf{e}_\theta$$

[1 POINT]

5.2 Draw a free body diagram showing the forces acting on the mass (assume $\theta > 0$). Include gravity.

[2 POINTS]



5.3 Use Newton's laws to show that (as long as the mass remains in contact with the drum) the equation of motion for θ and its time derivative $\omega = d\theta / dt$ can be expressed in a form that can be integrated with ode45 as follows

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ \alpha\end{bmatrix} \qquad \alpha = \begin{cases} \mu\left((g/R)\cos\theta + \omega^2\right) - (g/R)\sin\theta & \omega < \Omega \\ 0 & \omega = \Omega \end{cases}$$

Newton's law for the mass gives

$$\mathbf{F} = m\mathbf{a}$$

$$\{-N + mg\cos\theta\}\mathbf{e}_r + \{\mu N - mg\sin\theta\}\mathbf{e}_\theta = -mR\left(\frac{d\theta}{dt}\right)^2 \mathbf{e}_r + mR\frac{d^2\theta}{dt^2}\mathbf{e}_\theta$$

The two components of this vector equation show that

$$N = mR\left(\frac{d\theta}{dt}\right)^2 + mg\cos\theta$$

$$\mu N - mg\sin\theta = +mR\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = \mu\left(\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{R}\cos\theta\right) - \frac{g}{R}\sin\theta$$

Substituting $\omega = d\theta / dt$ reduces the equation to the form stated.

[3 POINTS]

5.4 Write a MATLAB script to solve the equations of motion. Add an 'Event' function that will stop the calculation when the mass loses contact with the drum.

Write your code to

(i) Plot a graph of θ as a function of the dimensionless time $t\sqrt{g/R}$

(ii) The dimensionless angular speed $\omega\sqrt{R/g}$ as a function of dimensionless time $t\sqrt{g/R}$

(iii) The dimensionless normal force N/mg acting between the drum and mass as a function of dimensionless time $t\sqrt{g/R}$

Run a series of simulations with $R = 0.5, m = 0.25, \Omega = 2\sqrt{g/R}$ for 0<t<5 and values of μ between 0.1 and 0.8 to explore the behavior of the system.

Write a short description of the behavior you observe (i.e. explain in a sentence or two the nature of the predicted motion of the mass), including suitable plots to illustrate your description. Try to determine (i) The influence of μ; and (ii) The influence (if any) of drum radius R

Observations:

(i) The predicted behavior (when plotted in the dimensionless form requested in the problem) is independent of *g* and *R*. You can easily prove this mathematically by either using dimensional analysis (if you know how to do that) or making a simple change of variables in the governing equations. Let $\tau = t\sqrt{g/R} \qquad \hat{\omega} = \omega\sqrt{R/g}$, then

$$\frac{1}{\sqrt{R/g}}\frac{d}{d\tau}\begin{bmatrix}\theta \\ \hat{\omega}\sqrt{g/R}\end{bmatrix} = \begin{bmatrix}\hat{\omega}\sqrt{g/R} \\ \alpha\end{bmatrix} \qquad \alpha = \begin{cases}\mu\left((g/R)\cos\theta + (g/R)\hat{\omega}^2\right) - (g/R)\sin\theta & \hat{\omega}\sqrt{g/R} < \Omega \\ 0 & \hat{\omega}\sqrt{g/R} \geq \Omega\end{cases}$$

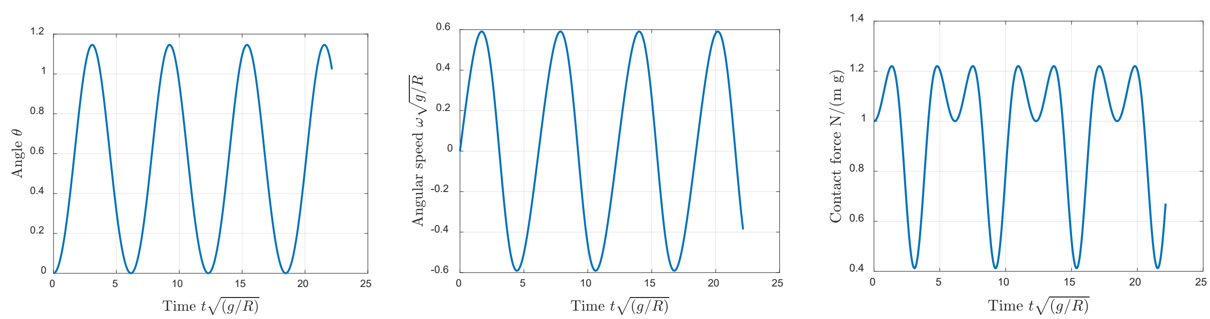
The *g* and *R* then cancel out giving

$$\frac{d}{d\tau}\begin{bmatrix}\theta \\ \hat{\omega}\end{bmatrix} = \begin{bmatrix}\hat{\omega} \\ \hat{\alpha}\end{bmatrix} \qquad \hat{\alpha} = \begin{cases}\mu\left(\cos\theta + \hat{\omega}^2\right) - \sin\theta & \hat{\omega} < \hat{\Omega} \\ 0 & \hat{\omega} \geq \hat{\Omega}\end{cases}$$

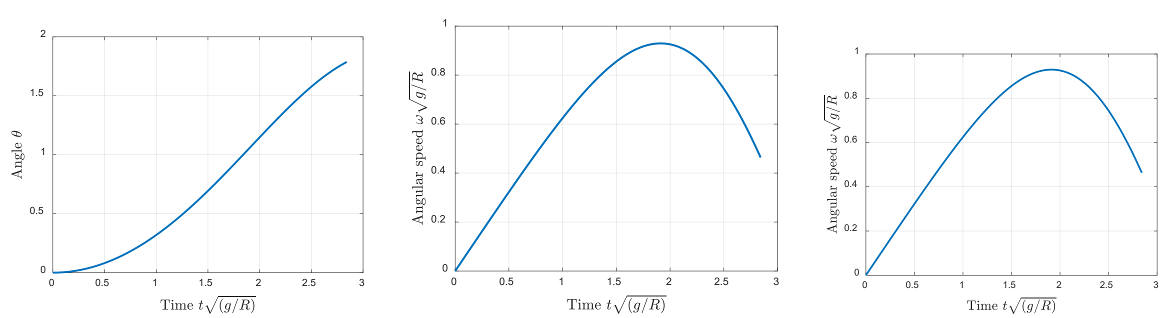
where $\hat{\alpha} = \alpha(R/g) \qquad \hat{\Omega} = \Omega\sqrt{R/g}$. In this form *g* and *R* don't appear in the EOM so can't affect the solution.

(ii) This means that behavior is determined solely by the values of μ and the dimensionless drum speed $\hat{\Omega}$. It is straightforward to show that if there is no slip between the drum and mass, the mass will remain in contact with the drum if dimensionless speed exceeds 1. For $\hat{\Omega} > 1$ Experimenting with different values of μ shows that:

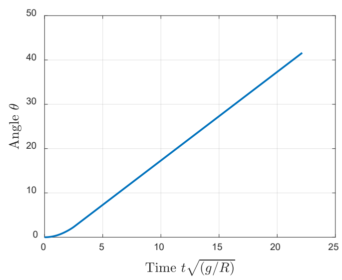
(a) For $\mu < 0.603$ the mass exhibits oscillatory behavior, sliding up and down the wall of the drum. The maximum value of θ is always less than $\pi/2$. Graphs showing this behavior are shown below (for $\mu = 0.5$)



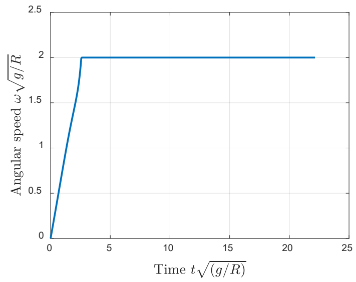
(b) For $0.603 < \mu < 0.714$ the mass rotates past $\theta = \pi/2$ and loses contact with the drum wall. Graphs showing behavior for $\mu = 0.65$ are shown below



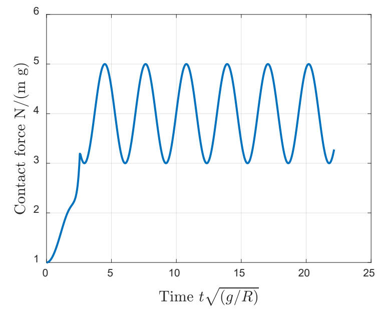
(c) $\mu > 0.715$ the mass remains in contact with the drum, and slip stops. In steady state the mass just rotates around with the drum. Graphs of this behavior (for $\mu = 0.75$) are shown below



Angle θ
Time $t\sqrt{(g/R)}$



Angular speed $\omega\sqrt{g/R}$
Time $t\sqrt{(g/R)}$



Contact force N/(m g)
Time $t\sqrt{(g/R)}$

[4 POINTS]