EN40: Dynamics and Vibrations

Homework 4: Conservation Laws for Particles
Due Friday Feb 28, 2020

School of Engineering
Brown University

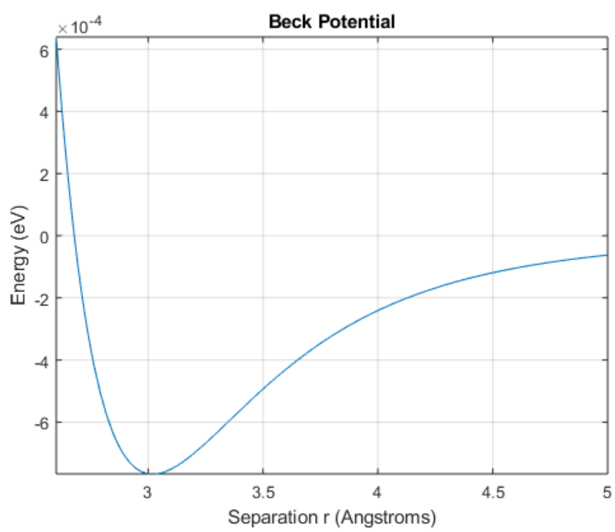
1. LAMMPS is an open-source molecular dynamics code maintained by Sandia National Labs. It is used to model material behavior, chemical reactions, and biochemical processes at molecular scales. It includes a large library of interatomic potentials: the 'Beck' potential is one example. It gives the potential energy of a pair of He atoms that are separated by a distance r as

$$U = E_0\left[\exp\left\{-\alpha r-(\beta r)^6\right\}-\frac{b^6}{\left(r^2+a^2\right)^3}\left\{1+\frac{c^2+3a^2}{r^2+a^2}\right\}\right]$$

Values for the constants are listed in the table

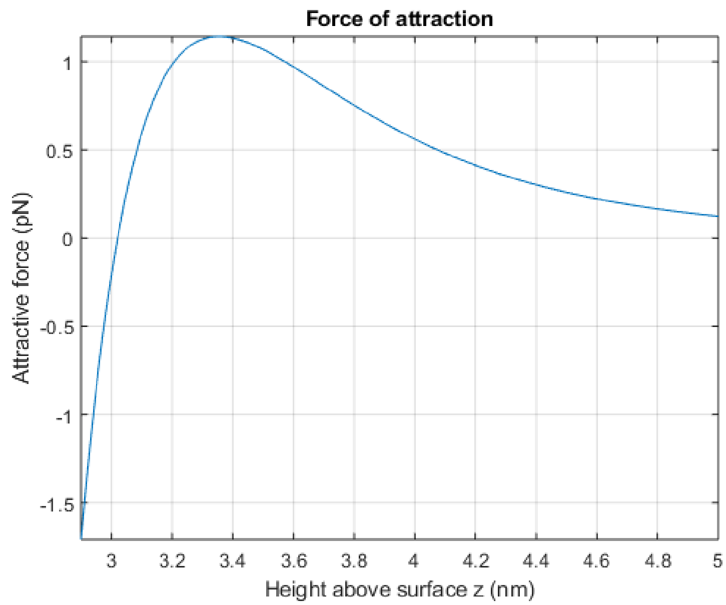
E_0	6.3878x10^{-17} J (398.7eV)
a	*0.675* Angstroms
α	4.39 Angstroms^{-1}
β	0.1829 Angstroms^{-1}
b	0.3616 Angstroms
c	1.6459 Angstroms

1.1 Plot the energy as a function of r (use units of eV for the energy, and Angstroms for the distance r. 2.6<r<5 Angstroms gives a clear plot) (Just hand in your plot; there is no need to submit MATLAB code)



[2 POINTS]

1.2 Plot a graph showing the force of attraction between the molecule and the surface as a function of its distance z from the surface. (use units of picoNewtons for the force, and angstroms for the distance r. *2.9*<r<5 Angstroms gives a clear plot) (Just hand in your plot; there is no need to submit MATLAB code)



[2 POINTS]

1.3 Find the (static) force required to detach the molecule from the surface (in picoNewtons). If you are using a MATLAB live script to do the calculation you will need to use the 'vpasolve' function to find the value of r that maximizes the force of attachment. See Sect 6.1 of the MATLAB tutorial for an example.

MATLAB gives $F=1.1422$ pN.

[2 POINTS]

1.4 Find the equilibrium separation between two He atoms (the separation when $F=0$) (in Angstroms)

From MATLAB r=3.02 Angstroms (you can see this from the graphs too)

[2 POINTS]

1.5 Find the stiffness of the bond (at the equilibrium separation) in N/m

From MATLAB $k=0.096$N/m

[2 POINTS]

2. Eviation Alice is a prototype 9+2 seat electric commuter aircraft.

The aircraft has the following specifications

- Max take-off weight 6350kg
- Max power output from each of three electric motors 260 kW (kiloWatts)
- Cruise speed 260 knots
- Battery capacity 900 kWh (the total energy in the battery , in kilo-Watt hours)
- Range, (at cruise speed, with 45 min reserve): 650 Nautical miles
- Approach speed 100 knots

2.1 Use the given cruise speed, battery capacity, and range – including the reserve – to estimate the power expenditure during level cruise.

At cruise speed it takes 650/260= 2.5 hrs to travel 650 miles; so the battery capacity will be exhausted after 2.5 hrs + 45 mins = 3.25 hrs (the reserve). The average power expenditure is 900/3.25 = 277 kW

[2 POINTS]

2.2 Hence, estimate the drag force on the aircraft during level cruise

At constant speed in level flight the power-KE relation for a particle tells us that $P_{Engine} + P_{Drag} = 0$, the drag power is

$$\mathbf{F} \cdot \mathbf{v} = -F_D V \Rightarrow F_D = P_{Engine} / V = 277 \times 10^3 / 133.8 = 2.07 kN$$

(Aside – if this is at max weight, it suggests the lift:drag ratio is 30. This seems high – see eg http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.539.1597&rep=rep1&type=pdf for typical lift:drag ratios (the paper is 20 years old, but still it's hard to believe lift/drag would have achieved a 50 improvement in 20 years….)

[3 POINTS]

2.3 If the aircraft descends from 10000ft to sea level with zero engine power at the same speed and lift:drag ratio as cruise, how far will the aircraft travel during the descent?

The total potential energy available is $U=mgh$; 10000ft is 3048m giving U=189.9 MJ. The glideslope is small so we can assume that the work done by drag is equal to the drag force multiplied by the horizontal distance traveled. The horizontal distance traveled is therefore $189.9 \times 10^6 / 2.07 \times 10^3 = 91.7$ km

[2POINTS]

2.4 Estimate the maximum (constant speed) climb rate of the aircraft (at max takeoff weight; assume the same power loss to drag as in level cruise)

During climb at constant speed we have that $P_{engine} + P_{Drag} + P_{Lift} + P_{Gravity} = 0$. The lift force (by definition) acts perpendicular to airflow and so is workless; the rate of work done by gravity is $-mg\mathbf{j}\cdot(v_x\mathbf{i} + v_y)\mathbf{j} = -v_y mg$. Therefore $v_y = (P_E + P_D)/(mg) = 8.07 m/s$ (1500 ft/min; a reasonable climb rate)

[2 POINTS]

2.5 Assuming that the engine produces a constant power, find a formula for the acceleration of the aircraft during take-off (use the power-KE relation for a particle – if you take a time derivative you get the acceleration – see class notes for an example. You can neglect drag to keep things simple). Hence, estimate the minimum length of runway required to take off (assume takeoff speed is the same as the approach speed; this is probably an overestimate.).

During takeoff $P_{Engine} = \dfrac{d}{dt}\left(\dfrac{1}{2}mv^2\right) \Rightarrow mv\dfrac{dv}{dt} = P_{Engine}$

Recall also the chain rule formula for acceleration $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$

Therefore

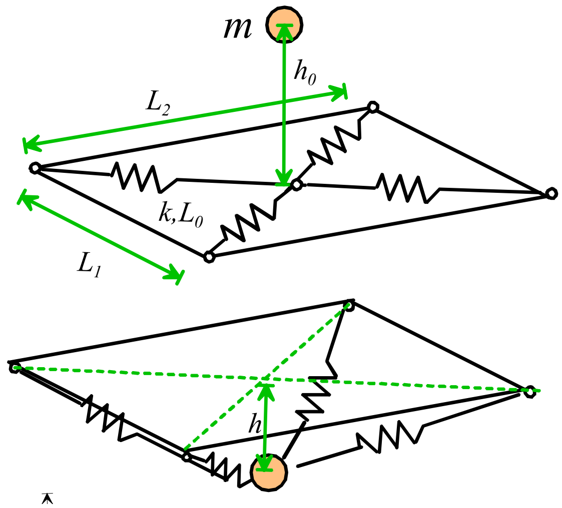
$$mv^2\frac{dv}{dx} = P_{Engine}$$

Integrate both sides

$$\int_0^{V_{TO}} mv^2 dv = \int_0^x P_{engine}dx \Rightarrow \frac{1}{3}mV_{TO}^3 = P_{engine}x$$

Therefore $x = \dfrac{1}{3}\dfrac{m}{P_{engine}}V_{TO}^3 = 369m$

[4 POINTS]



3. In this report a 7m x 5m fall protection net is tested by dropping a 106 kg sphere onto the net. The subsequent deflection of the sphere inside the net is measured with a high-speed camera. In a test with an initial drop height of 3m above the flat net, a deflection of h=2.6m was measured.

The net can be idealized as four springs, as shown in the sketch.

3.1 Use energy conservation and the known deflection and drop height to estimate the stiffness k of the springs (assume they are free of force before the sphere impacts the net)

Assume that the sphere is dropped from a height h_0 and is initially at rest. At the instant of maximum deflection, the system is again at rest. There is no KE in either the initial or final state. The system is conservative and the energy conservation equation shows that

$$U_1 = U_0 \Rightarrow mgh_0 = 2k(L-L_0)^2 - mgh$$

Where $L = \sqrt{(L_1/2)^2 + (L_2/2)^2 + h^2}$ and

$L_0 = \sqrt{(L_1/2)^2 + (L_2/2)^2}$

Hence $k = \dfrac{mg(h+h_0)}{2(L-L_0)^2}$ Substituting the numbers gives k=5.54 kN/m

[3 POINTS]

3.2 Hence, calculate the maximum acceleration of the sphere.

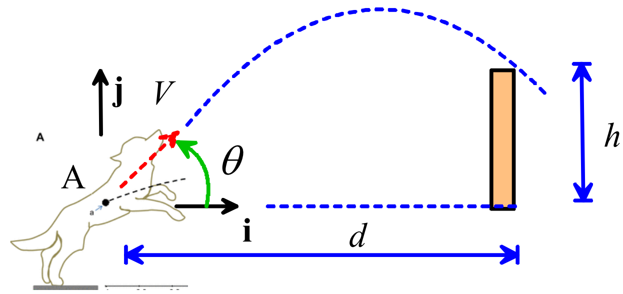
The max. acceleration occurs at the instant of max deflection. The force in each spring is $k(L-L_0)$. The vertical component of the force is $k(L-L_0)(h/L)$ (the h/L is the sin of the angle subtended by the spring and the horizontal). Newton's law vertically gives

$$ma_y = 4k(L-L_0)(h/L) - mg \Rightarrow a_y = 4(k/m)(L-L_0)(h/L) - g$$

Substituting numbers gives 68.61 m/s^2

[3 POINTS]

4 In this publication it is shown that when dogs jump over an obstacle, they select a jump trajectory that minimizes the energy required to clear the obstacle. The goal of this problem is to find formulas for the angle and energy of the optimal trajectory, as a function of the distance and height of the obstacle (presumably dogs do this calculation in their heads).



Suppose that the dog jumps so that its COM follows the trajectory shown: at time $t=0$ it has speed V and an angle θ , and just clears the obstacle at a distance d and height h an unknown time t later.

4.1 Use the trajectory equations to find a formula for the minimum jump speed V necessary to clear the obstacle, in terms of g,d,h and the jump angle θ (one way to do this is to use the **i** component of the trajectory equation to find the time required to reach the obstacle, and substitute this into the **j** component and then solve for V)

The trajectory must pass through $x=d$, $y=h$, so the trajectory equations show

$$V\cos\theta t = d$$

$$V\sin\theta t - \frac{1}{2}gt^2 = h$$

The first equation can be used to find t and then substituting into the second gives

$$t = d/(V\cos\theta)$$

$$d\tan\theta - \frac{1}{2}g\left(\frac{d}{V\cos\theta}\right)^2 = h$$

$$\Rightarrow d\sin\theta - h\cos\theta = \frac{gd^2}{2V^2\cos\theta} \Rightarrow V = \sqrt{\frac{gd^2}{2\cos\theta(d\sin - h\cos\theta)}}$$

[2 POINTS]

4.2 Hence, show that the energy (i.e. the KE of the dog just after jumping) required to clear the obstacle can be expressed as

$$\frac{E}{mgh} = \frac{(d/h)^2}{4\cos\theta((d/h)\sin\theta - \cos\theta)}$$

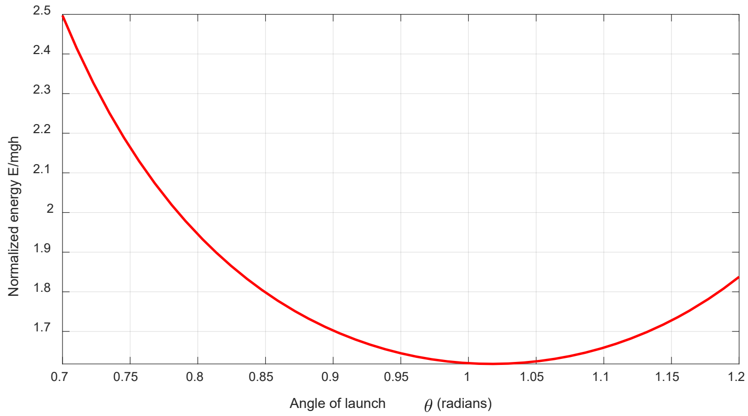
We can calculate the energy from the KE at the instant of launch:

$$E = \frac{1}{2}mV^2 = \frac{mgd^2}{4\cos\theta(d\sin\theta - h\cos\theta)}$$

$$\Rightarrow \frac{E}{mgh} = \frac{(d/h)^2}{4\cos\theta((d/h)\sin\theta - \cos\theta)}$$

[2 POINTS]

4.3 Plot a graphs of E / mgh as a function of θ for $d / h = 2$, with $0.7 < \theta < 1.2$. Note that there is a critical angle that minimizes the energy. (There is no need to submit MATLAB code, just submit the plot)



[2 POINTS]

4.4 Find a formula for the optimal angle in terms of *d/h*.

It is easiest to maximize mgh/E rather than minimize E/mgh – this gives

$$\frac{d}{d\theta}\left[\cos\theta((d/h)\sin\theta - \cos\theta)\right] = -\sin\theta((d/h)\sin\theta - \cos\theta) + \cos\theta((d/h)\cos\theta + \sin\theta)$$

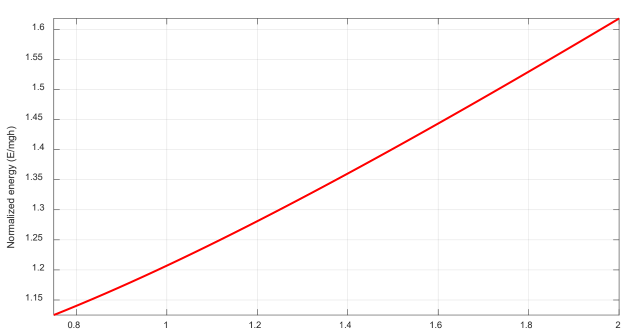
$$= 2\sin\theta\cos\theta + (d/h)(\cos^2\theta - \sin^2\theta)$$

$$= \sin 2\theta + (d/h)\cos 2\theta = 0$$

So $\theta = \frac{1}{2}\tan^{-1}(-d/h)$

[3 POINTS]

4.5 Hence, plot a graph of E / mgh as a function of *d/h*, for 0.7<d/h<2. Be careful with the range of θ : 4.4 has solutions with both positive and negative values of θ but of course only the positive values are relevant. There is no need to submit matlab code.

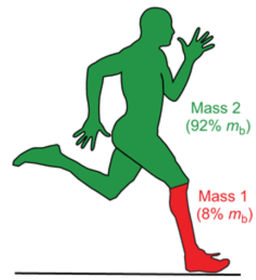


[2 POINTS]

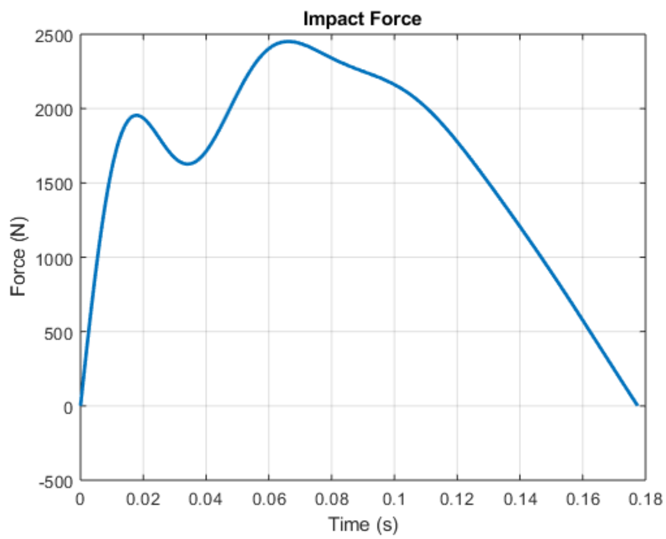
4.6 (optional) If you have a dog, please submit a picture (pictures of other pets may be submitted if you don’t have a dog)!

[0 POINTS]

5. This csv file stores the results of a force-plate measurement of the force acting on a runner's foot while it is in contact with the ground (estimated from this publication). The first column in the file is time (in seconds); the second column is the measured force (in Newtons). The runners mass is 70kg.



5.1 Write a MATLAB script to read the file and plot the force as a function of time, and also calculate the total impulse exerted by the contact force.



The data can be integrated using the MATLAB 'trapz' function which gives I=289.9 Ns

[2 POINTS]

5.2 Hence, calculate the vertical velocity of the runner just before and just after the impact with the ground (assume steady running, and neglect energy loss during the airborne phase of the stride)

We know that while airborne the runner's COM follows the trajectory equations – these predict that the take-off and landing speeds are the same (but in opposite directions). During the time in contact with the ground the runner is subjected to two forces: the reaction measured by the force-plate, and gravity. The reaction force exerts an upwards impulse of 289.9Ns; gravity exerts a downwards impulse of $mgt_{contact}$ where $t_{contact} = 0.18s$ is the time that the foot is in contact with the ground. The impulse-momentum relation relates the total impulse (reaction force upwards and gravity downwards) to the change in momentum. Taking **j** to be vertically upwards, the impulse-momentum relation for the period starting just before impact with the ground and ending just after gives

$$I^{Total}\mathbf{j} = mV\mathbf{j} - m(-V\mathbf{j}) = 2mV\mathbf{j} \Rightarrow V = I^{Total} / (2m) = (289.9 - 70 \times 9.81 \times 0.18) / (2 \times 70) = 1.188 m/s$$

[2 POINTS]

5.5 Calculate the number of foot strikes per second (assume each stride is identical). There is more than one way to solve this problem - try to do it with the impulse-momentum equations…

The momentum at the end of each stride is always the same (steady state; so the runner must have the same horizontal and vertical components of velocity) so the total impulse on the runner must be zero.

The runner experiences impulses from gravity and the impact force at the ground. The total is $I\mathbf{j} - mgT\mathbf{j} = 0$ where T is the time for one stride and I is the impulse measured from the force-plate. Therefore $T = I / mg = 0.422s$

Alternatively we can calculate the time that the runner is airborne using the solution to 5.2 and the straight line motion formulas. Just after take-off the runner has vertical speed 1.188 m/s and at the top of the trajectory has zero vertical speed, so the time to reach the top of the trajectory is
$1.188 / g = 0.1211s$

The total time airborne is twice this value (since the runner has to fall back to the ground). The time for 1 stride is then the time airborne plus the time on the ground $t_{contact}$. This gives
$T = 2 \times 0.1211 + 0.18 = 0.422s$

The number of steps per sec is therefore 1/0.422=2.36

[2 POINTS]

6. This website suggests that increasing the restitution coefficient between a golf club and ball from e=0.78 to e=0.822 will add 21 yards to a drive with a V=100mph club tip speed (just before impact). Check this estimate by calculating the flight distance. Use the following approach and data:

- Golf ball mass m_B=*45.93* grams
- Club head mass m_C = 200grams
- Idealize the collision between the club and head as a straight line collision; neglect any force exerted on the head of the club by the driver shaft during the collision
- Use the trajectory equations without air resistance to estimate the flight distance (this is not very accurate – it is not hard to incorporate lift and drag forces with MATLAB calculations but it's not worth the trouble….). Assume that the trajectory starts at a 45 degree angle.

Show that the additional distance resulting from changing the restitution coefficient from e_1 to e_2 is

$$\Delta x = \frac{\left[(1+e_2)^2 - (1+e_1)^2\right]}{\left(1+(m_B/m_C)\right)^2}\frac{V^2}{g}$$

We can use the straight-line collision formulas to calculate the speed of the golf ball after the impact. Let mass A denote the club head; mass B denote the ball. We have that $v_x^{A0} = 100mph \quad v_x^{B0} = 0$. For this case the linear momentum and the restitution formula are

$$m_A v_x^{A0} = m_A v_x^{A1} + m_B v_x^{B1} \qquad v_x^{A1} - v_x^{B1} = -e v_x^{A0}$$

To solve these for v_x^{B1} we can divide the first equation by m_A and subtract the second from the (modified) first equation

$$(1+e)v_x^{A0} = \left(1 + \frac{m_B}{m_A}\right)v_x^{B1} \qquad \Rightarrow v_x^{B1} = \frac{(1+e)}{1+(m_B/m_A)}v_x^{A0}$$

We can now calculate the distance traveled with the trajectory formulas – these give the position of the ball as

$$\mathbf{r} = v_x^{B1}\cos(45)t\mathbf{i} + \left(v_x^{B1}\sin(45)t - \frac{1}{2}gt^2\right)\mathbf{j}$$

At the end of the flight the height is zero, therefore

$$t = 2v_x^{B1}\sin(45)/g \Rightarrow x = 2\left(v_x^{B1}\right)^2\sin(45)\cos(45)/g = \left(v_x^{B1}\right)^2/g$$

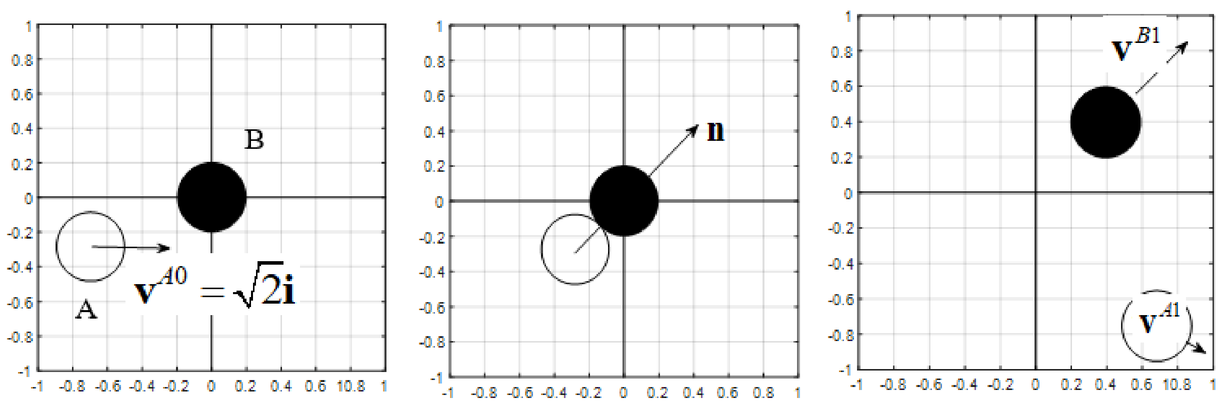
$$\Rightarrow x = \left[\frac{(1+e)}{1+(m_B/m_A)}\right]^2\frac{\left(v_x^{A0}\right)^2}{g}$$

The change in distance traveled resulting from the change in restitution coefficient follows as

$$\Delta x = \frac{\left[(1+e_2)^2-(1+e_1)^2\right]}{\left(1+(m_B/m_A)\right)^2}\frac{\left(v_x^{A0}\right)^2}{g}$$

Substituting numbers gives an additional distance of 20.4m, this is 21.8 yards. Our estimate is very close to what's advertised on the link.

[5 POINTS]

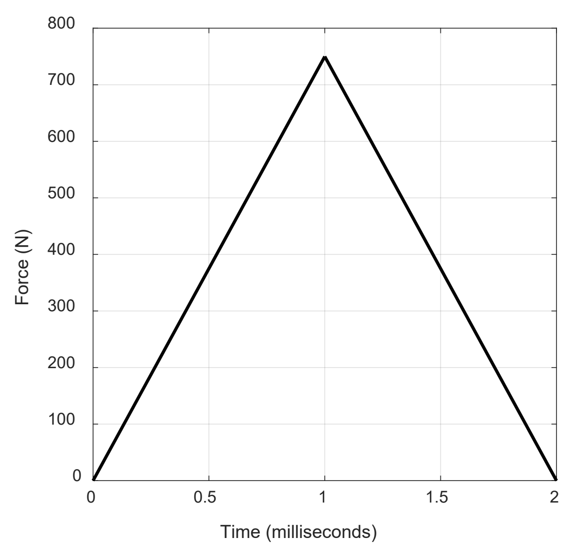


Before impact At impact After impact

7. The figure shows a frictionless oblique impact between two spheres. The spheres both have mass 1 kg. At time $t=0$ sphere B is at rest, and sphere A has velocity vector $\mathbf{v}^{A0} = \sqrt{2}\mathbf{i}$ m/s. The collision direction $\mathbf{n}$ is at 45 degrees to the $\mathbf{i}$ and $\mathbf{j}$ directions.

7.1 The figure shows the magnitude of the force acting on the particles at the point of contact during the collision. Calculate the magnitude of the impulse.

The impulse is the area under the curve, i.e. 0.75 Ns.



[1 POINT]

7.2 Explain why the direction of the impulse on each individual particle during the collision must be parallel to $(\mathbf{i}+\mathbf{j})$

The contact force during the impact acts parallel to the line connecting the two centers (because the impact is frictionless).

[1 POINT]

7.3 Hence, calculate the velocity vectors for particles A and B after the collision

We can use the impulse-momentum relations for a single particle.

For A $\mathbf{I}_A = m(\mathbf{v}^{A1} - \mathbf{v}^{A0}) \Rightarrow -\frac{3}{4}\frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}} = 1\times\left(\mathbf{v}^{A1} - \sqrt{2}\mathbf{i}\right) \Rightarrow \mathbf{v}^{A1} = -\frac{3}{4}\frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}} + \frac{2\mathbf{i}}{\sqrt{2}} = \frac{5\mathbf{i}-3\mathbf{j}}{4\sqrt{2}}$

For B $\mathbf{I}_B = m\left(\mathbf{v}^{B1} - \mathbf{v}^{B0}\right) \Rightarrow \frac{3}{4}\frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}} = 1\times\mathbf{v}^{B1}$

[3 POINTS]

7.4 Calculate the restitution coefficient for the collision.

We can calculate the restitution coefficient from the normal components of velocity

$$e = -\frac{\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right)\cdot\mathbf{n}}{\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}} = -\frac{\left(-2\mathbf{i}+6\mathbf{j}\right)/(4\sqrt{2})\cdot(\mathbf{i}+\mathbf{j})/\sqrt{2}}{(-\mathbf{i}\sqrt{2})\cdot(\mathbf{i}+\mathbf{j})/\sqrt{2}} = \frac{1}{2}$$

[2 POINTS]