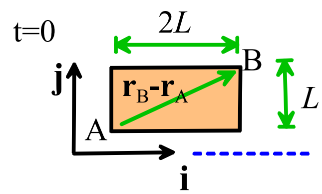
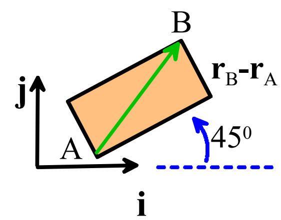
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 7: Rigid Body Kinematics, Inertial properties of rigid bodies
Due Friday April 17, 2020

1. The rigid body shown in the figure is at rest at time t=0, and rotates counterclockwise with constant angular acceleration of 8 rad/s^2. At the instant shown in the figure, find





1.1 The angular velocity vector

The block has rotated through $\pi/4$ radians and the angular acceleration is constant. The constant acceleration formula tells us that the time taken to reach the configuration shown is

$$\theta = \frac{1}{2}\alpha t^2 \Rightarrow t = \sqrt{2\theta/\alpha} = \sqrt{\pi}/4$$

Since the angular acceleration is constant, the angular velocity is $\boldsymbol{\omega} = \boldsymbol{\alpha} t = 8t\mathbf{k} = 2\sqrt{\pi}\mathbf{k}$

[1 POINT]

1.2 The spin tensor **W** (as a 2x2 matrix)

Using the formula

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} = 2\sqrt{\pi}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

[1 POINT]

1.3 The rotation tensor (a 2x2 matrix for a 2D problem) **R** that rotates the rectangle from its initial to its position at 2 sec.

The 2D rotation matrix is

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

[1 POINT]

1.4 Hence, find a formula for the vector $\mathbf{r}_B - \mathbf{r}_A$ in (**i**,**j**) components

$$\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2L \\ L \end{bmatrix} = \begin{bmatrix} L/\sqrt{2} \\ 3L/\sqrt{2} \end{bmatrix}$$

[1 POINT]

2. "Codman's paradox" is described in a 1935 medical textbook (p43). It's not really a paradox – but it does demonstrate the counter-intuitive nature of sequences of rotations. This paper has a clearer statement of the 'paradox,' than the original book, as follows: "first place your right arm hanging down along your side with your thumb pointing forward and your fingers pointing toward the ground. Next, elevate your arm horizontally so that your fingers point to the right, and then rotate your arm in the horizontal plane so that your fingers now point forward. Finally, rotate your arm downward so that your fingers eventually point toward the ground. After these three rotations, you will notice that your thumb points to the left. That is, your arm rotated by 90 degrees. The fact that you get this rotation without having performed a rotation about the longitudinal axis of your arm is known as Codman's paradox".

2.1 Take the **i,j,k** directions to be **i**: horizontal, pointing to your right; **j**: horizontal, pointing in front of you, and **k**: vertically upwards. Using this basis, write down the rotation matrices for the three rotations involved in the Codman maneuver.

The general formula for the rotation matrix corresponding to a rotation through θ about an axis parallel to **n** is

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

The first rotation is a 90 degree rotation with axis –**j**, which gives $\mathbf{R}^{(1)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

The second rotation is a 90 degree rotation with axis **k**, which gives $\mathbf{R}^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The third rotation is a 90 degree rotation with axis –**i**, which gives $\mathbf{R}^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

[3 POINTS]

2.2 Find the rotation matrix resulting from the sequence of three rotations.

The result of the sequence of three rotations is (you can use MATLAB)

$$\mathbf{R}^{(3)}\mathbf{R}^{(2)}\mathbf{R}^{(1)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

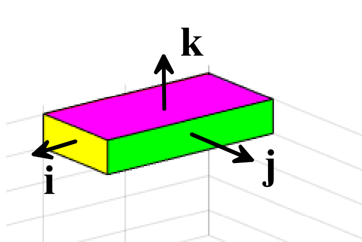
[1 POINT]

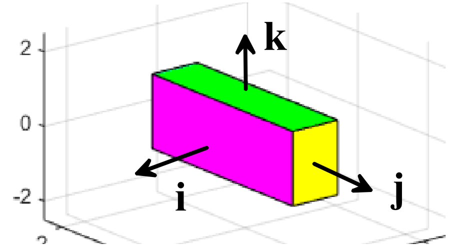
2.3 Show that 2.2 is equivalent to a 90 degree rotation about the **k** direction (the paradox)

We can see that 2.2 is identical to $\mathbf{R}^{(2)}$ and therefore is a 90 degree rotation with axis **k**. This is parallel to your arm (when it points down) and is therefore a rotation about the arm's axis.

[1 POINT]

3. Find a (single) rotation matrix that will rotate the prism from its initial to its final configuration shown in the figure. Find the axis and angle of the rotation.





We can accomplish the rotation in various ways – eg a 90 degree rotation about **i** followed by a 90 degree rotation about **k**. (you could do **k** first and then **j** as well).

From problem 2 we know a rotation about **i** is

$$\mathbf{R}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and similarly a rotation about **k** is

$$\mathbf{R}^{(2)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The combined effect is

$$\mathbf{R} = \mathbf{R}^{(2)}\mathbf{R}^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

[3 POINTS]

We can use the formula to find the axis and angle

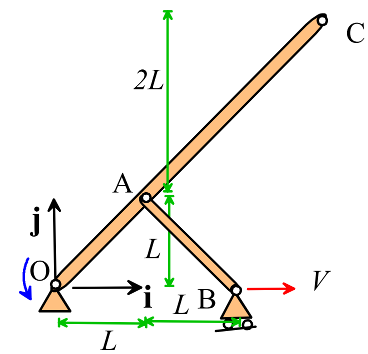
$$1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz}$$

$$\mathbf{n} = \frac{1}{2\sin\theta}\left[\left(R_{zy} - R_{yz}\right)\mathbf{i} + \left(R_{xz} - R_{zx}\right)\mathbf{j} + \left(R_{yx} - R_{xy}\right)\mathbf{k}\right]$$

This gives $\cos\theta = -(1/2) \Rightarrow \theta = 120^0$ $\quad \mathbf{n} = \frac{1}{2\sqrt{3}/2}\left[\mathbf{i} + \mathbf{j} + \mathbf{k}\right]$

[2 POINTS]

4. The figure shows a four-bar chain mechanism. Joint B moves horizontally with constant speed V. Calculate the angular velocities and angular accelerations of members OC and AB, and find the velocity and acceleration of C.



We can use the standard procedure – start at O and work around.

$$\mathbf{v}_A - \mathbf{v}_O = \omega_{OC}\mathbf{k}\times(L\mathbf{i}+L\mathbf{j}) = \omega_{OC}L(\mathbf{j}-\mathbf{i})$$
$$\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) = \omega_{AB}L(\mathbf{j}+\mathbf{i})$$
$$\Rightarrow \mathbf{v}_B - \mathbf{0} = (\omega_{OC}+\omega_{AB})L\mathbf{j} + (\omega_{AB}-\omega_{OC})L\mathbf{i} = V\mathbf{i}$$

Hence using the **i,j** components give two equations

$$\omega_{OC}+\omega_{AB}=0 \qquad \omega_{AB}-\omega_{OC}=V/L \quad \Rightarrow \omega_{AB}=-\omega_{OC}=V/2L$$

[3 POINTS]

And finally

$$\mathbf{v}_C - \mathbf{v}_O = \omega_{OC}\mathbf{k}\times(3L\mathbf{i}+3L\mathbf{j}) = 3L\omega_{OC}(\mathbf{j}-\mathbf{i})$$
$$\Rightarrow \mathbf{v}_C = (3V/2)(\mathbf{i}-\mathbf{j})$$

[1 POINT]

We can repeat the process for accelerations

$$\mathbf{a}_A - \mathbf{a}_O = \alpha_{OC}\mathbf{k}\times(L\mathbf{i}+L\mathbf{j}) - \omega_{OC}^2(L\mathbf{i}+L\mathbf{j}) = \alpha_{OC}L(\mathbf{j}-\mathbf{i}) - \omega_{OC}^2(L\mathbf{i}+L\mathbf{j})$$
$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{AB}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) - \omega_{AB}^2(L\mathbf{i}-L\mathbf{j}) = \alpha_{AB}L(\mathbf{j}+\mathbf{i}) - \omega_{AB}^2(L\mathbf{i}-L\mathbf{j})$$
$$\Rightarrow \mathbf{a}_B - \mathbf{0} = (\alpha_{OC}+\alpha_{AB}-\omega_{OC}^2+\omega_{AB}^2)L\mathbf{j} + (\alpha_{AB}-\alpha_{OC}-\omega_{OC}^2-\omega_{AB}^2)L\mathbf{i} = \mathbf{0}$$

Hence using the **i,j** components give two equations

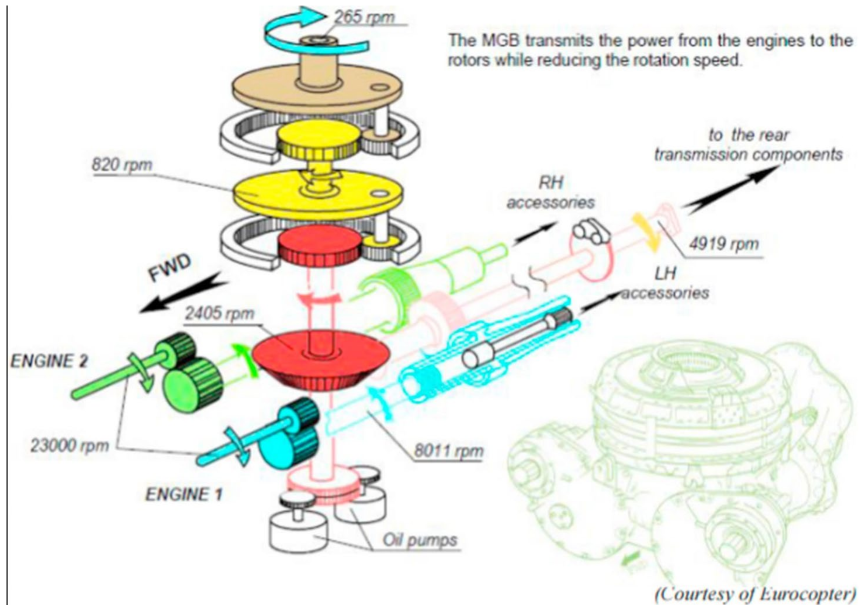
$$(\alpha_{OC}+\alpha_{AB})L - \omega_{OC}^2L^2 + \omega_{AB}^2L^2 = 0 \qquad (\alpha_{AB}-\alpha_{OC})L - \omega_{OC}^2L^2 - \omega_{AB}^2L^2 = 0$$
$$\Rightarrow \alpha_{AB} = \omega_{OC}^2 = V^2/4L^2 \qquad \alpha_{OC} = -V^2/4L^2$$

[3 POINTS]

And finally

$$\mathbf{a}_C - \mathbf{a}_O = \alpha_{OC}\mathbf{k}\times(3L\mathbf{i}+3L\mathbf{j}) - \omega_{OC}^2(3L\mathbf{i}+3L\mathbf{j}) = 3L\alpha_{OC}(\mathbf{j}-\mathbf{i}) - \omega_{OC}^2(3L\mathbf{i}+3L\mathbf{j})$$
$$= -(3V^2/(4L^2))(\mathbf{j}-\mathbf{i}) - (V^2/(4L^2))(3L\mathbf{i}+3L\mathbf{j})$$
$$= -(3V^2/(2L))\mathbf{j}$$

[2 POINTS]



5. This paper describes a planetary gear system in a helicopter (see the figure). The ring gear in both epicyclic gears are stationary. The figure shows the angular speeds of the sun and planet carrier on the two systems. Select numbers of teeth for the ring, sun and planet gears in the two epicyclics (find the lowest possible numbers of teeth on each gear).

The formulas relating angular speeds to numbers of teeth on the gears are

$$\frac{\omega_{zP}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=-\frac{N_S}{N_P} \qquad \frac{\omega_{zR}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=-\frac{N_S}{N_R} \qquad N_R=N_S+2N_P$$

The gears must have integer numbers of teeth.

For the upper epicyclic $\omega_{ZPC}=265 \quad \omega_{zS}=820 \quad \omega_R=0$ rpm, so

$$\frac{-265}{820-265}=-\frac{N_S}{N_R} \qquad N_R=N_S+2N_P$$

So we can choose $N_R=555,\ N_S=265,\ N_P=145$ these are all divisible by 5 so $N_R=111,\ N_S=53,\ N_P=29$; 29 is prime so this is the lowest possible # teeth.

[3 POINTS]

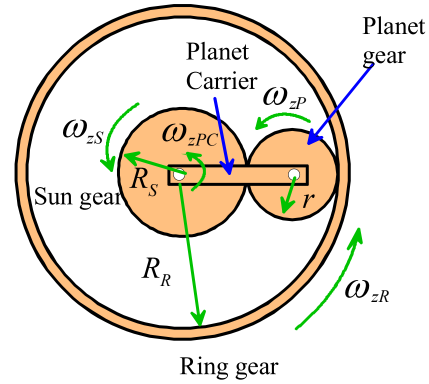
For the lower epicyclic $\omega_{ZPC}=820 \quad \omega_{zS}=2405 \quad \omega_R=0$ rpm, so

$$\frac{-820}{2405-820}=-\frac{N_S}{N_R} \qquad N_R=N_S+2N_P$$

So we can choose $N_R=3170,\ N_S=1640,\ N_P=765$ these are all divisible by 5 so $N_R=634,\ N_S=328,\ N_P=153$; these have no common factors so are the lowest possible # teeth.

[3 POINTS]

6 In the figure shown, the planet carrier (the bar) rotates counterclockwise with angular speed $\omega_{zPC} = \omega_0$. The planet gear has zero angular speed. Find the angular speed of the sun gear.

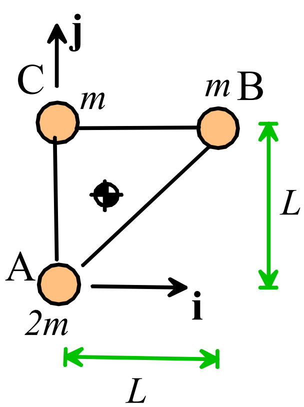


Use method from class – consider motion of system in a reference frame rotating with planet carrier. This means we subtract the angular speed of the carrier off everything – the planet gear now has angular speed $\omega_{zP} - \omega_{zPC}$ and the sun has angular speed $\omega_{zS} - \omega_{zPC}$. In our rotating reference frame the centers of the sun and planet gear are both stationary, and they look like a standard pair of gears. We can use the usual gear ratio formula for two gears

$(\omega_{zP} - \omega_{zPC})/(\omega_{zS} - \omega_{zPC}) = -R_S/r \Rightarrow \omega_{zPC}(1 + R_S/r) = \omega_{zS} R_S/r$ and so $\omega_{zS} = \omega_0 (R_S + r)/R_S$ (positive, so counterclockwise)

[3 POINTS]

7. The figure shows three particles connected by rigid massless links. The particle at A has mass 2*m*; those at B and C have mass *m*. The assembly rotates at constant angular speed ω about an axis parallel to **k** passing through the center of mass. The point of this problem is to demonstrate that the rigid body formula for the kinetic energy of the system gives the same answer as calculating the kinetic energy of each mass separately, and summing them. The rigid body formulas for angular momentum and kinetic energy are just fast ways of summing the total angular momentum and KE of a system of particles.



7.1 Calculate the position of the center of mass of the assembly

$$\mathbf{r}_G = \frac{1}{M}\sum_i m_i \mathbf{r}_i = \frac{1}{4m}\left(mL\mathbf{j} + mL(\mathbf{i}+\mathbf{j})\right) = \left(\frac{L}{4}\mathbf{i} + \frac{L}{2}\mathbf{j}\right)$$

[1 POINT]

7.2 Calculate the 2D mass moment of inertia of the system about the center of mass

$$I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2\right)$$

where $\mathbf{d}_i = d_{xi}\mathbf{i} + d_{yi}\mathbf{j} = \mathbf{r}_i - \mathbf{r}_G$ is the position vector of the *i*th particle with respect to the center of mass.

Use the formula

$$I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2\right) = 2m\left[\left(\frac{L}{4}\right)^2 + \left(\frac{L}{2}\right)^2\right] + m\left[\left(\frac{L}{4}\right)^2 + \left(\frac{L}{2}\right)^2\right] + m\left[\left(\frac{3L}{4}\right)^2 + \left(\frac{L}{2}\right)^2\right] = \frac{7}{4}mL^2$$

[1 POINT]

7.3 Suppose that the assembly rotates about its center of mass with angular velocity $\omega\mathbf{k}$ (the center of mass is stationary). What are the speeds of the particles A,B and C?

We can use the circular motion formula. The distances of A, B and C from the COM are

$$d_A = \sqrt{\left(\frac{L}{4}\right)^2 + \left(\frac{L}{2}\right)^2} = \frac{\sqrt{5}}{4}L$$

$$d_B = \sqrt{\left(\frac{3L}{4}\right)^2 + \left(\frac{L}{2}\right)^2} = \frac{\sqrt{13}}{4}L$$

$$d_C = \sqrt{\left(\frac{L}{4}\right)^2 + \left(\frac{L}{2}\right)^2} = \frac{\sqrt{5}}{4}L$$

The speeds are

$$V_A = \frac{\sqrt{5}}{4}L\omega L \qquad V_B = \frac{\sqrt{13}}{4}\omega L \qquad V_C = \frac{\sqrt{5}}{4}\omega L$$

[3 POINTS]

7.4 Calculate the total kinetic energy of the system (a) using your answer to 6.2; and (b) using your answer to 6.3. (The point of this problem is to demonstrate that the rigid body formula $(1/2)I\omega^2$ is just a quick way of summing the kinetic energies of the 3 masses. For the simple 2D system here it is quite simple to prove the equivalence for any arrangement of masses. For 3D the derivation is more complicated, but the idea is the same.)

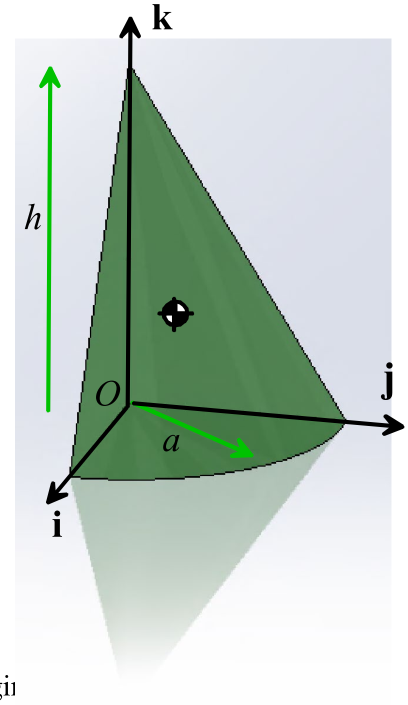
We can use the rigid body formula: $KE = \frac{1}{2}I_G\omega^2 = \frac{7}{8}mL^2\omega^2$

Or we can sum the KEs of the three masses

$$KE = \frac{1}{2}2mV_A^2 + \frac{1}{2}mV_B^2 + \frac{1}{2}mV_C^2 = m\left(\frac{\sqrt{5}}{4}\omega L\right)^2 + \frac{1}{2}m\left(\frac{\sqrt{13}}{4}\omega L\right)^2 + \frac{1}{2}m\left(\frac{\sqrt{5}}{4}\omega L\right)^2 = \frac{7}{8}mL^2\omega^2$$

[2 POINTS]

8. The figure shows a 1/4 segment of a cone with radius a, height h and uniform mass density ρ Using a Matlab 'Live Script', calculate



8.1 The total mass M (you will need to do the relevant integrals using cylindrical-polar coordinates)

$$M = \int_0^h \int_0^{\pi/2} \int_0^{a(1-z/h)} \rho R dR d\theta dz = \rho \pi a^2 h / 12$$

You can do the integrals in different orders, of course, but the limits may change.

(see below for Live Script)

[1 POINT]

8.2 The position vector of the center of mass (with respect to the origin

$$\mathbf{r}_{COM} = \frac{1}{M} \int_0^h \int_0^{\pi/2} \int_0^{a(1-z/h)} \rho (R\cos\theta \mathbf{i} + R\sin\theta \mathbf{j} + z\mathbf{k}) R dR d\theta dz = (a/\pi)\mathbf{i} + (a/\pi)\mathbf{j} + (h/4)\mathbf{k}$$

(see below for Live Script)

[2 POINTS]

8.3 The inertia tensor (matrix) about the center of mass, in the basis shown

$$\mathbf{I}_{COM} = \int_0^h \int_0^{\pi/2} \int_0^{a(1-z/h)} \rho \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix} R dR d\theta dz$$

$$d_x = R\cos\theta - (a/\pi) \qquad d_y = R\sin\theta - (a/\pi) \qquad d_z = z - (h/4)$$

(See live script for the solution – it's messy)

[3 POINTS]

8.4 Using the parallel axis theorem, calculate the mass moment of inertia about the origin O.

The general formula is

$$\mathbf{I}_O = \mathbf{I}_G + M \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix}$$

[2 POINTS]

Please upload your 'Live script' solution to Canvas.

Graders – only the Live Script upload is required….

```
clear all
syms x y z R r dx dy dz h theta a M rho mass real
dm = rho*R
M = simplify(int(int(int(dm,R,[0,a*(1-z/h)]),theta,[0,pi/2]),z,[0,h]))
r = [R*cos(theta),R*sin(theta),z]
rG = simplify(int(int(int(r*dm,R,[0,a*(1-z/h)]),theta,[0,pi/2]),z,[0,h])/M)
dx = r(1)-rG(1); dy = r(2)-rG(2); dz = r(3)-rG(3);
integrand = dm*[dy^2+dz^2,-dx*dy,-dx*dz;...
                -dx*dy,dx^2+dz^2,-dy*dz;...
                -dx*dz,-dy*dz,dx^2+dy^2];
IG = simplify(int(int(int(integrand,R,[0,a*(1-z/h)]),theta,[0,pi/2]),z,[0,h]))
IGwithmass = simplify(mass*IG/M)
dx = -rG(1); dy = -rG(2); dz = -rG(3);
Io = simplify(IG + M*[dy^2+dz^2,-dx*dy,-dx*dz;...
                      -dx*dy,dx^2+dz^2,-dx*dz;...
                      -dx*dz,-dy*dz,dx^2+dy^2])
Iowithmass = simplify(mass*Io/M)
```

```
IG =
```

$$\begin{pmatrix} \sigma_1 & -\frac{a^4\,h\,\rho\,(3\,\pi - 10)}{120\,\pi} & \sigma_2 \\ -\frac{a^4\,h\,\rho\,(3\,\pi - 10)}{120\,\pi} & \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_2 & \frac{a^4\,h\,\rho\,(3\,\pi^2 - 20)}{120\,\pi} \end{pmatrix}$$

where

$$\sigma_1 = \frac{\pi\,a^2\,h\,\rho\,(12\,a^2 + 3\,h^2)}{960} - \frac{a^4\,h\,\rho}{12\,\pi}$$

$$\sigma_2 = \frac{a^3\,h^2\,\rho}{240}$$

```
IGwithmass =
```

$$\begin{pmatrix} \sigma_1 & -\frac{a^2\,\mathrm{mass}\,(3\,\pi - 10)}{10\,\pi^2} & \sigma_2 \\ -\frac{a^2\,\mathrm{mass}\,(3\,\pi - 10)}{10\,\pi^2} & \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_2 & \frac{a^2\,\mathrm{mass}\,(3\,\pi^2 - 20)}{10\,\pi^2} \end{pmatrix}$$

where

$$\sigma_1 = \frac{\mathrm{mass}\,(12\,a^2\,\pi^2 - 80\,a^2 + 3\,h^2\,\pi^2)}{80\,\pi^2}$$

$$\sigma_2 = \frac{a\,h\,\mathrm{mass}}{20\,\pi}$$

```
Io =
```

$$\begin{pmatrix} \frac{\pi a^2 h \rho (3 a^2 + 2 h^2)}{240} & -\frac{a^4 h \rho}{40} & \sigma_1 \\ -\frac{a^4 h \rho}{40} & \frac{\pi a^2 h \rho (3 a^2 + 2 h^2)}{240} & \sigma_1 \\ \sigma_1 & \sigma_1 & \frac{\pi a^4 h \rho}{40} \end{pmatrix}$$

where

$$\sigma_1 = -\frac{a^3 h^2 \rho}{60}$$

```
Iowithmass =
```

$$\begin{pmatrix} \frac{\text{mass} (3 a^2 + 2 h^2)}{20} & -\frac{3 a^2 \text{mass}}{10 \pi} & \sigma_1 \\ -\frac{3 a^2 \text{mass}}{10 \pi} & \frac{\text{mass} (3 a^2 + 2 h^2)}{20} & \sigma_1 \\ \sigma_1 & \sigma_1 & \frac{3 a^2 \text{mass}}{10} \end{pmatrix}$$

where

$$\sigma_1 = -\frac{a\, h\, \text{mass}}{5 \pi}$$