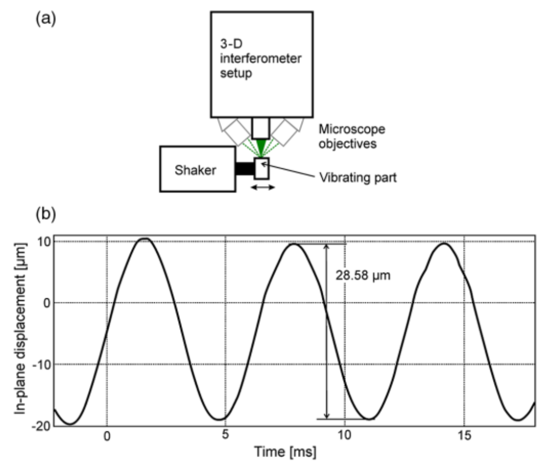
EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday March 20, 2020

School of Engineering
Brown University

1. The figure (from this publication) shows a vibration measurement from vibrometer microscope. Use the figure to estimate



1.1 The amplitude of the displacement

From the graph, $X_0 = 26.58/2 = 13.29\mu m$

[1 POINT]

1.2 The period of the vibration

There are 2 cycles in 12.5 millisec, so the period is 6.25 millisecs.

[1 POINT]

1.3 The frequency (in Hertz) and angular frequency (in rad/s)

The frequency is 1/*T*=160Hz, or $\frac{2\pi}{T} = 1005.3 rad/s$ rad/s

[1 POINT]

1.4 The amplitude of the velocity

The simple harmonic motion formulas give $V = \omega X_0 = 1005.3 \times 13.29 \times 10^{-6} = 0.01336 m/s$

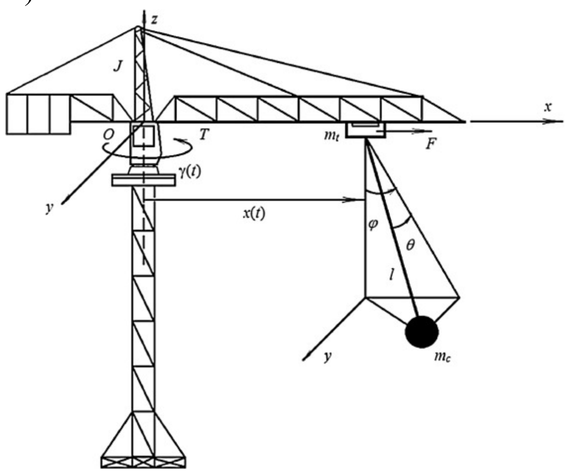
[1 POINT]

1.5 The amplitude of the acceleration

The simple harmonic motion formulas give
$A = \omega V = 0.008/2\pi = 13.43 m/s^2$

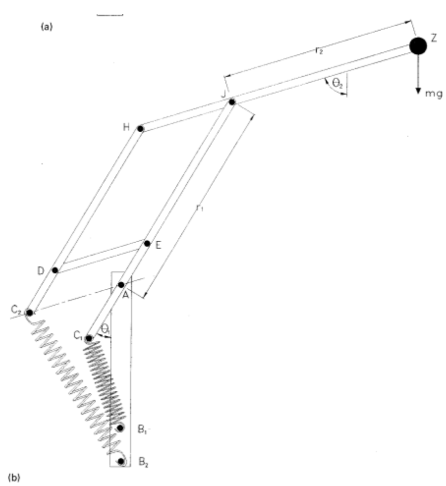
[1 POINT]

2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures (you may need to consult the publications to understand the system)

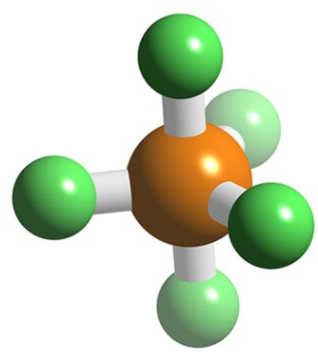


(a) Theoretical model of a tower crane (you may need to check the publication for a clearer description of the system than the figure

(b) Universal Stage



(c) AnglePoise Lamp (treat the system as 2D)



(d) Phosphorus pentafluoride molecule

(a) The DOF are indicated by the coordinates on the figure – (1) rotation of the boom about the slewing bearing; (2) Motion of the trolley along the boom; (3) extending the cable (4,5) Two angles defining the orientation of the cable, for a total of 5DOF (the length of the cable could be an additional DOF but the paper seems to make l constant – but 6DOF is an acceptable answer). There are either no rigid body modes in this system or one or two (the crane is anchored to the ground and so cant translate indefinitely but the slewing bearing may allow the boom to rotate in one direction forever, however, and the traveler thing on the boom could also keep moving horizontally without vibration. It's more likely there is a motor attached to these though.).

(b) Each bearing can rotate about its axis, and no other motion is possible. According to the product description online the specimen (the triangular thing) can also rotate in the holder about the horizontal axis. So 3 or 4 DOF. Or 3 (or 4) rigid bodies (the 3 curved members possibly plus specimen) and 3 or 4 joints with 5 constraints each – So #DOF = 3*6 - 3*5= 3. Or if you include the specimen/additional joint 4DOF. Either no rigid body modes if the joints don't permit steady rotation forever (ie there's a torsional spring of some sort restraining motion) or possibly 3 or 4 rigid body modes if each joint allows rotation about its axis forever. So either 3,4 or no vibration modes… All these answers are fine for grading purposes as long as the reasoning is clear.
(c) There are 4 2D rigid bodies with 3DOF each (12 DOF) 5 pin joints with 2 constraints each (10 constraints) so 2DOF total (they are the two angles shown in the figure). No rigid body modes so 2 vibration modes.

(d) 6 atoms (particles), 3 DOF each = 18 DOF. There are 6 rigid body modes so 12 vibration modes.

[8 POINTS]

3. Solve the following differential equations (please solve them by hand, using the tabulated solutions to differential equation – you can check the answers with matlab if you like)

3.1 $\frac{1}{5}\frac{d^2y}{dt^2}+5y=0 \qquad y=1 \qquad \frac{dy}{dt}=0 \qquad t=0$

3.2 $\frac{d^2y}{dt^2}-4y=-4 \qquad y=1 \qquad \frac{dy}{dt}=-1 \qquad t=0$

3.3 $\frac{1}{4}\frac{d^2y}{dt^2}+2\frac{dy}{dt}+4y=32\sin(t)+64\cos(t) \qquad y=0 \qquad \frac{dy}{dt}=0 \qquad t=0$

3.1
Rearrange in standard form

$$\frac{1}{25}\frac{d^2y}{dt^2}+y=0$$

This is a Case I equation - compare with the standard form to see that $\omega_n=5 \qquad C=0$

The solution is

$$x(t)=C+(x_0-C)\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t$$

We are given $x_0=1 \qquad v_0=0$ so

$$y(t)=\cos 5t$$

[3 POINTS]

3.2

Rearrange in standard form

$$\frac{1}{4}\frac{d^2y}{dt^2} - y = -1$$

This is a Case II equation - compare with the standard form to see that $\alpha = 2 \qquad C = 1$

. The solution is

$$x(t) = C + \frac{1}{2}\left(\left(x_0 - C\right) + \frac{v_0}{\alpha}\right)\exp(\alpha t) + \frac{1}{2}\left(\left(x_0 - C\right) - \frac{v_0}{\alpha}\right)\exp(-\alpha t)$$

We are given $x_0 = 1 \quad v_0 = -1$ so

$$y(t) = 1 - \frac{1}{4}\exp(2t) + \frac{1}{4}\exp(-2t)$$

[3 POINTS]

3.3 $\frac{1}{4}\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = 32\sin(t) + 16\cos(t) \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

We can rearrange this as a Case 6 equation

$$\frac{1}{4^2}\frac{d^2y}{dt^2} + \frac{2\times 1}{4}\frac{dy}{dt} + y = 8(\sin t + \frac{2\times 1}{4}\cos(t))$$

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = C + K\left(z + \frac{2\zeta}{\omega_n}\frac{dz}{dt}\right) \qquad z = \sin t$$

It appears that $\omega = 1,\ K = 8,\ \omega_n = 4,\ \zeta = 1\ C = 0$.

The steady-state solution follows as

$$x_p(t) = X_0 \sin\left(\omega t + \phi\right) \qquad X_0 = KY_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{\left\{1 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi = \tan^{-1}\frac{-2\varsigma\omega^3/\omega_n^3}{1 - (1 - 4\varsigma^2)\omega^2/\omega_n^2} \quad (-\pi < \phi < 0)$$

The homogeneous solution is

$$x_h(t) = \left\{x_0^h + \left[v_0^h + \omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

with

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin\phi$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0 \omega \cos\phi$$

$$\text{Substituting numbers gives} \quad \begin{matrix} M = 1.0523 & \phi = -0.0263 & X_0 = 8.4181 \\ x_0^h = 0.2215 & v_0^h = -8.4152 & \end{matrix}$$

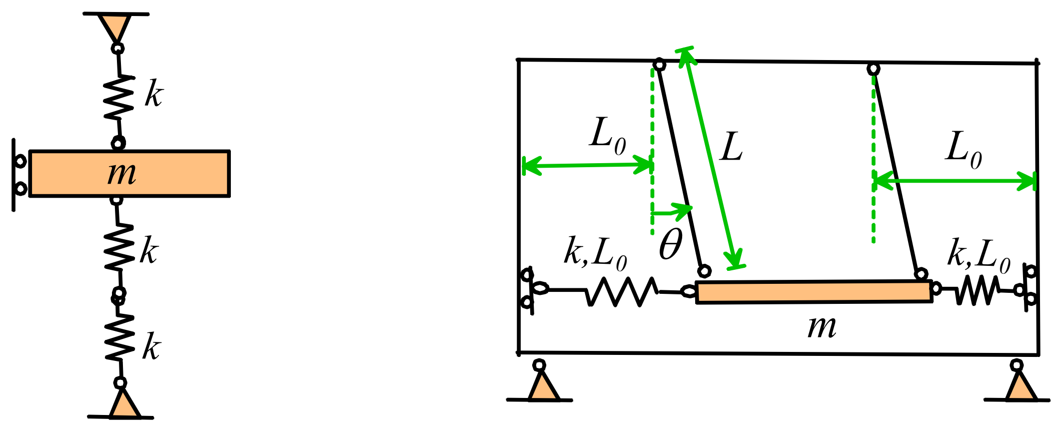
The total solution is therefore

$$y(t) = 8.4181\sin(t - 0.0263) + \exp(-4t)\{0.2215 - 7.5294t\}$$

We can check that this is correct by substituting it into the differential equation, and by substituting *t=0* into *y* and *dy/dt* and checking that initial conditions are satisfied.

[3 POINTS]

4. Find formulas for the natural frequency of vibration for the systems shown in the figure



For the first system, we can replace the springs with an equivalent single spring. The two springs connected end to end are in series, so

$$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{eff} = k/2$$

The single spring is in parallel the pair of springs, so the effective stiffness of the entire assembly is $3k/2$.

The formula for natural frequency gives $\omega = \sqrt{3k/(2m)}$

[2 POINTS]

We can get an EOM for the second system using the energy method. The platform is in circular motion, so its speed (from the circular motion formula) is

$$v = L\left(\frac{d\theta}{dt}\right)$$

and therefore the KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$

The PE includes gravity and the energy of the springs. Geometry shows that the spring lengths are $L_0 + L\sin\theta,\ \ L_0 - L\sin\theta$ so

$$U = -mgL\cos\theta + \frac{1}{2}k\left(L_0 + L\sin\theta - L_0\right)^2 + + \frac{1}{2}k\left(L_0 - L\sin\theta - L_0\right)^2$$

$$T + U = const \Rightarrow \frac{d}{dt}(T+U) = 0$$

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) + mgL\sin\theta\frac{d\theta}{dt} + kL^2\sin 2\theta\frac{d\theta}{dt} = 0$$

$$mL^2\left(\frac{d^2\theta}{dt^2}\right) + mgL\sin\theta + kL^2\sin 2\theta = 0$$

(here we used the formula $2\sin\theta\cos\theta = \sin 2\theta$ to make finding the small angle approximation easier but its fine to leave this term as just $2\sin\theta\cos\theta$)

To linearize the equation just set $\sin\theta \approx \theta,\ \ \sin 2\theta \approx 2\theta$, which gives

$$mL^2\left(\frac{d^2\theta}{dt^2}\right) + \left(2kL^2 + mgL\right)\theta = 0$$

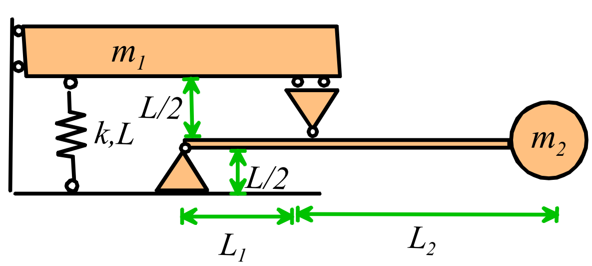
$$\Rightarrow \frac{mL^2}{\left(2kL^2 + mgL\right)}\left(\frac{d^2\theta}{dt^2}\right) + \theta = 0$$

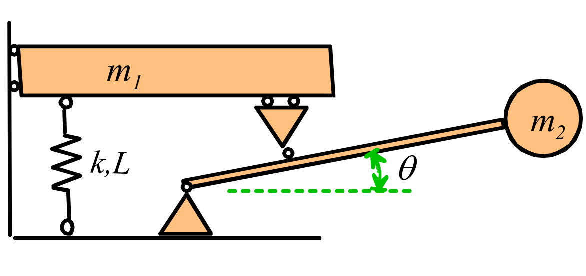
and compare to the standard case I EOM to see that

$$\omega_n = \sqrt{\frac{2kL^2 + mgL}{mL^2}}$$

[3 POINTS]

5. The figure shows an 'anti-resonant' vibration isolation system (we'll analyze its behavior in more detail in next week's HW).





5.1 Find formulas for the kinetic and potential energy of the system in terms of the angle θ and its time derivatives. **Neglect gravity**.

The length of the spring is (simple geometry) $y = L + L_1 \sin\theta$.

The velocity of the mass m_1 follows as $\frac{dy}{dt} = L_1 \cos\theta \frac{d\theta}{dt}$

Mass m_2 is in circular motion about the left most pivot, so its speed is $v = (L_1 + L_2)\left(\frac{d\theta}{dt}\right)$

Therefore

$$U = \frac{1}{2}k(L + L_1 \sin\theta - L)^2 = \frac{1}{2}kL_1^2 \sin^2\theta$$

$$T = \frac{1}{2}m_1\left(L_1 \cos\theta \frac{d\theta}{dt}\right)^2 + \frac{1}{2}m_2\left((L_1 + L_2)\frac{d\theta}{dt}\right)^2$$

[2 POINTS]

5.3 Hence, find the equation of motion for θ

The system is conservative, so

$$T + U = const \Rightarrow \frac{d}{dt}(T + U) = 0$$

$$m_1 L_1^2\left(-\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^3 + \cos^2\theta\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right) + m_2(L_1 + L_2)^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + kL_1^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0$$

We can simplify this to

$$\left(m_1 L_1^2\cos^2\theta + m_2(L_1 + L_2)^2\right)\frac{d^2\theta}{dt^2} - m_1 L_1^2\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2 + kL_1^2\sin\theta\cos\theta = 0$$

[2 POINTS]

5.4 Linearize the equation for small θ, and hence find the natural frequency of the system. (Your equation of motion will include a term that is multiplied by $\sin\theta(d\theta/dt)^2$ - for small harmonic θ , the angular velocity $d\theta/dt$ is also small, so when the equation of motion is linearized this term is neglected)

Using the usual trig approximations for small angles and neglecting the quadratic term in $d\theta / dt$

$$\left(m_1 L_1^2 + m_2 (L_1 + L_2)^2\right)\frac{d^2\theta}{dt^2} + kL_1^2\theta = 0$$

Hence

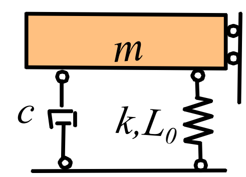
$$\frac{\left(m_1 L_1^2 + m_2 (L_1 + L_2)^2\right)}{kL_1^2}\frac{d^2\theta}{dt^2} + \theta = 0$$

So the natural frequency is

$$\sqrt{\frac{kL_1^2}{\left(m_1 L_1^2 + m_2 (L_1 + L_2)^2\right)}}$$

[2 POINTS]

6. The spring-mass-dashpot system shown in the figure has an undamped natural frequency $\omega_n = 100 rad / s$ and damping factor $\zeta = 0.1$. The spring has stiffness $k = 10^3 N / m$.



6.1 Calculate the mass *m* and dashpot coefficient *c*.

Using the standard formulas

$$\omega_n = \sqrt{\frac{k}{m}} = 100 rad / s \quad \Rightarrow m = \frac{k}{\omega_n^2} = \frac{10^3}{10^4} = 0.1kg$$

$$\zeta = \frac{c}{2\sqrt{km}} = 0.1 \Rightarrow c = 2\zeta\sqrt{km} = 2Ns / m$$

[2 POINTS]

6.2 The system is released from rest in the vertical configuration shown. At time *t=0* the spring is free of force. Plot a graph of the subsequent vertical displacement of the mass from its initial position (in mm) as a function of time (for 0<t<0.5s). Include the effects of gravity. You only need to submit relevant calculations and the plot, a MATLAB upload is not required.

The equation of motion for the spring mass system (with gravity) is

$$\frac{m}{k}\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x = -\frac{mg}{k}$$

Comparing this to the case V EOM we see that

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\sqrt{km}} \qquad C = -\frac{mg}{k}$$

The system is underdamped, therefore the solution is

$$x(t) = C + \exp(-\varsigma\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \varsigma\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$$

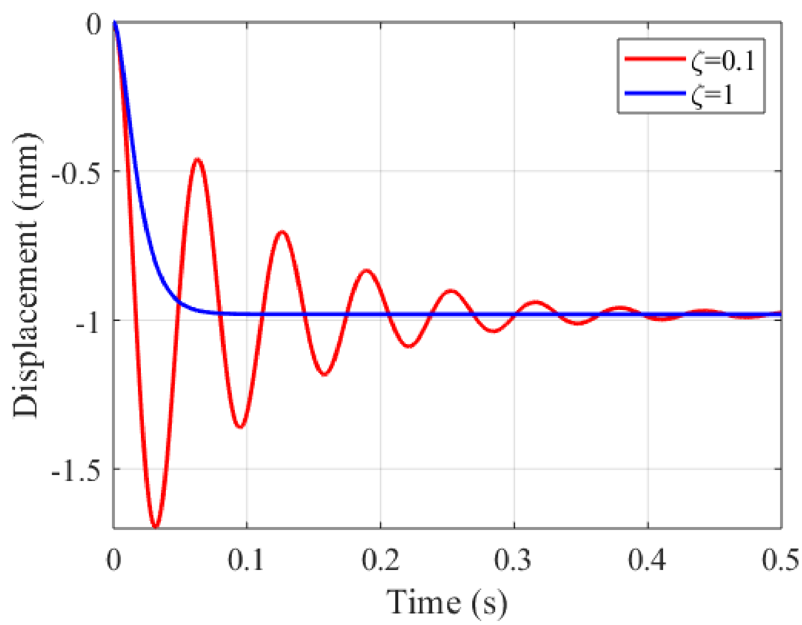
where $\omega_d = \omega_n\sqrt{1-\varsigma^2}$

The plot is shown below.

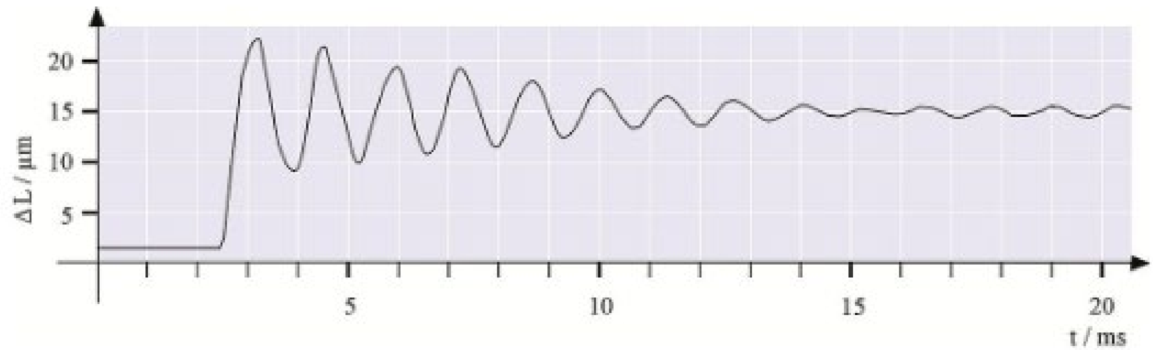
6.3 What value of the dashpot coefficient c is required to make the system critically damped?

For critical damping $\zeta = 1 \Rightarrow c = 20$ Ns/m .

The figure below shows the critically damped solution along with the solution for part 6.2 – this was not asked for in the problem.



[2 POINTS]



7. The figure (from this product spec) shows the measured damped vibration response of a piezoelectric actuator.

7.1 Calculate the period and log decrement for the signal

There are 3 cycles in 4 milliseconds – the period is (4/3) milliseconds.

We have to measure the displacement from the steady-state position (15 microns) for the formulas to work. So the first peak has height 7.5 microns; the 6th peak is about 2 microns. The log decrement follows as

$$\delta = \frac{1}{5}\ln\left(\frac{7.5}{2}\right) = 0.2644$$

[2 POINTS]

7.2 Hence determine the undamped natural frequency ω_n and damping coefficient ζ for the system

We can use the formulas from class -

$$\omega_n = \frac{\sqrt{4\pi^2+\delta^2}}{T} = 4.716\times 10^3 \quad rad/s$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = 0.042$$

[2 POINTS]