School of Engineering
Brown University

# EN40: Dynamics and Vibrations

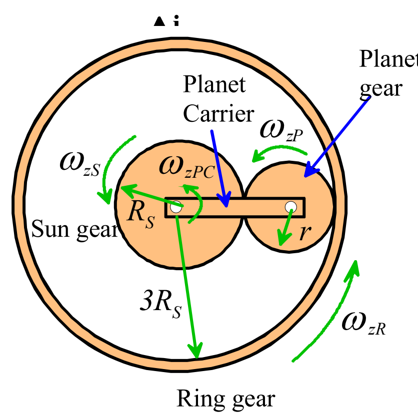
## Homework 8: Rigid Body Dynamics
## Due Friday April 24, 2020

**1** A two bladed wind-turbine with total mass 120kg (60kg per blade) and rotor diameter 20m is spun up from rest to an angular speed of 10 radians per second in 100 sec. Calculate the torque acting on the turbine.

The mass moment of inertia of the blades about the COM at the axle is $I_G = mD^2/12 = 4000kgm^2$ and the moment-d**h**/dt formula gives $Q = I_G\alpha = 4000\times 10/100 = 400Nm$

**[3 POINTS]**

**2.** In the planetary gear system shown, the sun gear has radius $R_S$ and mass $m$, the ring gear has radius $3R_S$, while the planet gear has mass $m$ and the planet carrier has mass $m/2$. The sun gear rotates with angular speed $\omega_{zS}$ and the ring gear is stationary.




Find a formula for the total angular momentum of the assembly about the center of the sun gear, in terms of $\omega_{zS}$, $R_S$ and $m$. Treat the gears as disks, the planet carrier as a 1D rod and assume there's only one planet gear as shown to keep things simple; this would be a rather unusual gear system but adding more gears just makes the problem tedious without illustrating any new concepts...

The 2D formula for angular momentum of a rigid body (about the origin) is

$$\mathbf{h} = \mathbf{r}_G \times m\mathbf{v}_G + I_{Gzz}\omega_z\mathbf{k}$$

where $\mathbf{r}_G$ is the position vector of the COM of the body relative to the origin.

We need to find the angular speed of all the moving parts: using the gear formulas

$$\frac{\omega_{zP}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=-\frac{R_S}{R_P} \qquad \frac{\omega_{zR}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=-\frac{R_S}{R_R} \qquad R_R=R_S+2R_P$$

we see that

$$\frac{0-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=-\frac{R_S}{R_R}\Rightarrow \omega_{zS}\frac{R_S}{R_R}=\omega_{zPC}\left(1+\frac{R_S}{R_R}\right)$$

$$\Rightarrow \omega_{zS}\frac{1}{3}=\omega_{zPC}\left(1+\frac{1}{3}\right)\Rightarrow \omega_{zPC}=\frac{1}{4}\omega_{zS}$$

and

$$\frac{\omega_{zP}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=-\frac{2R_S}{R_R-R_S}\Rightarrow \frac{\omega_{zP}-\omega_{zPC}}{\omega_{zS}-\omega_{zPC}}=\omega_{zPC}-(\omega_{zS}-\omega_{zPC})=-\frac{1}{2}\omega_{zS}$$

**[2 POINTS]**

The COM of the planet carrier is half way along its length; its COM is in circular motion with speed $V=\omega_{zPC}R_S$

Similarly the COM of the planet gear is in circular motion with speed $V=\omega_{zPC}2R_S$

**[2 POINTS]**

Now we can add up all the angular momenta:

Sun $\mathbf{h}_S=\frac{1}{2}mR^2\omega_{zS}\mathbf{k}$

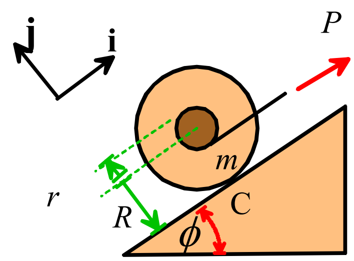
Planet carrier $\mathbf{h}_{PC}=R_S\mathbf{i}\times(\frac{1}{2}mR_S\frac{1}{4}\omega_{Zs})\mathbf{j}+\frac{1}{12}\frac{1}{2}m(2R_S)^2\frac{1}{4}\omega_{zS}\mathbf{k}=\frac{1}{6}mR_S^2\omega_{zS}\mathbf{k}$

Planet gear $\mathbf{h}_P=2R_S\mathbf{i}\times(m2R_S\frac{1}{4}\omega_{Zs})\mathbf{j}+\frac{1}{2}m(R_S)^2(-\frac{1}{2}\omega_{zS})\mathbf{k}=\frac{3}{4}mR_s^2\omega_{zs}\mathbf{k}$

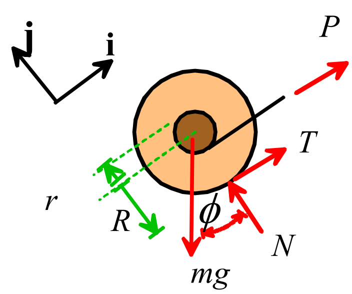
Sum everything $\mathbf{h}_P=\frac{17}{12}mR_S^2\omega_{zS}\mathbf{k}$

**[4 POINTS]**

**3.** The point of this problem is to illustrate the choices you can make when you apply the moment-angular momentum equation to calculate the acceleration and angular acceleration of a rigid body subjected to forces. The figure shows a spool with radius $R$ and mass moment of inertia $mR^2/2$ that is being pulled up a ramp by a cable subjected to a (known) force $P$. Assume that the spool rolls without slip.




3.1 Draw a free body diagram showing the forces acting on the spool




**[3 POINTS]**

(OK to draw the friction force $T$ acting down the ramp instead, since there is no slip. It is *not* OK to label the friction force as $\mu N$ . No slip means that $T < \mu N$ ; $T$ could have any value between $\pm \mu N$ )

3.2 Write down the rigid body kinematics equation that relates the angular acceleration $\alpha_z$ and linear acceleration $a_{Gx}$ of the center of mass of the spool

The rigid body kinematics equation gives $\mathbf{v}_O - \mathbf{v}_C = \omega_z \mathbf{k} \times (\mathbf{r}_O - \mathbf{r}_C) = -\omega_z R \mathbf{i}$
Taking the time derivative gives

$$\mathbf{a}_G = a_{Gx}\mathbf{i} = -\alpha_z R \mathbf{i} \Rightarrow a_{Gx} = -\alpha_z R$$

These are just the standard rolling wheel formulas of course, and can be written down directly.

Another approach is to use the acceleration formula for rigid bodies

$$\mathbf{a}_O - \mathbf{a}_C = \alpha_z \mathbf{k} \times (\mathbf{r}_O - \mathbf{r}_C) - \omega_z^2 (\mathbf{r}_O - \mathbf{r}_C) = -\alpha_z R \mathbf{i} - \omega_z^2 R \mathbf{j}$$

but this equation is a bit harder to understand – firstly it's not easy to know what the acceleration of point C on the wheel is (it is not zero), and the extra **j** component on the right hand side is mysterious. The equation is actually fine – the constraint at the contact point requires only the **tangential** component of acceleration (i.e. in the **i** direction) to be the same as the wedge (zero). The normal (**j**) component of acceleration of point C on the wheel is $\omega_z^2 R$;

**[1 POINT]**

3.3 Write down the equation relating forces and the time derivative of linear momentum of the spool (i.e. Newton's law, but the version for rigid bodies)

**F**=m**a** gives $(T + P - mg\sin\phi)\mathbf{i} + (N - mg\cos\phi)\mathbf{j} = ma_{Gx}\mathbf{i}$

**[2 POINTS]**

3.4 Write down the equation that relates the total moment acting on the spool to the time derivative of its angular momentum. Take moments (and angular momentum) about the center of the spool. Hence, solve the equations in 3.3. and 3.4 to calculate the angular acceleration of the spool and the acceleration of its COM $a_{Gx}$

Applying the moment-angular momentum equation about O gives

$$(TR + Pr)\mathbf{k} = \mathbf{0} \times ma_{Gx}\mathbf{i} + \frac{1}{2}mR^2\alpha_z$$

To find $\alpha_z$ we need to eliminate $T$ and $a_{Gx}$ from 3.2, 3.3 and 3.4;

$$T + P - mg\sin\phi = ma_{Gx} = -m\alpha_z R$$

$$TR + Pr = \frac{1}{2}mR^2\alpha_z$$

$$\Rightarrow mg\sin\phi - P = m\alpha_z R + \frac{1}{2}mR\alpha_z - P\frac{r}{R} \Rightarrow \alpha_z = \frac{2(mg\sin\phi - P(1 - r/R))}{3mR}$$

The acceleration then follows as

$$a_{Gx} = -\frac{2(mg\sin\phi - P(1 - r/R))}{3m}$$

**[3 POINTS]**

3.5 Repeat 3.4, but this time apply the moment – d**h**/dt relation by taking moments about the contact point *C*. Notice that (just like when you do statics) you can simplify the algebra by choosing to take moments about a convenient point – it doesn't change the answer, but

The moment – mass moment of inertia formula about C gives

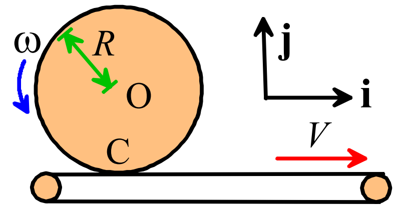
$$(mgR\sin\phi - P(R - r))\mathbf{k} = R\mathbf{j} \times ma_{Gx}\mathbf{i} + \frac{1}{2}mR^2\alpha_z\mathbf{k} = \left(-Rma_{Gx} + \frac{1}{2}mR^2\alpha_z\right)\mathbf{k}$$

$$\alpha_z = \frac{2(mg\sin\phi - P(1 - r/R))}{3mR}$$

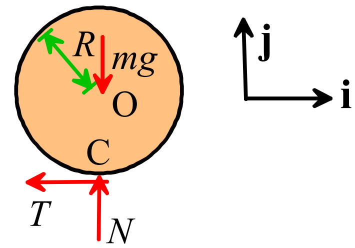
$$a_{Gx} = -\frac{2(mg\sin\phi - P(1 - r/R))}{3m}$$

**[2 POINTS]**

**4.** A solid cylinder with mass $m$ and radius $R$ rests on a conveyor belt. At time $t=0$ the belt moves with speed $V$ to the right, and the cylinder has zero angular velocity (so its center is also moving with speed $V$. The belt is then suddenly reversed, so it moves with speed $V$ to the left. The goal of this problem is to analyze the subsequent motion of the cylinder.




4.1 Draw a free body diagram showing the forces acting on the cylinder just after the belt is reversed




**[3 POINTS]**

(Note that the angular velocity of the cylinder cannot change instantly, since the forces acting on it are finite. That means slip must occur at the contact. The belt is moving to the left, and point C on the cylinder moves to the right, so friction will try to drag C to the left to prevent relative motion between the belt and the cylinder. Since slip occur it's OK to label the friction force with $\mu N$ , but a FBD with the friction acting to the right is incorrect)

4.2 Write down the equations of translational and rotational motion for the cylinder (use the 2D equations)

Translational motion $-T\mathbf{i}+(N-mg)\mathbf{j}=ma_x\mathbf{i}$

Rotational motion (moments about O) $-TR\mathbf{k}=I_{Gzz}\alpha_z\mathbf{k}$

Or you could take moments about C

$$\mathbf{0}=\mathbf{r}_G\times m\mathbf{a}_G=R\mathbf{j}\times(ma_{Gx}\mathbf{i})+I_{Gzz}\alpha_z\mathbf{k}=\left(I_{Gzz}\alpha_z-Rma_{Gx}\right)\mathbf{k}$$

**[2 POINTS]**

4.3 Use the friction law and the solution to 4.2 to calculate the linear and angular acceleration of the cylinder during the period of slip

$T=\mu N$ because we have slip so solving 4.2 gives $a_x=-\mu g \quad \alpha_z=-\dfrac{\mu mgR}{I_{Gzz}}$

**[2 POINTS]**

4.4 Hence, calculate how long it takes before the cylinder begins to roll without slip on the belt. Find the linear and angular velocity at this time. Does the cylinder end up moving to the left or to the right?

The velocity of O follows as $v=v_{x0}+a_xt=V-\mu gt$ The angular velocity is

$$\omega_z=\alpha_zt=-\frac{\mu mgR}{I_{Gzz}}t$$

At the onset of rolling the contact point C has to have the same velocity as the belt. The velocity of C is

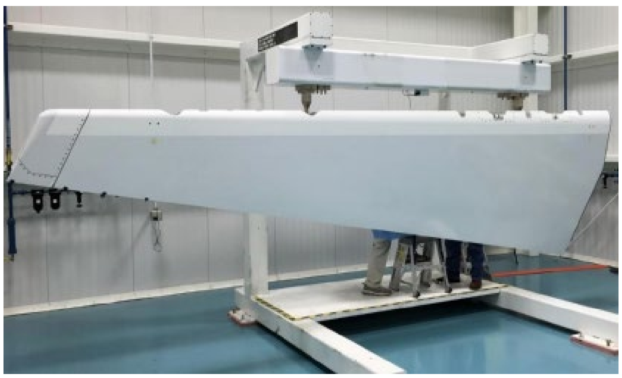
$$v_c = v_O + \omega_z R = V - \mu g t - \frac{\mu m g R^2}{I_{Gzz}} t$$

So, at onset of rolling $V - \mu g \left(1 + \frac{mR^2}{I_{Gzz}}\right) t = -V \Rightarrow t = \frac{2V}{\mu g} \frac{I_{Gzz}}{mR^2 + I_{Gzz}} = \frac{2V}{3\mu g}$ for $I_{Gzz} = \frac{1}{2} m R^2$

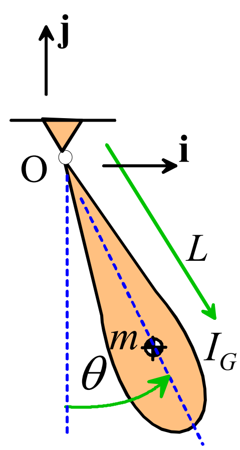
Velocity of O and angular velocity follow as $v_O = \frac{V}{3} \qquad \omega_z = -\frac{\mu m g R}{mR^2/2} \frac{2V}{3\mu g} = -\frac{4}{3} \frac{V}{R}$

The cylinder still moves to the right, even though the belt has reversed….

**[3 POINTS]**



**5.** This product specification describes a device that is designed to measure the position of the center of mass and mass moment of inertia of large aircraft parts. Their approach is to suspend the part of interest as a pendulum, and then to measure the amplitude and period of free oscillation of the part. They also measure the reaction forces at the pivots. This information is sufficient to determine the mass moment of inertia, as well as the distance of the center of mass of the part from the pivot (the remaining two coordinates of the COM can be found from static measurements).




The goal of this problem is to find the formulas that can be used to determine the COM position and inertia from measurements of the period and reaction forces on a suspended rigid body.

5.1 Write down a formula for the mass moment of inertia of the pendulum about O, in terms of $I_G, m, L$ .

Using the parallel axis theorem $I_O = I_G + mL^2$

**[1 POINT]**

5.2 Use the energy method discussed in class to find the equation of motion for the pendulum, and hence find the formula for its natural frequency of vibration, in terms of $I_G, m, L, g$

The kinetic and potential energy are

$$T = \frac{1}{2} I_O \left(\frac{d\theta}{dt}\right)^2 \qquad V = -mgL\cos\theta$$

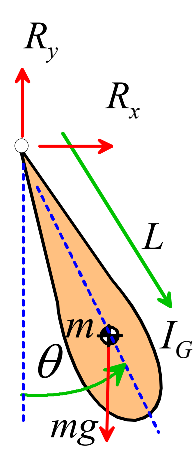
We can use $\frac{d}{dt}(T+V) = I_O \left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + mgL\sin\theta\frac{d\theta}{dt} = 0$

Using the small angle approximation and rearranging into standard form $\frac{I_O}{mgL}\frac{d^2\theta}{dt^2} + \theta = 0$

The natural frequency follows as $\omega_n = \sqrt{\frac{mgL}{I_G + mL^2}}$

**[3 POINTS]**

5.3 Draw a free body diagram showing the forces acting on pendulum




**[3 POINTS]**

5.4 Find a formula for the acceleration of the center of mass in terms of $\theta$ and its time derivatives, and use Newton's laws to find formulas for the reactions at the pivot.

We can use the rigid body kinematics equation

$$\mathbf{a}_G - \mathbf{a}_0 = \boldsymbol{\alpha} \times (\mathbf{r}_G - \mathbf{r}_0) - \omega^2 (\mathbf{r}_G - \mathbf{r}_O) = \frac{d^2\theta}{dt^2}\mathbf{k} \times (L\sin\theta\mathbf{i} - L\cos\theta\mathbf{j}) - \left(\frac{d\theta}{dt}\right)^2 (L\sin\theta\mathbf{i} - L\cos\theta\mathbf{j})$$

$$\Rightarrow \mathbf{a}_G = L\left\{\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2 \sin\theta\right\}\mathbf{i} + L\left\{\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2 \cos\theta\right\}\mathbf{j}$$

**F**=*m***a** then gives

$$R_x = mL\left\{\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2 \sin\theta\right\} \qquad R_y = mg + mL\left\{\frac{d^2\theta}{dt^2}\sin\theta + \left(\frac{d\theta}{dt}\right)^2 \cos\theta\right\}$$

**[3 POINTS]**

5.5 Suppose that the pendulum is set swinging with a very small amplitude $\theta_0$ . The amplitude $\theta_0$; the natural frequency $\omega_n$ and the amplitude of vibration of the horizontal force $H_0$ are measured. Show that (provided $\theta_0 << 1$) $H_0 \approx mL\theta_0\omega_n^2$ , and hence that the mass moment of inertia $I_G$ and the distance of the COM from the pivot can be calculated from these measurements as

$$L = \frac{H_0}{m\omega_n^2\theta_0} \qquad I_G = \frac{H_0 g}{\omega_n^4\theta_0}\left\{1 - \frac{H_0}{mg\theta_0}\right\}$$

We know that $\theta = \theta_0 \sin\omega_n t$ , so substituting into the formula for the horizontal reaction we see that

$$R_x = mL\left\{-\theta_0\omega_n^2 \sin\omega_n t\cos\theta - \left(\theta_0\omega_n \cos\omega_n t\right)^2 \sin\theta\right\}$$

For $\theta_0 << 1$ we can use the approximations $\cos\theta \approx 1, \quad \sin\theta \approx \theta$ ; also $\theta_0^3 \leq \theta_0$ so the second term can be neglected. Thus

$$R_x = -\theta_0 mL\omega_n^2 \sin\omega_n t$$

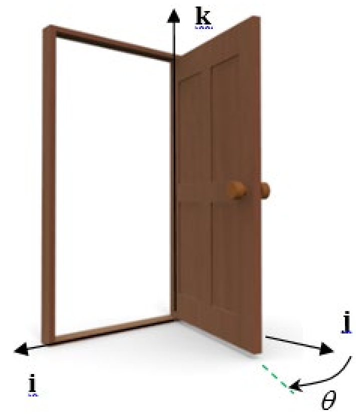
The reaction is harmonic, with amplitude $H_0 = \theta_0 mL\omega_n^2$

It follows that $L = \dfrac{H_0}{m\omega_n^2\theta_0}$

We also know that $\omega_n = \sqrt{\dfrac{mgL}{I_G + mL^2}} \Rightarrow I_G = \dfrac{mgL}{\omega_n^2} - mL^2 = \dfrac{H_0 g}{\omega_n^4\theta_0}\left\{1 - \dfrac{H_0}{mg\theta_0}\right\}$

**[3 POINTS]**

**6.** A wooden door with dimensions 7'x4'x1.5" is hung on hinges with a helical thread. The thread has pitch $p$ – i.e. if the door is rotated through 360 degrees its height will decrease by a distance $p$




Design the pitch $p$ so that if the door is released from rest and swings through a 90 degree angle, it will close in 10 seconds. (Start by using energy conservation to find the angular acceleration of the door in terms of $p$)

If we neglect friction at the hinge (and air resistance) we can use the energy conservation equation
$T + U = C$

The kinetic energy is $\frac{1}{2} I_{Ozz}\omega_z^2$ and the potential energy is

$U = -mg\frac{p\theta}{2\pi}$ (this is just the gravitational PE: we know that the door's height decreases by $p$ if it turns through a full rotation;

**[2 POINTS]**

Take the time derivative of the energy equation to see that

$$I_{Ozz}\omega_z \frac{d\omega_z}{dt} - mg\frac{p}{2\pi}\omega_z = 0 \Rightarrow \frac{d\omega_z}{dt} = \frac{mg}{I_{Ozz}}\frac{p}{2\pi}$$

**[1 POINT]**

This shows that the door has a constant angular acceleration. We can use the constant acceleration formula to calculate the angle turned as a function of time

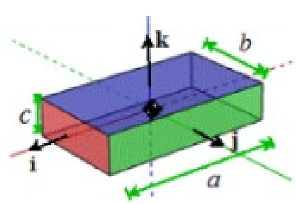
$$\theta = \frac{1}{2}\frac{mg}{I_{Ozz}}\frac{p}{2\pi}t^2$$

We want $\theta = \pi/2$ at $t=10$s so

$$p = \frac{2\pi^2}{100}\frac{I_{Ozz}}{mg}$$

**[2 POINTS]**

**Prism**
$M = \rho abc$




$$\frac{M}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

Finally we need to find the mass moment of inertia of the door about the **k** axis – we can treat it as a prism, in which case the parallel axis theorem gives

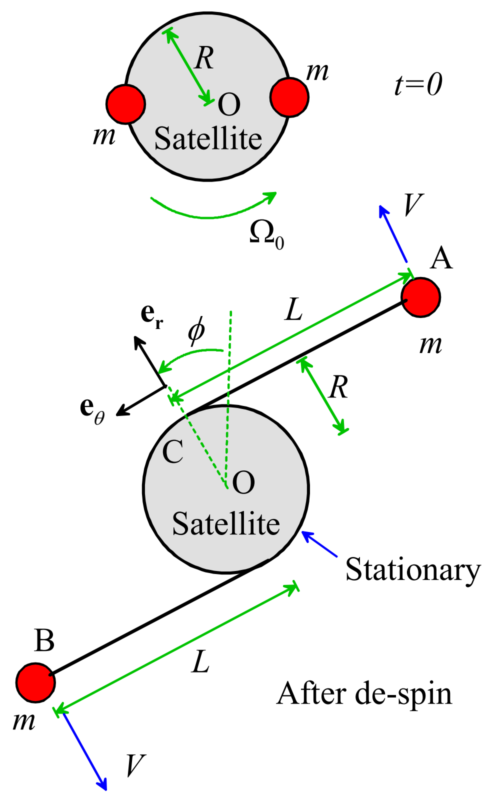
$$I_{Ozz} = \frac{m}{12}\left(a^2+b^2\right) + m(b/2)^2 = m\left(a^2/12 + b^2/3\right)$$

Where $a$ is the door thickness (1.5") (the hinge is assumed to be at mid-thickness) and $b$ is its width (4'). Hence

$$p = \frac{2\pi^2}{100}\frac{a^2/12 + b^2/3}{g} = 0.395" \quad (\sim 10mm)$$

**[ 2 Points]**

**7** The figure shows a so called 'yoyo de-spin' of a satellite. The satellite is cylindrical, with radius $R$ and mass moment of inertia about its center $I_{Gzz}$. Two masses $m$ (with negligible mass moment of inertia about their COM) are attached to the cylinder by inextensible, massless tethers.




At time $t=0$ the tethers are wound tightly around the cylinder and the assembly spins with angular speed $\Omega_0$. To de-spin the satellite, the masses are released, and the rotational motion of the assembly causes the tethers to unwind from the cylinder. This slows the rotation of the cylinder. When the rotation of the cylinder stops, the tethers are cut. The goal of this problem is to determine the length of the tethers necessary to stop the rotation of the satellite.

7.1 Consider the system at time $t=0$. Assume that the center of the satellite at O is stationary and the satellite (with masses attached) spins at angular speed $\Omega_0$. Find the total kinetic energy of the system (small masses + cylinder together) at time $t=0,$ in terms of $\Omega_0, m, R, I_{Gzz}$. Treat the small masses $m$ as particles.

The masses are in circular motion about O and so move at speed $V = \Omega_0 R$ and have KE $\frac{1}{2} m V^2$. The total KE is thus

$$T = \frac{1}{2} I_{Gzz} \Omega_0^2 + m R^2 \Omega_0^2$$

**[2 POINTS]**

7.2 Find a formula the total angular momentum of the system (masses + cylinder) about the center of the cylinder at time $t=0$ in terms of $\Omega_0, m, R, I_{Gzz}$.

The masses have angular momentum $\mathbf{h} = RmV\mathbf{k} = mR^2\Omega_0\mathbf{k}$. The total angular momentum is therefore

$$\mathbf{h} = \left[ I_{Gzz}\Omega_0 + 2mR^2\Omega_0 \right]\mathbf{k}$$

**[1 POINT]**

7.3 Consider the assembly at the instant that the cylinder just comes to rest (after de-spin). Write down the position vector of the mass at A at this instant, (taking the origin to be at O) in the $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ basis in terms of $R$, $L$.

$$\mathbf{r} = R\mathbf{e}_r - L\mathbf{e}_\theta$$

**[1 POINT]**

7.4 Suppose that at the instant the satellite stops rotating, the tether CA has angular speed $\omega = d\phi / dt$ and length $L$. Note that the cylinder (i.e. the satellite) and tether must have the same velocity where they touch at point C. Use the rigid body kinematics formula to find the velocity vector of the mass at A in the $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ basis in terms of $\omega, L$.

Since the satellite is not rotating, point C is stationary. The rigid body kinematics formula for CA gives

$$\mathbf{v} = \mathbf{v}_C + \omega\mathbf{k} \times (-L\mathbf{e}_\theta) = \omega L \mathbf{e}_r$$

**[1 POINT]**

7.5 Hence, find formulas for the total kinetic energy and angular momentum about O of the system (i.e. the satellite and both masses combined) at the instant that the cylindrical satellite comes to rest, in terms of $\omega, L, m$.

By symmetry both A and B must have the same KE and angular momentum so the total is

$$T = 2 \times \frac{1}{2} m(\omega L)^2$$

$$\mathbf{h} = \mathbf{r} \times m\mathbf{v} = 2 \times \left(R\mathbf{e}_r - L\mathbf{e}_\theta\right) \times m\omega L \mathbf{e}_r = 2m\omega L^2 \mathbf{k}$$

**[2 POINTS]**

7.6 Finally, by considering the energy and angular momentum of the system show that the cylinder comes to rest when the tether length reaches

$$L = \sqrt{\frac{\left(I_{Gzz} + 2mR^2\right)}{2m}}$$

No external forces act on the system consisting of the two masses and cylinder, so its total energy and angular momentum must be conserved. The cylinder is stationary and so has neither KE nor angular momentum. Therefore (since there are two masses)

$$m(\omega L)^2 = \frac{1}{2}\left(I_{Gzz} + 2mR^2\right)\Omega_0^2$$

$$2m\omega L^2 \mathbf{k} = \left[I_{Gzz} + 2mR^2\right]\Omega_0 \mathbf{k}$$

We can solve these equations for $\omega, L$ to see that

$$\omega L = \sqrt{\frac{\left(I_{Gzz} + 2mR^2\right)}{2m}}\Omega_0$$

$$\omega L^2 = \frac{\left[I_{Gzz} + 2mR^2\right]}{2m}\Omega_0 \Rightarrow L = \sqrt{\frac{\left(I_{Gzz} + 2mR^2\right)}{2m}}$$

**[3 POINTS]**