



School of Engineering  
Brown University

## EN40: Dynamics and Vibrations

### Homework 2: Kinematics and Dynamics of Particles Due Friday June 4, 2021

**1. Straight Line Motion.** The CTF Finance Center in Guanzhou has the world's fastest elevator, with the following specifications:

- Height: 400m
- Maximum speed 20 m/s
- Maximum acceleration (and deceleration) 2 m/s<sup>2</sup>

Assume that the elevator (i) starts at rest on the ground floor; (ii) accelerates with constant acceleration up to the maximum speed, then (iii) travels at constant speed; then (iv) decelerates to rest with constant acceleration and comes to rest on the top floor.

1.1 How long does the elevator take to reach the maximum speed?

Using the straight-line motion formulas

$$v = v_0 + at \Rightarrow t = (v - v_0) / a = 20 / 2 = 10s$$

[1 POINT]

1.2 What distance does the elevator travel while accelerating?

Another straight line motion formula

$$x = x_0 + v_0t + \frac{1}{2}at^2 \Rightarrow x = \frac{1}{2}2(10)^2 = 100m$$

[1 POINT]

1.3 What distance does the elevator travel at constant speed?

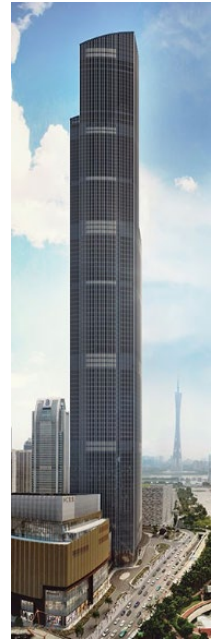
The acceleration/deceleration amount to a total distance of 200m, leaving 200m for the constant speed phase.

[1 POINT]

1.4 What is the total time required to reach the top floor at 400m ?

The time taken to travel 200m at 20m/s is 10 sec. It therefore takes a total of 30s to reach the top of the tower.

[2 POINTS]



**2. Calculus review** (apologies for inflicting this on you but hopefully it will be helpful. We suggest doing the problems by hand rather than MATLAB; the problems are meant to help you brush up on calculus that sometimes appears on exams. The calculus in this problem appears again in later parts of the course, so try to remember it!): A particle starts at position  $x = 0$  with speed  $v = 0$  at time  $t = 0$ . For each case below, please find formulas for the speed  $v$  and position  $x$  of the particle as functions of time.

2.1 The acceleration of the particle depends on time  $a(t) = A_0 \exp(-t) - 2A_0 \exp(-2t)$  where  $A_0, \omega$  are constants and  $\exp(x) \equiv e^x$

For the speed

$$\begin{aligned} \frac{dv}{dt} &= A_0 \exp(-t) - 2A_0 \exp(-2t) \Rightarrow \int_0^v dv = \int_0^t (A_0 \exp(-t) - 2A_0 \exp(-2t)) dt \\ \Rightarrow v &= -A_0 \exp(-t) + A_0 \exp(-2t) \end{aligned}$$

And hence position

$$\begin{aligned} \frac{dx}{dt} &= -A_0 \exp(-t) + A_0 \exp(-2t) \Rightarrow \int_0^x dx = \int_0^t [-A_0 \exp(-t) + A_0 \exp(-2t)] dt \\ x &= \left[ A_0 \exp(-t) - \frac{A_0}{2} \exp(-2t) \right]_0^t = A_0 (\exp(-t) - 1) + \frac{A_0}{2} (1 - \exp(-2t)) \end{aligned}$$

**[3 POINTS]**

2.2 The acceleration of the particle depends on its speed as  $a(v) = 1 - c^2 v^2$ , where  $c$  is a constant

$$\frac{dv}{dt} \Rightarrow \int_0^v \frac{dv}{1 - c^2 v^2} = \int_0^t dt$$

We can evaluate the integral on the left with the substitution

$$cv = \tanh(u) \Rightarrow cdv = (1 - \tanh^2 u) du \text{ whence}$$

$$\int_0^{\tanh^{-1}(cv)} \frac{1}{c} du = t \Rightarrow v = \frac{1}{c} \tanh(ct)$$

The position follows as

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{c} \tanh(ct) \Rightarrow \int_0^x dx = \int_0^t \frac{1}{c} \tanh(ct) dt \\ \Rightarrow x &= \frac{1}{c^2} \log(\cosh(ct)) \end{aligned}$$

(If you do this integral with MATLAB you might get a different answer – which might depend on MATLAB version. With 2020b I get

$$\int \tanh(ct) dt = t - \frac{1}{c} \log(\tanh(ct) + 1)$$

With a bit of fiddling about you can show that this is the same as the answer we get by hand

$$\int \tanh(ct) dt = t - \frac{1}{c} \log(\tanh(ct) + 1) = \frac{1}{c} \{-\log(\exp(-ct)) - \log((\tanh(ct) + 1))\}$$

$$= -\frac{1}{c} \log \left\{ \left( \frac{e^{ct} - e^{-ct}}{e^{ct} + e^{-ct}} + 1 \right) e^{-ct} \right\} = -\frac{1}{c} \log \left\{ \left( \frac{2e^{ct}}{e^{ct} + e^{-ct}} \right) e^{-ct} \right\} = \frac{1}{c} \log \left\{ \left( \frac{e^{ct} + e^{-ct}}{2} \right) \right\} = \frac{1}{c} \log(\cosh(ct))$$

**Graders – note that there are other ways to do the integrals too, eg using difference of two perfect squares and partial fractions. The answers might also be given in terms of exponentials instead of hyperbolic functions. Anything convincing that gives the right answer should get credit.**

[3 POINTS]

2.3 The acceleration of the particle depends on its position as

$$a(x) = 2(1 - 3\sqrt{x} + 2x)$$

(You can solve this problem by finding  $v$  as a function of  $x$  first, then find  $x$  as a function of  $t$ , and finally use your answer to find  $v$  as a function of  $t$ . There are other ways too, if you know how to solve differential equations).

$$a(x) = v \frac{dv}{dx} = 2(1 - 3\sqrt{x} + 2x) \Rightarrow \int_0^v v dv = \int_0^x 2(1 - 3\sqrt{x} + 2x) dx$$

$$\Rightarrow \frac{1}{2} v^2 = 2(x - 2x^{3/2} + x^2) = 2x(1 - \sqrt{x})^2$$

$$\Rightarrow v = 2\sqrt{x}(1 - \sqrt{x})$$

Note that choosing the correct sign for the square root of  $v$  is a bit tricky. We chose this one to make sure  $v > 0$  for  $0 < x < 1$ . If we choose the other sign, then  $v < 0$ , which means that the particle will move backwards and have a negative  $x$ . This is a problem, since its acceleration is then complex valued. This makes no physical sense. If we don't care about the physics and just wanted to do the calculus anyway we could forge ahead, but we'd have to be a bit careful to do the calculus correctly in the complex plane, and things like square roots and logs end up being multiple valued for complex arguments as well. We don't really want to go there.... So choose the branch with  $v > 0$ . If you didn't see this don't stress over it – it's a rather nasty and unusual issue...

Hence

$$\frac{dx}{dt} = 2\sqrt{x}(1 - \sqrt{x}) \Rightarrow \int_0^x \frac{dx}{2\sqrt{x}(1 - \sqrt{x})} = \int_0^t dt$$

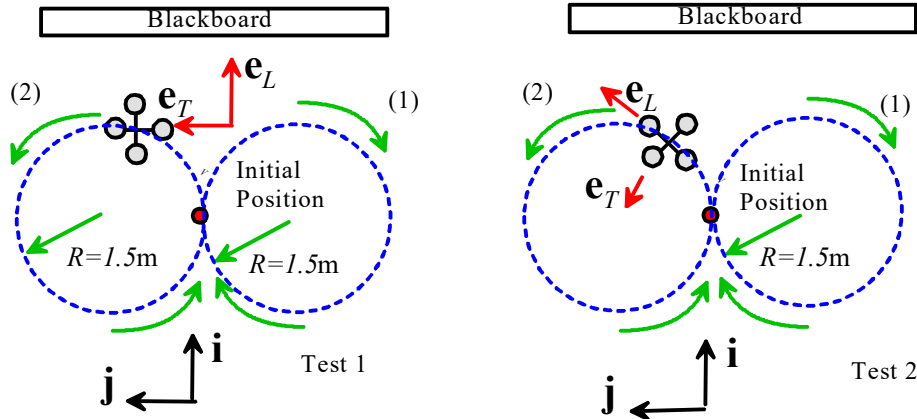
$$\Rightarrow -\log(1 - \sqrt{x}) = t$$

$$\Rightarrow x = (1 - e^{-t})^2$$

and finally

$$v = \frac{dx}{dt} = 2e^{-t}(1 - e^{-t})$$

[4 POINTS]



**3. Using MATLAB to process position / acceleration / velocity measurements:** This video demonstration shows a small quadcopter flying around a ‘figure of eight’ path, while recording its position, velocity and acceleration.

1. Position was measured using a radio system located in a fixed position in the room. The coordinate system is shown in the figure (the origin is just some random position determined by the location of the radio anchors;  $\mathbf{i}$  is towards the blackboard, and  $\mathbf{j}$  is to the left of the screen in the video).
2. Velocity, acceleration, and the yaw of the quadcopter were measured using on-board sensors. The sensors measure data in the  $\{\mathbf{e}_L, \mathbf{e}_T, \mathbf{e}_N\}$  basis that rotates with the quadcopter (to keep things simple you can assume that the quadcopter remains approximately level).

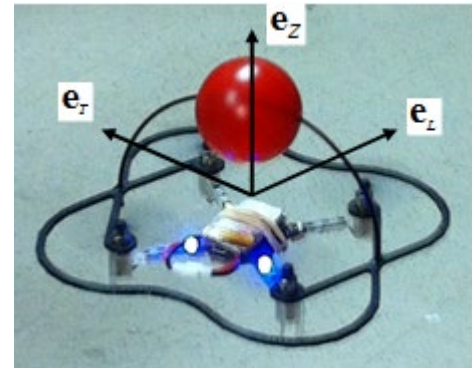
The following flight was programmed into the controller:

- (0) The quadcopter started on its flight stand, with the  $\{\mathbf{e}_L, \mathbf{e}_T, \mathbf{e}_N\}$  quadcopter axes parallel to  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
- (1) Ascend to the target altitude with constant vertical speed in 5 seconds
- (2) Fly two circles with 1.5m radius at constant speed, in 15 seconds, i.e. the controller was programmed to fly the path

$$\mathbf{r} = \begin{cases} \mathbf{r}_0 + R \sin \omega(t-5)\mathbf{i} - R \{1 - \cos \omega(t-5)\}\mathbf{j} & 5 < t < 12.5 \\ \mathbf{r}_0 + R \sin \omega(t-5)\mathbf{i} + R \{1 - \cos \omega(t-5)\}\mathbf{j} & 12.5 < t < 20 \end{cases}$$

$$\omega = 4\pi / 15 \text{ rad / s} \quad R = 1.5 \text{ m}$$

- (3) Descend to land on the ground in front of the lecture bench in 5 seconds

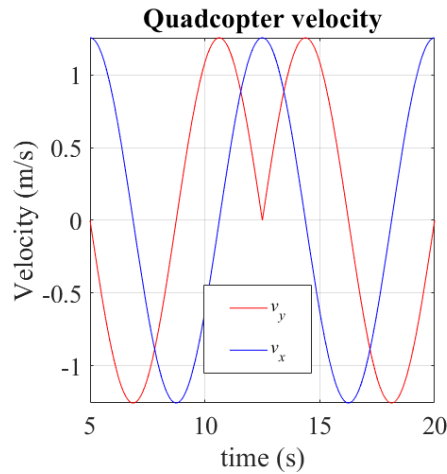


Two tests were run. In the first test, the quadcopter was programmed to keep a fixed orientation, with the  $\{\mathbf{e}_L, \mathbf{e}_T, \mathbf{e}_N\}$  quadcopter axes parallel to  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  throughout the flight. Data from this test are stored in the file called Figure-eight-fixed.csv. In the second test, the quadcopter turned so that the  $\mathbf{e}_L$  direction was parallel to the (desired) direction of motion around the circle. Data from this test are stored in the file called Figure-eight-with-turn.csv. In both files, the first column of data is the time (s); the second-4<sup>th</sup> columns are  $(x, y, z)$  position, in m; the 5<sup>th</sup>-7<sup>th</sup> columns are velocity  $(v_L, v_T, v_N)$ , in m/s; the 8<sup>th</sup>-10<sup>th</sup> columns are acceleration  $(a_L, a_T, a_N)$ , in m/s<sup>2</sup>, and the last column is the yaw (in degrees).

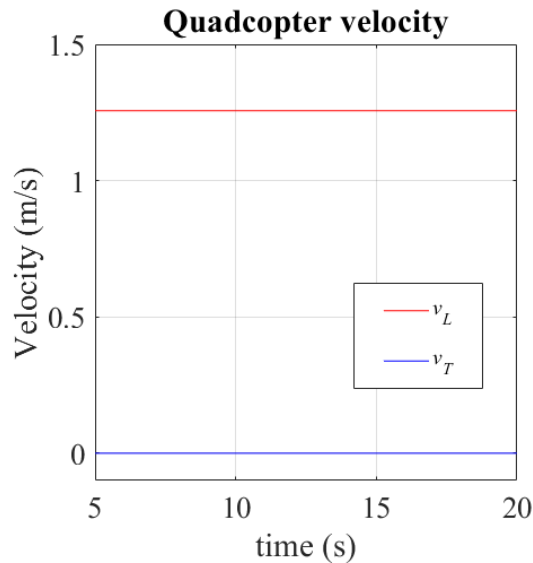
3.1 Before trying to process the data, sketch graphs (you can do this by hand, or use MATLAB, or some other software is fine too if you are averse to MATLAB) showing:

(a) The expected variation of the velocity components ( $v_L, v_T$ ) for the two tests during the time interval  $5 < t < 20$  (assuming the path is a perfect circle, which of course it is not)....

For the first test, the  $L$  and  $T$  components are the same as the  $x$  and  $y$  components of velocity, respectively. We can directly differentiate the position vector to find velocity and acceleration



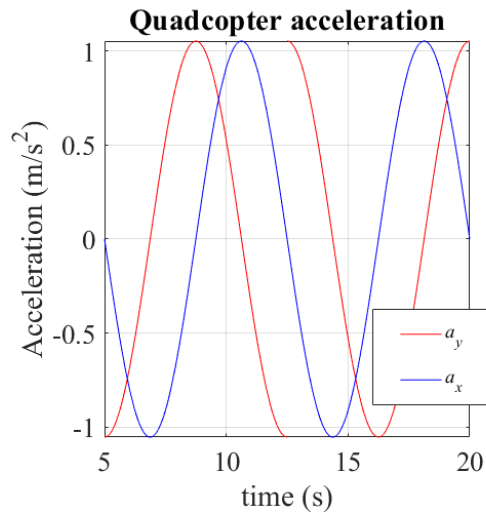
For the second test, the  $L, T$  components are the same as  $\mathbf{t}, \mathbf{n}$  components. The  $\mathbf{n}$  component of velocity is always zero, and the  $\mathbf{t}$  component is equal to the speed. The formula gives  $V = R\omega = 6\pi / 15 \text{ m/s}$



**[4 POINTS]**

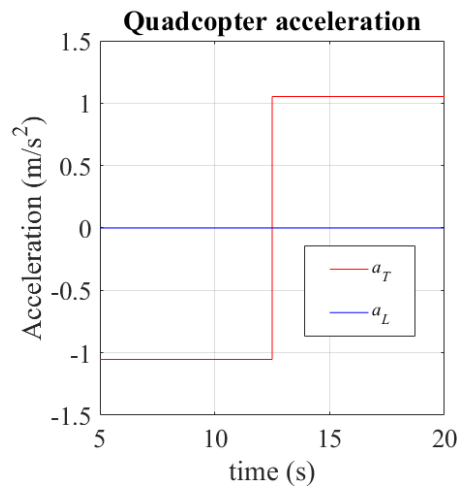
(b) The expected variation of the acceleration components ( $a_L, a_T$ ) for the two tests during the time interval  $5 < t < 20$  (again, assuming perfectly circular motion)

For the first test, the  $L$  and  $T$  components are the same as the  $x$  and  $y$  components of acceleration, respectively. We can directly differentiate the velocity vector to find acceleration



For the second test, the  $L$  component is the  $\mathbf{t}$  component of acceleration, and the  $T$  is the  $\mathbf{n}$  component if the quadcopter is turning left, and  $-T$  is the  $\mathbf{n}$  component if the quadcopter is turning right.

$$\text{Therefore } a_L = 0, \quad a_T = \pm V^2 / R = \begin{cases} -(6\pi/15)^2 / 1.5 & 5 < t < 12.5 \\ (6\pi/15)^2 / 1.5 & 12.5 < t < 20 \end{cases}$$

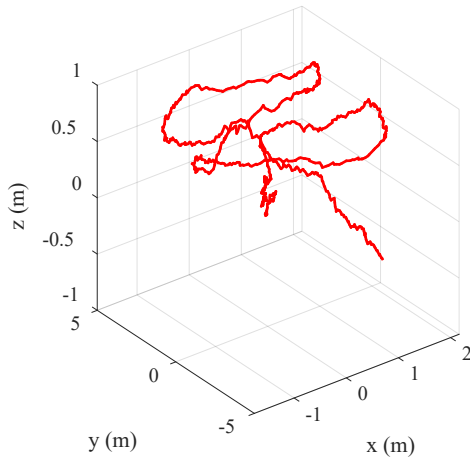


[4 POINTS]

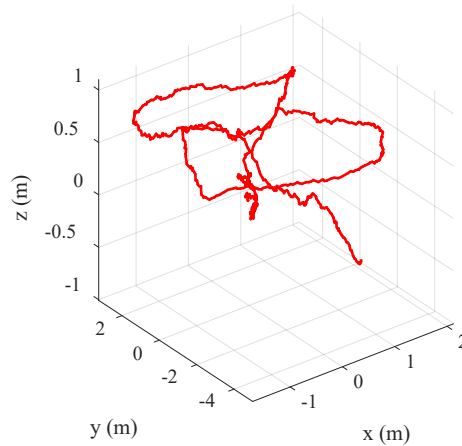
3.2 Write a MATLAB script that accomplishes the following tasks (Your MATLAB code should be uploaded to CANVAS as a submission to this problem): For each data file

- (a) Read the data into a matrix using the MATLAB 'csvread' command
- (b) Plot the trajectory (on a 3D plot).

**Trajectory (fixed orientation)**



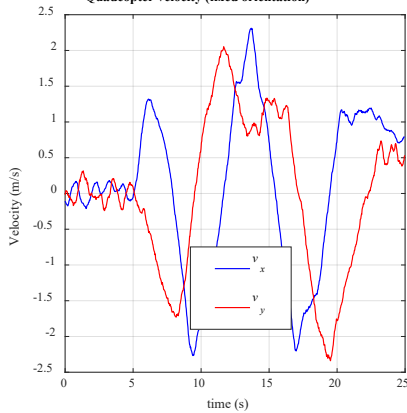
**Trajectory (with turn)**



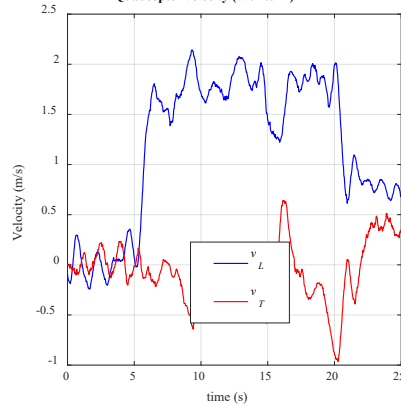
**[2 POINTS]**

- (c) Plot the measured ( $v_L, v_T$ ) as a function of time

**Quadcopter velocity (fixed orientation)**

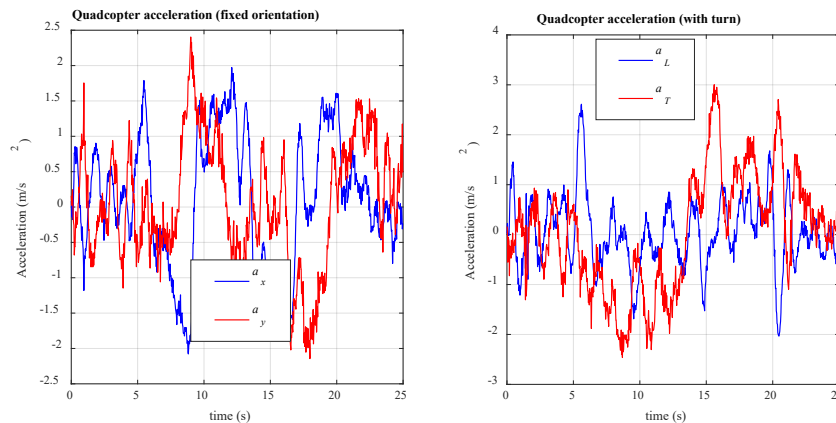


**Quadcopter velocity (with turn)**



**[2 POINTS]**

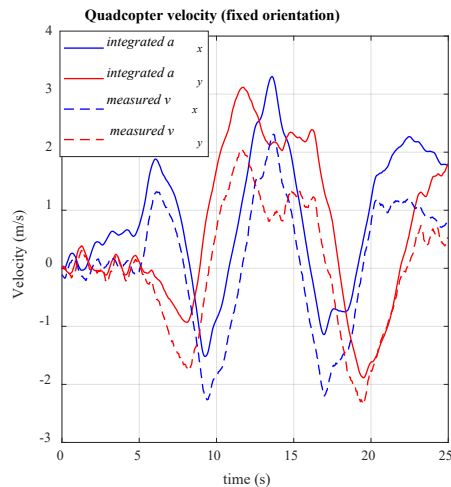
(d) Plot the measured  $(a_L, a_T)$  as a function of time



[2 POINTS]

3.3 Add MATLAB code to check the consistency of the data. For the data recorded with fixed quadcopter orientation, calculate the velocity by integrating the accelerometer readings  $(a_L, a_T)$  using the 'cumtrapz' function, and (on the same graph) plot the measured  $(v_L, v_T)$ .

Can this procedure be used to calculate the velocity from the acceleration with the data for the rotating quadcopter? If not, how would you determine the velocity components from the measured acceleration components? (you don't actually have to do the calculation, unless you are curious to see what happens)



[2 POINTS]

For the dataset with the quadcopter rotating, it would be possible (provided the orientation is maintained accurately) to calculate the speed (the tangential velocity) from the measured longitudinal acceleration

$$\frac{dV}{dt} = \frac{dv_T}{dt} = a_L$$

Integrating the transverse component of acceleration does not produce anything meaningful. The transverse component of velocity should be zero.

Another approach would be to change the basis to a fixed global frame using the yaw data, and then the global components of acceleration can be integrated to obtain the global velocities. These could be transformed back to the local quadcopter frame.

[2 POINTS]



**4. Simple circular motion problem** The figure shows the dimensions of the rotor from an ‘ultracentrifuge’ sold by [Thermo-Fisher Scientific](#).

4.1 During operation, the rotor spins about its axis at a maximum speed of 100,000 rpm. Calculate the magnitude of the acceleration of a point at the ‘maximum radius’ of the rotor. Give your answer both in  $m/s^2$  and also as the ‘relative centrifugal force (rcf)’ typically quoted by centrifuge manufacturers – which is the acceleration divided by the gravitational acceleration.

We can use the circular motion formulas from the notes

$$\mathbf{a} = R\alpha\mathbf{t} + R\omega^2\mathbf{n}$$

$$\alpha = 0 \quad \omega = 100000 \times 2\pi / 60 \quad R = 0.0718$$

$$\Rightarrow |a| = R\omega^2 = 7.87 \times 10^6 \text{ m/s}^2$$

The ‘rcf’ is

$$|a| / g = 0.802 \times 10^6$$

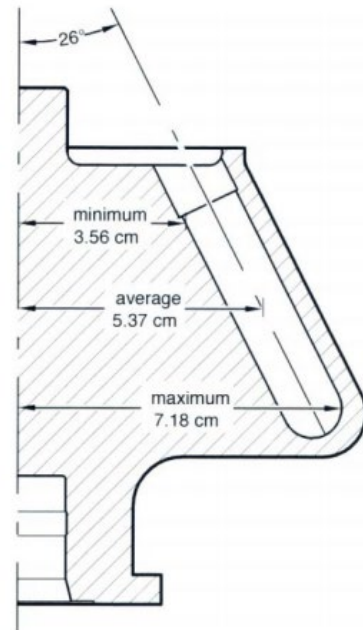


Figure 2-1. Rotor Radii

[2 POINTS]

4.2 In one mode of operation the centrifuge spins up to a speed of 500rpm in one minute (with constant acceleration). Calculate the angular acceleration of the rotor, and hence calculate the magnitude of the acceleration of a point at the ‘maximum radius’ on the rotor at the instant the rotor reaches 500rpm.

This time

$$\mathbf{a} = R\alpha\mathbf{t} + R\omega^2\mathbf{n}$$

$$\omega = \alpha t \Rightarrow \alpha = 500 \times 2\pi / 60^2 \quad \omega = 500 \times 2\pi / 60 \quad R = 0.0718$$

$$\Rightarrow |a| = \sqrt{(R\alpha)^2 + (R\omega^2)^2} = 197 \text{ m/s}^2$$

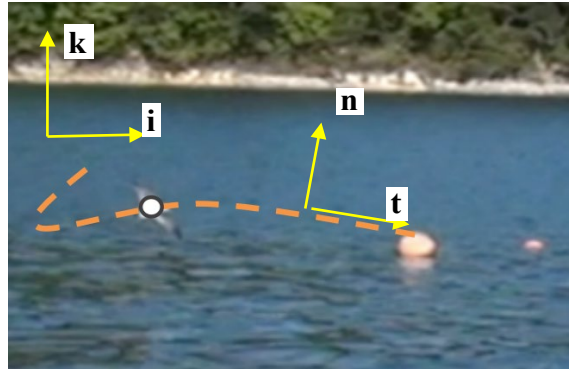
[2 POINTS]

**5. Normal-Tangential Coordinates.** The  $x, z$  coordinates ( $x$  is horizontal,  $z$  is vertical) of a seagull landing on a buoy as a function of time (from [this paper](#)) can be fit by polynomials

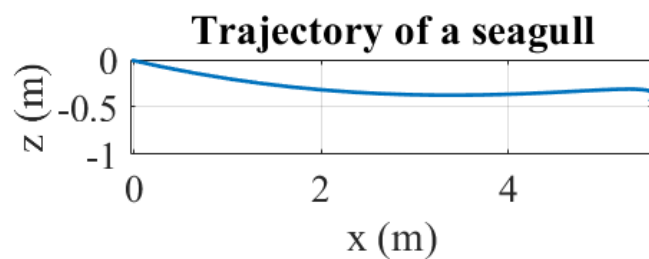
$$x = -0.6471t^3 - 1.0593t^2 + 6.7421t - 0.0394$$

$$z = -0.8033t^3 + 2.0538t^2 - 1.5713t + 0.0039$$

with  $x$  and  $z$  in meters, and  $t$  in seconds. Use a MATLAB live script to:



5.1 Plot the trajectory ( $z$  vs  $x$ ) for a time interval  $0 < t < 1.5$ s



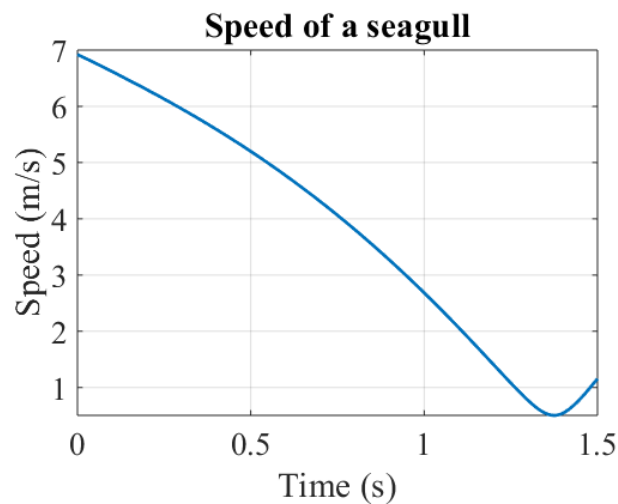
[2 POINTS]

5.2 Plot the speed as a function of time, for a time interval  $0 < t < 1.5$ s

The speed can be found by

(1) Calculating the  $x, z$  components of velocity  $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dz}{dt}\mathbf{k}$

(2) Taking the magnitude of the vector



[3 POINTS]

5.3 Let  $\mathbf{n}, \mathbf{t}$  be unit vectors normal and tangent to the path (take  $\mathbf{n}$  to be the vector with a positive  $\mathbf{k}$  component, as shown in the figure). Plot the normal and tangential components of acceleration (on the same plot), for a time interval  $0 < t < 1.5$ s.

We can find the acceleration vector in  $\mathbf{i}, \mathbf{k}$  components by differentiating the position vector with respect to time twice

$$\mathbf{a} = \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2z}{dt^2} \mathbf{k}$$

We and also find the  $\mathbf{i}, \mathbf{k}$  components of a unit vector  $\mathbf{t}$  tangent to the path using the velocity formula

$$\mathbf{v} = V\mathbf{t} \Rightarrow \mathbf{t} = \mathbf{v} / V$$

and of course we can get the velocity vector from

$$\mathbf{v} = \frac{dx}{dt} \mathbf{i} + \frac{dz}{dt} \mathbf{k}$$

We also need to find a unit vector  $\mathbf{n}$  perpendicular to the path. We could do this by using  $\mathbf{n} \cdot \mathbf{t} = 0 \Rightarrow [n_x, n_z] \cdot [t_x, t_z] = 0 \Rightarrow n_x t_x + n_z t_z = 0$  so if we choose  $\mathbf{n} = [-t_z, t_x]$  we end up with the vector we want. Or we can note that  $\mathbf{n}$  is perpendicular to both  $\mathbf{t}$  and  $\mathbf{j}$  (which points into the page), so if we cross  $\mathbf{t}$  and  $\mathbf{j}$  we get  $\mathbf{n}$ .  $\mathbf{n} = \mathbf{t} \times \mathbf{j}$

Finally we can find the  $\mathbf{t}$  and  $\mathbf{n}$  components of acceleration by noting that  $\mathbf{a} = a_t \mathbf{t} + a_n \mathbf{n}$  so if we dot both sides of this equation with  $\mathbf{t}$  and  $\mathbf{n}$  (and remember that  $\mathbf{t} \cdot \mathbf{n} = 0$   $\mathbf{t} \cdot \mathbf{t} = 1$   $\mathbf{n} \cdot \mathbf{n} = 1$ )

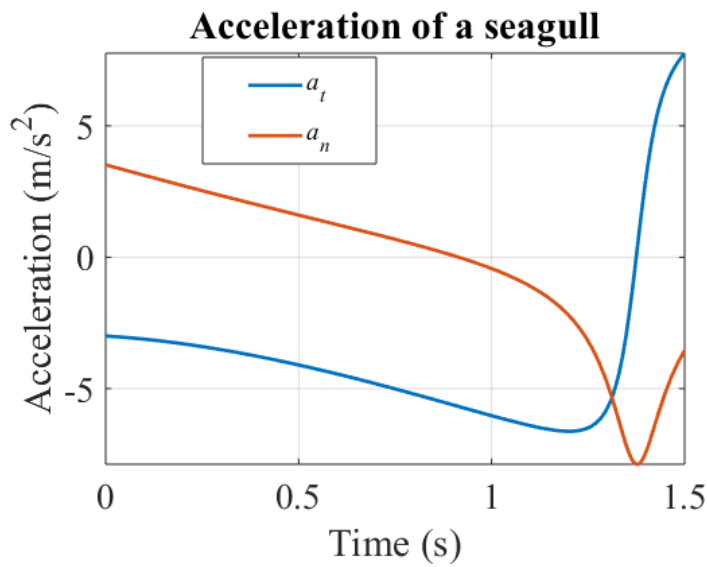
$$\mathbf{a} = a_t \mathbf{t} + a_n \mathbf{n} \Rightarrow \mathbf{a} \cdot \mathbf{t} = a_t \mathbf{t} \cdot \mathbf{t} + a_n \mathbf{n} \cdot \mathbf{t}$$

$$\Rightarrow a_t = \mathbf{a} \cdot \mathbf{t}$$

$$\mathbf{a} = a_t \mathbf{t} + a_n \mathbf{n} \Rightarrow \mathbf{a} \cdot \mathbf{n} = a_t \mathbf{t} \cdot \mathbf{n} + a_n \mathbf{n} \cdot \mathbf{n}$$

$$\Rightarrow a_n = \mathbf{a} \cdot \mathbf{n}$$

(and since we know  $\mathbf{a}$ ,  $\mathbf{n}$  and  $\mathbf{t}$  in  $\{\mathbf{i}, \mathbf{k}\}$  components we can just go ahead and find the dot product.



[3 POINTS]

5.4 The lift force on the seagull acts in the  $\mathbf{n}$  direction, the drag force acts in the  $-\mathbf{t}$  direction (note the negative sign!). The seagull has a mass of 0.37kg. Plot the lift and drag forces as a function of time for  $0 < t < 1.5$ s. (You will need to (i) draw an FBD for the gull; and (ii) use Newton's law and the calculations in 5.3 to find the lift and drag forces).

We know that

$$\mathbf{F} = m\mathbf{a}$$

$$-F_D \mathbf{t} + F_L \mathbf{n} - mg\mathbf{k} = m\mathbf{a}$$

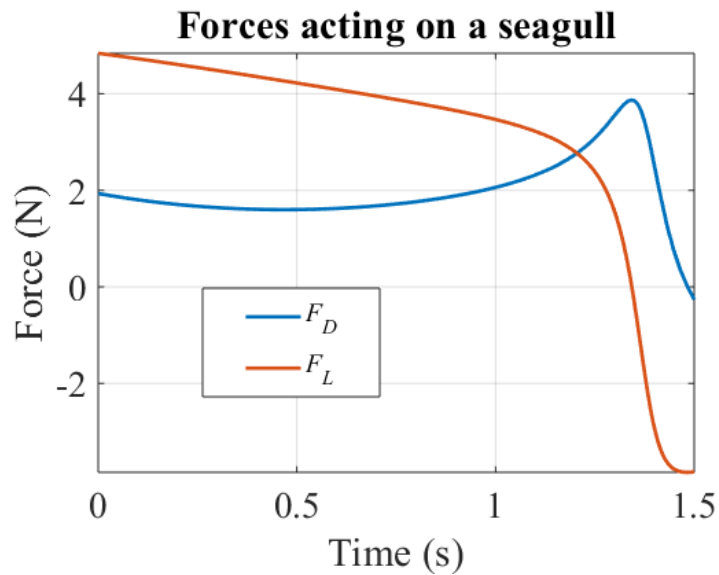
So we can do the same trick as in 5.3

$$-F_D \mathbf{t} \cdot \mathbf{t} + F_L \mathbf{n} \cdot \mathbf{t} - mg\mathbf{k} \cdot \mathbf{t} = m\mathbf{a} \cdot \mathbf{t}$$

$$\Rightarrow F_D = -mgt_z - ma_t$$

$$-F_D \mathbf{t} \cdot \mathbf{n} + F_L \mathbf{n} \cdot \mathbf{n} - mg\mathbf{k} \cdot \mathbf{n} = m\mathbf{a} \cdot \mathbf{n}$$

$$\Rightarrow F_L = mgn_z + ma_n$$

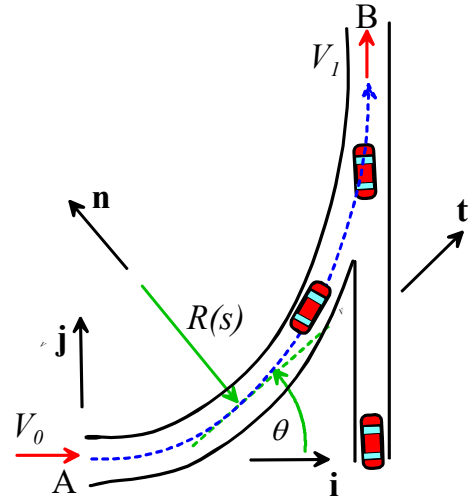


[3 POINTS]

6. The goal of this problem is to calculate the optimum shape of a highway entry ramp. Vehicles enter the ramp with speed  $V_0$  at A and merge with traffic moving with higher speed  $V_1$  at B. They will travel with constant tangential acceleration between A and B.

The magnitude of the vehicle's acceleration between A and B cannot not exceed  $a_{\max}$ , otherwise the car will skid.

6.1 Find a formula for the car's speed  $V$  in terms of the distance (arc length)  $s$  travelled from A, the tangential acceleration  $a_t$  and  $V_0$



Using the formula for tangential acceleration

$$\frac{dV}{dt} = V \frac{dV}{ds} = a_t \Rightarrow \int_{V_0}^{V_1} V dV = \int_0^s a_t ds \Rightarrow V_1^2 = V_0^2 + 2a_t s$$

[2 POINTS]

6.2 The highway ramp is to be designed so that cars can travel with (i) a constant tangential acceleration, and (ii) with acceleration magnitude equal to the maximum possible acceleration all the way from A to B. Show that the radius of curvature of the car's path must satisfy

$$\frac{d\theta}{ds} = \frac{1}{R(s)} = \frac{\sqrt{a_{\max}^2 - a_t^2}}{(V_0^2 + 2a_t s)}$$

(you can assume that  $1/R = d\theta/ds$ )

The acceleration vector (in normal-tangential coordinates) is

$$\mathbf{a} = a_t \mathbf{t} + \frac{V^2}{R} \mathbf{n}$$

Setting the magnitude of the vector equal to  $a_{\max}$  gives

$$\begin{aligned} \sqrt{a_t^2 + \frac{V^4}{R^2}} &= a_{\max} \\ \Rightarrow \frac{1}{R^2} &= \frac{a_{\max}^2 - a_t^2}{V^4} \Rightarrow \frac{1}{R} = \frac{\sqrt{a_{\max}^2 - a_t^2}}{(V_0^2 + 2a_t s)} \end{aligned}$$

[2 POINTS]

6.3 Hence, find a formula for the angle  $\theta$  between the tangent to the curve and the  $\mathbf{i}$  direction in terms of distance travelled  $s$ , ( as well as  $a_{\max}, a_t, V_0$  )

$$\int_0^\theta d\theta = \int_0^s \frac{\sqrt{a_{\max}^2 - a_t^2}}{(V_0^2 + 2a_t s)} ds \Rightarrow$$

$$\theta = \frac{\sqrt{a_{\max}^2 - a_t^2}}{2a_t} \log \left\{ \frac{(V_0^2 + 2a_t s)}{V_0^2} \right\}$$

[2 POINTS]

6.4 Find formulas for (a) the velocity vector and (b) the acceleration vector of a car traveling along the ramp as a function of time,  $V_0, a_t, a_{\max}$  in  $\{\mathbf{n}, \mathbf{t}\}$  coordinates

We know that the car's speed is  $V = V_0 + a_t t$  .

Note also that  $\mathbf{t} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$        $\mathbf{n} = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$

The velocity vector follows as

$$\mathbf{v} = V\mathbf{t} = (V_0 + a_t t)\mathbf{t}$$

The acceleration vector is

$$\mathbf{a} = a_t \mathbf{t} + \sqrt{a_{\max}^2 - a_t^2} \mathbf{n}$$

[2 POINTS]

6.5 Assume that the entry ramp must turn through an angle of 90 degrees. Find formulas for the tangential acceleration  $a_t$  and the total arc-length of the ramp  $L$ , in terms of  $V_1, V_0, a_{\max}$  . Calculate values for these quantities for  $V_1 = 55\text{mph}$ ,  $V_0 = 25\text{mph}$ ,  $a_{\max} = 0.2g$  .

$$\frac{\pi}{2} = \frac{\sqrt{a_{\max}^2 - a_t^2}}{2a_t} \log \left\{ \frac{V_1^2}{V_0^2} \right\} \Rightarrow \frac{a_{\max}^2}{a_t^2} = 1 + \frac{\pi^2}{[\log(V_1^2 / V_0^2)]^2} \Rightarrow a_t = a_{\max} \sqrt{\frac{1}{1 + \pi^2 / \{\log(V_1^2 / V_0^2)\}^2}}$$

The length of the ramp follows as

$$V_1^2 - V_0^2 = 2a_t L \Rightarrow L = \frac{(V_1^2 - V_0^2)}{2a_{\max}} \sqrt{1 + \frac{\pi^2}{[\log(V_1^2 / V_0^2)]^2}}$$

MATLAB gives numbers  $a_t = 0.8802\text{m} / \text{s}^2$      $L_0 = 272.5\text{m}$

[3 POINTS]

6.6 Note that the  $(x,y)$  coordinates of the point at a distance  $s$  along the ramp are related to the angle  $\theta$  by

$$\frac{dx}{ds} = \cos \theta \quad \frac{dy}{ds} = \sin \theta$$

By finding formulas for the parametric equations  $x(s), y(s)$  of the curve of the ramp, plot the shape of the curve for  $V_1 = 55\text{mph}$ ,  $V_0 = 25\text{mph}$ ,  $a_{\max} = 0.2g$ . You only need to submit your plot, there is no need to submit MATLAB code. There is also no need to write down the formulas for  $x(s), y(s)$  – they are very messy. If you do the plot in a ‘Live Script’ using the ‘fplot’ function, you may (depending on your version of MATLAB) have to tell MATLAB to convert x and y to real numbers (the formulas may have a very small imaginary part because of rounding errors). You can do this with, eg `fplot(real(x),real(y),[0,L])`

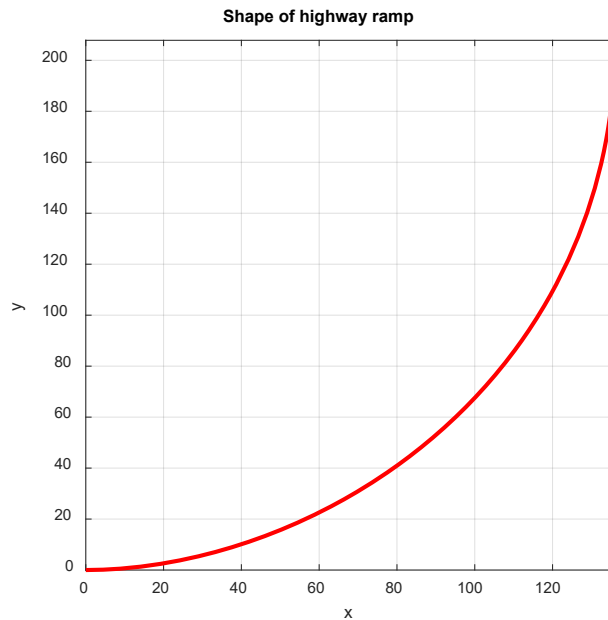
Substituting for  $a_t$  in (6.3) gives

$$\theta = \frac{\sqrt{a_{\max}^2 - a_t^2}}{2a_t} \log \left\{ \frac{(V_0^2 + 2a_t s)}{V_0^2} \right\} = \frac{\pi}{2} \frac{1}{\log(V_1^2 / V_0^2)} \log \left\{ \frac{(V_0^2 + 2a_t s)}{V_0^2} \right\}$$

We can then calculate the parametric curve as

$$x = \int_0^s \cos \theta ds \quad y = \int_0^s \sin \theta ds$$

The resulting plot is shown.



**[3 POINTS]**