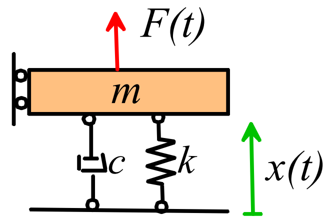
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 3, 2020

1. The spring-mass system shown in the figure has

- m=40kg,
- $k = 38.6kN/m$
- $c = 0.2518kNs/m$



The mass vibrates harmonically with an amplitude of 1mm and angular frequency $\omega = 100rad/s$.

1.1 What is the amplitude of the force?

This is a standard spring-mass system so there is no need to derive the EOM; we can use the solution from the notes. The formula for vibration amplitude is

$$X_0 = KF_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}$$

$$\omega_n = \sqrt{k/m} \qquad \zeta = c/(2\sqrt{km}) \qquad K = 1/k$$

Substituting numbers (see Live script) gives $F_0 = 362N$

```
clear all
m = 40; k = 38.6e03; c = 0.2518e03;
wn = sqrt(k/m);
z = c/(2*sqrt(k*m));
K = 1/k;
w = 100;
X0 = 1.e-03;
M = 1/sqrt((1-w^2/wn^2)^2 + (2*z*w/wn)^2);
F0 = X0/(K*M)
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F0 = 362.2761
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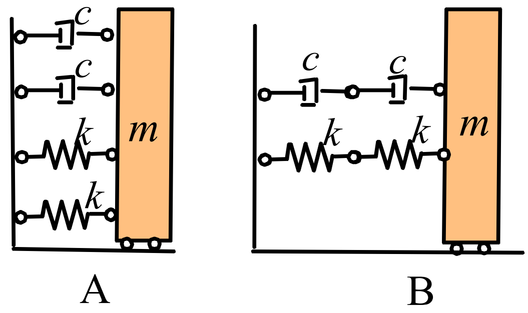
[3 POINTS]

1.2 What is the frequency of the force that yields the maximum vibration amplitude? If the force amplitude is the same as the value computed in 1.1, what is the resulting vibration amplitude?

The ζ value (from 1.1) is 0.1 - this is small enough to use the approximations $\omega = \omega_n$ $M = 1/(2\zeta)$ at the resonant peak. Substituting numbers gives $X0 = 46.3mm$

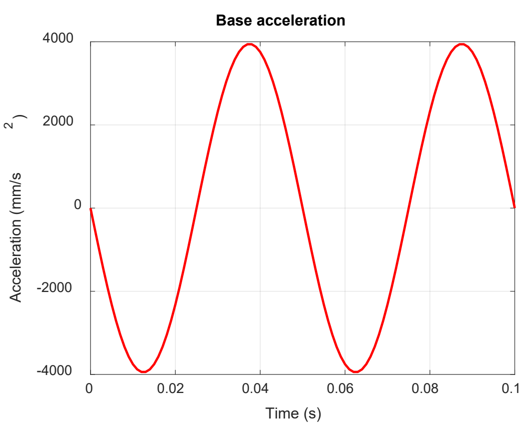
[1 POINT]

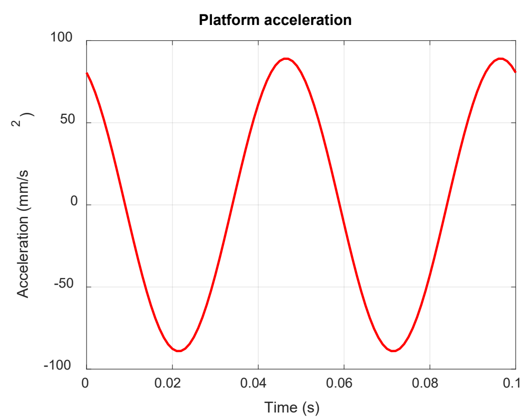
2. System A has natural frequency and damping coefficient ω_n, ζ. Find the natural frequency and damping coefficient of system B



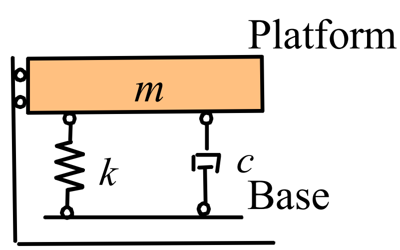
The springs/dashpots in A are in parallel; in B they are in series. Therefore $\omega_{nA} = \sqrt{2k/m}$ $\zeta_A = 2c/(2\sqrt{2km})$ and $\omega_{nB} = \sqrt{k/2m}$ $\zeta_B = c/4\sqrt{km/2}$ giving $\omega_{nB} = \omega_n/2$, $\zeta_B = \zeta/2$

[3 POINTS]





3 The figure shows readings from accelerometers attached to the base and platform of a vibration isolation table.



3.1 Find

(1) The period of the vibration;

There are 2 cycles in 0.1s so the period is 0.05s

[1 POINT]

(2) The frequency of the vibration (in both Hz and radians per second);

The frequency is 20Hz, or 126 rad/s

[1 POINTS]

(3) The amplitude of the displacement (in mm) of the base

The acceleration amplitude of the base is 4 m/s^2. The amplitude of the displacement is $Y_0 = A_0 / \omega^2 = 0.25mm$

[1 POINT]

(4) The amplitude of the displacement (in mm) of the platform

The acceleration amplitude of the platform is 85 mm/s^2. The amplitude of the displacement is $X_0 = A_0 / \omega^2 = 0.0054mm$

[1 POINT]

(5) The magnification.

The magnification is $X_0 / Y_0 = 0.0054 / 0.25 = 0.022$

[1 POINT]

3..2 In a separate experiment, the natural frequency of vibration of the table is measured to be 2Hz. Use this information and your solution to 21.1 to calculate the damping coefficient ζ for the platform

We know that the magnification is $M = \dfrac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2+(2\zeta\omega/\omega_n)^2}}$

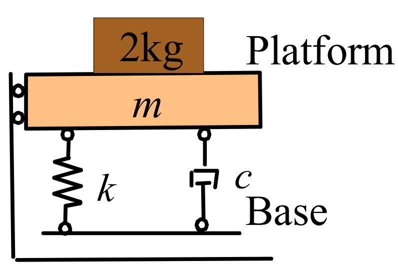
$$M^2\left[\left(1-\omega^2/\omega_n^2\right)^2+(2\zeta\omega/\omega_n)^2\right]=1+(2\zeta\omega/\omega_n)^2$$

Hence $\Rightarrow (2\zeta\omega/\omega_n)^2\left[1-M^2\right]=M^2\left(1-\omega^2/\omega_n^2\right)^2-1$

$$\Rightarrow \zeta = \frac{\omega_n}{2\omega}\sqrt{\frac{M^2\left(1-\omega^2/\omega_n^2\right)^2-1}{\left[1-M^2\right]}} \approx \frac{M\omega}{2\omega_n} = 0.1$$

[3 POINTS]

3.3 A 2kg mass is placed on the table. It is found that the natural frequency of the table with the mass on its surface decreases to 1Hz. Calculate the value of the table mass *m*, the spring constant *k* and the dashpot coefficient *c*.



Using the formulas for natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow 2\times 2\pi = \sqrt{\frac{k}{m}} \qquad 2\pi = \sqrt{\frac{k}{m+2}}$$

$$\Rightarrow 4 = \frac{m+2}{m} \Rightarrow m = \frac{2}{3}kg$$

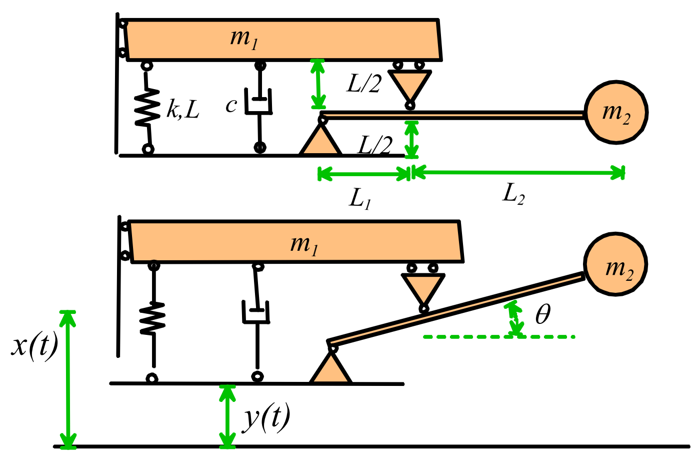
$$k = 16\pi^2 m = 105N/m$$

The formula for damping coefficient gives

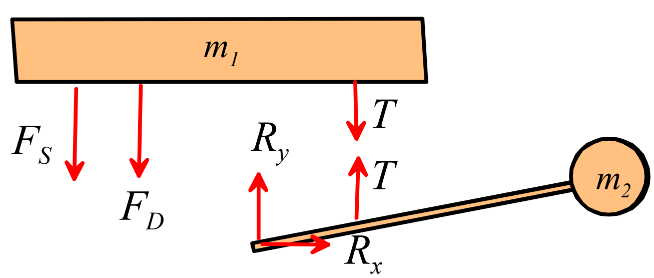
$$\zeta = \frac{c}{2\sqrt{km}} \Rightarrow c = 0.1\times 2\sqrt{105\times 2/3} = 1.7Ns/m$$

[2 POINTS]

4. In this problem we will analyze the behavior of the 'anti-resonant' vibration isolation system introduced in Homework 5. The system is illustrated in the figure. Assume that the base vibrates vertically with a displacement $y(t) = Y_0 \sin \omega t$. Our goal is to calculate a formula for the steady-state vertical motion $x(t)$ of the platform, and to compare the behavior of this system with the standard base excited spring-mass-damper design for an isolation system.



4.1 Draw free body diagrams showing the forces acting on the mass m_1 and the pendulum assembly (see the figure).



(Graders - OK to omit the horizontal reaction)

[3 POINTS]

4.2 Using geometry, find an expression for the acceleration of mass m_2 in terms of θ and y (and their time derivatives as well as relevant geometric constants) (You could write down a formula for the position vector relative to a fixed origin and differentiate it). Show that if θ and its time derivatives are small the result can be approximated by

$$\mathbf{a} \approx \left[\frac{d^2y}{dt^2} + (L_1+L_2)\frac{d^2\theta}{dt^2} \right]\mathbf{j}$$

Show also that

$$\frac{d^2x}{dt^2}-\frac{d^2y}{dt^2}\approx L_1\frac{d^2\theta}{dt^2}$$

The position vector of m_2 is

$$\mathbf{r}=(L_1+L_2)\cos\theta\mathbf{i}+(y+L/2+(L_1+L_2)\sin\theta)\mathbf{j}$$

Differentiate twice with respect to time to get

$$\mathbf{a}=-(L_1+L_2)\left\{\left(\frac{d\theta}{dt}\right)^2\cos\theta+\frac{d^2\theta}{dt^2}\sin\theta\right\}\mathbf{i}+\left[\frac{d^2y}{dt^2}+(L_1+L_2)\left\{-\left(\frac{d\theta}{dt}\right)^2\sin\theta+\frac{d^2\theta}{dt^2}\cos\theta\right\}\right]\mathbf{j}$$

Using the approximations $\cos\theta\approx 1$ $\quad\sin\theta\approx\theta$ and neglecting squared or higher order products of θ and its time derivatives we get the result stated.

Similarly

$$x-y=L+L_1\sin\theta$$

Differentiate this twice

$$\frac{d^2x}{dt^2}-\frac{d^2y}{dt^2}=L_1\left(-\left(\frac{d\theta}{dt}\right)^2\sin\theta+\frac{d^2\theta}{dt^2}\cos\theta\right)$$

Use the same approximation to get the stated result.

[3 POINTS]

4.3 For the pendulum, write down $\mathbf{F}=m\mathbf{a}$ and $\mathbf{M}=0$ about the center of mass, in terms of reaction forces shown in your FBD. Use the approximation in 3.2 for the acceleration.

For m_2 we have

$$R_x\mathbf{i}+(R_y+T)\mathbf{j}=+m_2\left[\frac{d^2y}{dt^2}+(L_1+L_2)\frac{d^2\theta}{dt^2}\right]\mathbf{j}$$

$$R_x(L_1+L_2)\sin\theta-R_y(L_1+L_2)\cos\theta-TL_2\cos\theta=0$$

(Graders – setting $\theta=0$ in the moment equation is OK since its obvious $R_x=0$ the trig terms cancel

[2 POINTS]

4.4 Write down $\mathbf{F}=m\mathbf{a}$ for mass m_1 , and hence use 4.3 and the second of 4.2 to show that (if θ and its time derivatives are small) then

$$\left(m_1+m_2\frac{(L_1+L_2)^2}{L_1^2}\right)\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=kL+m_2\frac{(L_1+L_2)L_2}{L_1^2}\frac{d^2y}{dt^2}+c\frac{dy}{dt}+ky$$

F=m**a** gives

$$m_1 \frac{d^2x}{dt^2} = -T - k(x-y-L) - c\frac{d}{dt}(x-y)$$

We can solve (4.3) for T to get

$$T = \frac{L_1+L_2}{L_1} m_2 \left(\frac{d^2y}{dt^2} + (L_1+L_2)\frac{d^2\theta}{dt^2} \right)$$

From 4.2 $\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} \approx L_1 \frac{d^2\theta}{dt^2}$ so we can rewrite this as

$$T = \frac{L_1+L_2}{L_1} m_2 \left(\frac{d^2y}{dt^2} + \frac{(L_1+L_2)}{L_1}\left[\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} \right] \right) = \frac{L_1+L_2}{L_1} m_2 \left(\frac{(L_1+L_2)}{L_1}\frac{d^2x}{dt^2} - \frac{L_2}{L_1}\frac{d^2y}{dt^2} \right)$$

Hence substitute for T and rearrange to get

$$\left(m_1 + m_2 \frac{(L_1+L_2)^2}{L_1^2} \right) \frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = kL + m_2 \frac{(L_1+L_2)L_2}{L_1^2}\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky$$

[3 POINTS]

4.5 Show that the equation can be re-arranged into the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = \frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y$$

and show that

$$\omega_n = \sqrt{\left(\frac{kL_1^2}{L_1^2 m_1 + m_2(L_1+L_2)^2} \right)} \qquad \lambda = \sqrt{\left(\frac{m_2(L_1+L_2)L_2}{L_1^2 m_1 + m_2(L_1+L_2)^2} \right)}$$

$$\zeta = \frac{c}{2\sqrt{k(m_1L_1^2 + m_2(L_1+L_2)^2)/L_1^2}}$$

We can re-write the equation as

$$\left(\frac{L_1^2 m_1 + m_2(L_1+L_2)^2}{kL_1^2} \right)\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x = \left(\frac{L_1^2 m_1 + m_2(L_1+L_2)^2}{kL_1^2} \right)\left(\frac{m_2(L_1+L_2)L_2}{L_1^2 m_1 + m_2(L_1+L_2)^2} \right)\frac{d^2y}{dt^2} + \frac{c}{k}\frac{dy}{dt} + y$$

Hence

$$\omega_n = \sqrt{\left(\frac{kL_1^2}{L_1^2 m_1 + m_2(L_1+L_2)^2}\right)} \qquad \lambda = \sqrt{\left(\frac{m_2(L_1+L_2)L_2}{L_1^2 m_1 + m_2(L_1+L_2)^2}\right)}$$

$$\zeta = \frac{c}{2\sqrt{k(m_1L_1^2 + m_2(L_1+L_2)^2)/L_1^2}}$$

[3 POINTS]

4.6 The solution to the EOM in part 4.5 can be found What is the frequency corresponding to the anti-resonance (the minimum value of *M*), in terms of λ, ω_n (give an approximate solution for $\zeta << 1$) ? What is (approximately) the smallest vibration amplitude (in terms of λ, ζ)?

The magnification (from the handout) is $M(\omega/\omega_n, \zeta) = \dfrac{\left\{\left(1-\lambda^2\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}$

The minimum will occur (approximately) when $\left(1-\dfrac{\lambda^2\omega^2}{\omega_n^2}\right) = 0 \Rightarrow \omega = \omega_n/\lambda$. The corresponding vibration amplitude is

$$M = \frac{\left(\frac{2\zeta}{\lambda}\right)}{\sqrt{\left(1-\frac{1}{\lambda^2}\right)^2 + \left(\frac{2\zeta}{\lambda}\right)^2}}$$

If λ is not close to 1, then

$$M \approx \frac{2\zeta}{\left|\left(\lambda - \frac{1}{\lambda}\right)\right|}$$

[2 POINTS]

4.7 For what range of frequency (in terms of λ, ω_n) does the antiresonant system give better performance than the simpler spring-mass-damper system?

The magnification for the anti-resonant isolator is equal to that of the conventional system when

$$\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2} = \sqrt{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

This gives

$$\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2=1$$

$$\Rightarrow \omega=0 \quad \text{or} \quad \frac{\lambda^2\omega^2}{\omega_n^2}=2\Rightarrow \omega=\frac{\sqrt{2}}{\lambda}\omega_n$$

The anti-resonant system is better than the conventional system for ω below this value. It only isolates vibrations if $\omega/\omega_n>\sqrt{2}$, however.

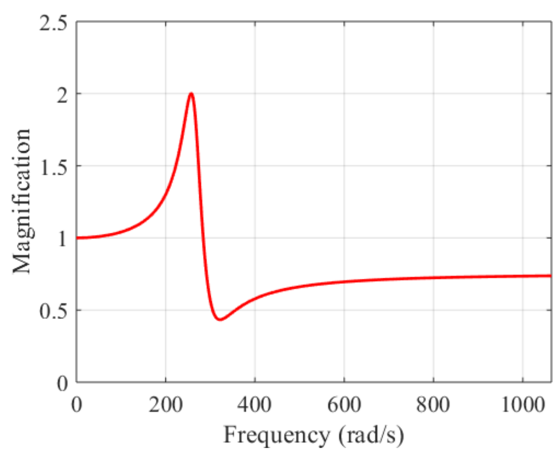
[2 POINTS]

4.8 This paper analyzes a design for an anti-resonant isolator to be used in a submarine. The isolator has the following specifications:

- $m_1=345.37kg \qquad m_2=23.65kg$
- $(L_1+L_2)/L_1=9$
- $k=160$ MN/m

The authors do not state the value they used for the dashpot coefficient *c*. Select a value for *c* that will give *M<2* for all frequencies, and gives the best vibration isolation at the anti-resonance (you can use any method you like to do this – trial and error, or if you want to be super-precise use more fancy math. For example you can find the frequency that maximizes *M* and then solve $\max(M)=2$ for *c*). For this value of *c* plot a graph showing the predicted magnification for the isolator.

A super-precise MATLAB calculation (see Live Script for details) gives $c=95.4kNs/m$. The resulting plot is shown below



[2 POINTS]