EN40: Dynamics and Vibrations

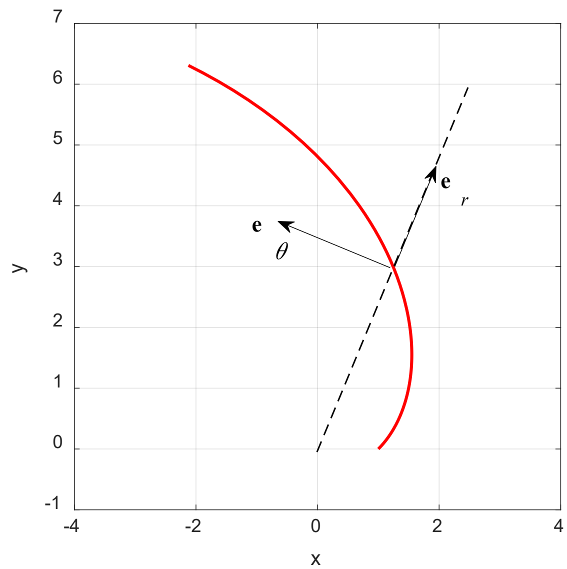
Homework 3: Kinematics and Dynamics of Particles
Due Friday June 11, 2021

School of Engineering
Brown University

Please submit your solutions to the MATLAB coding problems 5, 6 by uploading a single file, with a .m extension, to Canvas.

1. Polar Coordinates: The trajectory of a particle is specified in polar coordinates as a function of time as

$$r = 1 + \frac{V}{\sqrt{2}}t \qquad \theta = \log\left(1 + \frac{V}{\sqrt{2}}t\right)$$



1.1 Find formulas for the velocity and acceleration of the particle as a function of V and t using the $\mathbf{e}_r, \mathbf{e}_\theta$ coordinate system. Find a formula for the speed of the particle in terms of V

The polar coordinate formula for velocity (use the chain rule to differentiate θ) gives

$$\mathbf{v} = (dr/dt)\mathbf{e}_r + r(d\theta/dt)\mathbf{e}_\theta$$

$$\mathbf{v} = \frac{V}{\sqrt{2}}\mathbf{e}_r + \frac{V}{\sqrt{2}}\mathbf{e}_\theta$$

[1 POINT]

The polar coordinate formula for acceleration

$$\mathbf{a} = \left\{(d^2r/dt^2) - r(d\theta/dt)^2\right\}\mathbf{e}_r + \left\{rd^2\theta/dt^2 + 2(dr/dt)(d\theta/dt)\right\}\mathbf{e}_\theta \text{ gives}$$

$$\mathbf{a} = -\left(1 + \frac{Vt}{\sqrt{2}}\right)\left(\frac{V/\sqrt{2}}{1 + Vt/\sqrt{2}}\right)^2 \mathbf{e}_r + \left\{-\left(1 + \frac{Vt}{\sqrt{2}}\right)\frac{V^2/2}{\left(1 + Vt/\sqrt{2}\right)^2} + 2\frac{V}{\sqrt{2}}\left(\frac{V/\sqrt{2}}{1 + V/\sqrt{2}}\right)\right\}\mathbf{e}_\theta$$

$$= \frac{V^2}{2\left(1 + Vt/\sqrt{2}\right)}\left(-\mathbf{e}_r + \mathbf{e}_\theta\right)$$

[2 POINTS]

The speed of the particle is the magnitude of the velocity, and is equal to V.

1.2 Hence, find the normal and tangential components of acceleration of the particle

$$a_t = \mathbf{t}\cdot\mathbf{a} \qquad \mathbf{t} = \mathbf{v}/|\mathbf{v}| = (\mathbf{e}_r + \mathbf{e}_\theta)/\sqrt{2} \quad \Rightarrow \quad a_t = 0$$

$$a_n = |\mathbf{n}\cdot\mathbf{a}| \qquad \mathbf{n} = \pm\mathbf{k}\times\mathbf{t} = (\mathbf{e}_r - \mathbf{e}_\theta)/\sqrt{2} \;\Rightarrow\; a_n = \frac{V^2}{\sqrt{2}\left(1 + Vt/\sqrt{2}\right)}$$

[2 POINTS]

1.3 Hence, find a formula for the radius of curvature of the path, as a function of *t*.

From the formula for acceleration in normal-tangential coordinates

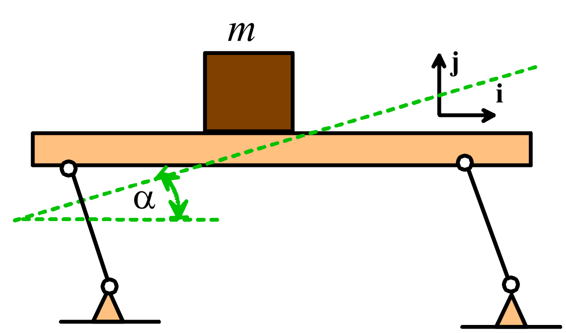
$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n} \Rightarrow \frac{1}{R} = \frac{1}{\sqrt{2}+Vt} \Rightarrow R = \sqrt{2} + Vt = \sqrt{2}r$$

[2 POINTS]

2. Simple Newton's law problem. A vibrating conveyor has a horizontal platform that moves along a straight line at an angle α to the horizontal. The position vector of an arbitrary point on the platform can therefore be expressed as

$$\mathbf{r} = (X_0 + L_0\cos\alpha\sin\Omega t)\mathbf{i} + (Y_0 + L_0\sin\alpha\sin\Omega t)\mathbf{j}$$

where X_0, Y_0, L_0, Ω are constants (L_0, Ω are the amplitude and angular frequency of the oscillatory motion of the platform, respectively).



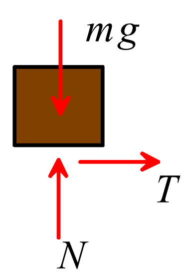
2.1 Find a formula for the acceleration vector of the platform

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -L_0\Omega^2\cos\alpha\sin\Omega t)\mathbf{i} - L_0\Omega^2\sin\alpha\sin\Omega t\mathbf{j}$$

[1 POINT]

2.2 Draw a free body diagram showing the forces acting on the object on the platform. Assume no slip (so $T \neq \mu N$). Hence, find a formula for the critical value of Ω that is required for an object of mass *m* to slip on the platform (You will need to calculate the reaction forces *T,N* at the contact. the mass will slip if $|T| > \mu N$ at any time during a cycle $0 < t < 2\pi/\Omega$).

Assume no slip. A FBD is shown (*T* could also be drawn acting to the left, since there is no slip)



Newton's law gives

$$T\mathbf{i} + (N - mg)\mathbf{j} = -mL_0\Omega^2\cos\alpha\sin\Omega t\mathbf{i} - mL_0\Omega^2\sin\alpha\sin\Omega t\mathbf{j}$$

And hence

$$T\mathbf{i} + (N - mg)\mathbf{j} = -mL_0\Omega^2 \cos\alpha \sin\Omega t\mathbf{i} - mL_0\Omega^2 \sin\alpha \sin\Omega t\mathbf{j}$$

$$T = -mL_0\Omega^2 \cos\alpha \sin\Omega t$$

$$N = mg - mL_0\Omega^2 \sin\alpha \sin\Omega t$$

The smallest value of N and the largest (absolute) value of T occur when $\sin\Omega t = 1$ so slip (if it occurs at all) will first occur at this instant. At the critical instant and angular speed

$$mL_0\Omega^2 \cos\alpha = \mu\left(mg - mL_0\Omega^2 \sin\alpha\right)$$

$$\Rightarrow \Omega = \sqrt{\frac{\mu g}{L_0(\cos\alpha + \mu\sin\alpha)}}$$

[3 POINTS]

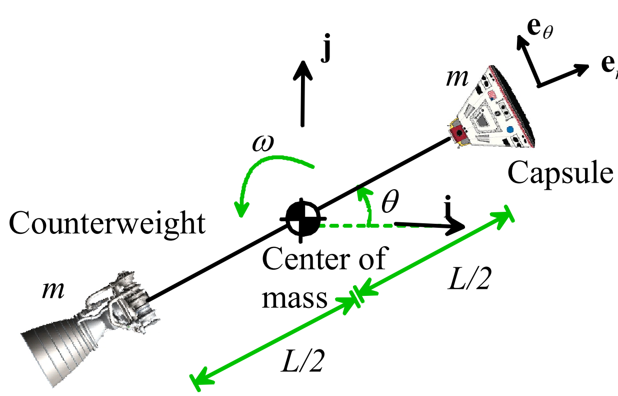
2.3 Find a formula for the value of α that minimizes Ω

To minimize Ω we must maximize $(\cos\alpha + \mu\sin\alpha)$. This is a standard calculus operation (there are other ways to do it too)

$$\frac{d}{d\alpha}(\cos\alpha + \mu\sin\alpha) = -\sin\alpha + \mu\cos\alpha = 0 \Rightarrow \alpha = \tan^{-1}\mu$$

[1 POINT]

3. The figure shows a proposed mechanism to generate artificial gravity inside a space capsule. The capsule is attached to a counterweight by a tether with length L, and the assembly rotates around its center of mass (located midway between the capsule and counterweight) with angular speed $\frac{d\theta}{dt} = \omega$. You can take the center of mass to be stationary.



3.1 Assume that the angular speed of the tether is constant, with $\omega = 0.1$ rad/sec. Use the circular motion formulas to find the cable length L that causes the capsule to have an acceleration with magnitude 2.5ms^{-2}.

Using the circular motion formula $a = L\omega^2 / 2$. Hence $L = 2\times 2.5 / 0.1^2 = 500m$

[1 POINT]

3.2 Because of a medical emergency, it becomes necessary to increase the acceleration of the capsule. This will be accomplished by shortening the tether from its initial length L_0 to a new length L_1. After the cable reaches its new length, the tether will rotate with a higher angular speed ω_1, hence increasing the capsule's acceleration.

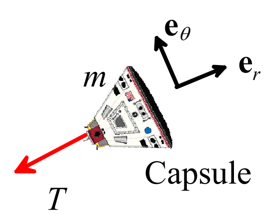
Suppose that the tether length is reduced at constant rate $\frac{dL}{dt} = \beta$, where $\beta < 0$ is a constant.

Write down a formula for the acceleration of the capsule as the cable length is being reduced, using the polar coordinate system shown in the figure, in terms of L, β, ω and $d\omega / dt$. Do not assume that ω is constant.

$$\mathbf{a} = -\frac{L}{2}\omega^2 \mathbf{e}_r + \left(\beta\omega + \frac{1}{2} L \frac{d\omega}{dt} \right) \mathbf{e}_\theta$$

[1 POINT]

3.3 Draw a free body diagram showing the forces acting on the capsule. (**There is no gravity – the capsule is in space!)**



[1 POINT]

3.4 Use your solutions to 3.2 and 3.3 to show that the angular acceleration of the tether as the cable length is being reduced is

$$\frac{d\omega}{dt} = -2\beta \frac{\omega}{L}$$

Use

$$\mathbf{F} = m\mathbf{a} \Rightarrow -T\mathbf{e}_r = -m\frac{L}{2}\omega^2 \mathbf{e}_r + m\left(\beta\omega + \frac{1}{2} L \frac{d\omega}{dt} \right) \mathbf{e}_\theta$$

The $\mathbf{e}_\theta$ components of this equation show that

$$m\left(\beta\omega + \frac{1}{2} L \frac{d\omega}{dt} \right) = 0$$

which rearranges to the required result.

[2 POINTS]

3.5 Hence, show that the angular speed is related to the tether length by

$$\frac{d\omega}{dL} = -2\frac{\omega}{L}$$

Use the chain rule $\frac{d\omega}{dt} = \frac{d\omega}{dL}\frac{dL}{dt} = \frac{d\omega}{dL}\beta = -2\beta\frac{\omega}{L}$, which gives the required solution

[2 POINTS]

3.6 Assume that at time *t*=0 the cable has length L_0 and spins with angular speed ω_0 . By separating variables in the differential equation given in 3.5 and integrating (or using angular momentum conservation) find a formula for the value of ω after the cable has been reduced to its new length L_1

Separate variables and integrate

$$\int_{\omega_0}^{\omega_1} \frac{d\omega}{\omega} = -2\int_{L_0}^{L_1} \frac{dL}{L} \Rightarrow \log\left(\frac{\omega_1}{\omega_0}\right) = -2\log\frac{L_1}{L_0}$$

$$\Rightarrow \frac{\omega_1}{\omega_0} = \frac{L_0^2}{L_1^2}$$

(you can get the same result using angular momentum conservation)

[2 POINTS]

3.7 Finally, use a value of $\omega_0 = 0.1$ rad/sec for the initial angular speed, and assume that the initial cable length L_0 has the value you calculated in 3.1. Find the length L_1 of the cable that will raise the magnitude of the acceleration to 10 ms^{-2}, and the corresponding new (constant) value of the angular speed ω_1

We require the acceleration to be 10m/s after shortening, so $L_1\omega_1^2/2 = 10$ and we know that $L_0\omega_0^2/2 = 2.5$

Therefore $\frac{\omega_1^2}{\omega_0^2} = \frac{10}{2.5}\frac{L_0}{L_1}$. Combining

$$\left(\frac{L_0}{L_1}\right)^4 = \frac{10}{2.5}\frac{L_0}{L_1} \Rightarrow L_1 = L_0\left(\frac{2.5}{10}\right)^{1/3} = 500\times\left(\frac{2.5}{10}\right)^{1/3} = 315m$$

The corresponding angular speed is

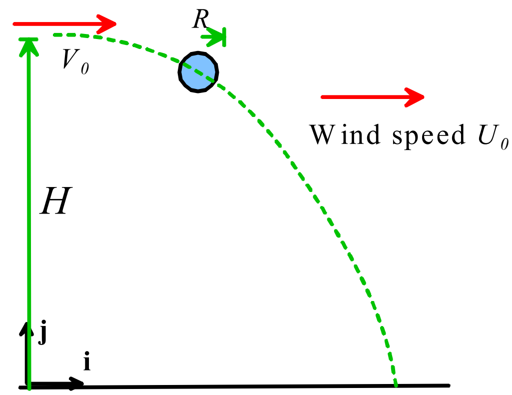
$$\omega_1 = \omega_0\frac{L_0^2}{L_1^2} = 0.1\left(\frac{500}{314.98}\right)^2 = 0.252 rad/s$$

[2 POINTS]

4. A small droplet with mass density ρ and radius R is launched horizontally with a speed V_0 from a height H above the ground (eg by a sneeze). It is subjected to the force of gravity and an air drag force

$$\mathbf{F}_D = 6\pi R\eta\left\{(U_0 - v_x)\mathbf{i} - v_y\mathbf{k}\right\}$$

where U_0 is the (horizontal) wind speed, η is the viscosity of air, and $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$ is the velocity vector of the droplet. The goal of this problem is to calculate the horizontal distance traveled by the droplet before it hits the ground.



4.1 Show that the horizontal and vertical components of velocity satisfy differential equations

$$\frac{dv_x}{dt} = \frac{9}{2}\frac{\eta}{\rho R^2}\left(U_0 - v_x\right) \qquad \frac{dv_y}{dt} = -g - \frac{9}{2}\frac{\eta}{\rho R^2}v_y$$

The droplet is subjected to the force of gravity and the drag force. Newton's law gives

$$\mathbf{F} = m\mathbf{a} \Rightarrow \mathbf{F}_D - mg\mathbf{k} = m\left(\frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{k}\right)$$

$$6\pi R\eta\left\{(U_0 - v_x)\mathbf{i} - v_y\mathbf{k}\right\} - \frac{4\pi R^3\rho}{3}g\mathbf{k} = \frac{4\pi R^3\rho}{3}\left(\frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{k}\right)$$

The **i**,**j** components give the two equations stated.

[2 POINTS]

4.2 Find formulas for the velocity and position vectors of the droplet as a function of time (you can do the integrals by hand or use MATLAB).

$$\frac{dv_x}{dt} = \frac{9}{2}\frac{\eta}{\rho R^2}\left(U_0 - v_x\right) \Rightarrow \int_{V_0}^{v_x}\frac{dv}{\left(U_0 - v_x\right)} = \int_0^t \frac{9}{2}\frac{\eta}{\rho R^2}dt$$

$$\Rightarrow -\log\frac{U_0 - v_x}{U_0 - V_0} = \frac{9}{2}\frac{\eta t}{\rho R^2}$$

$$\Rightarrow v_x = U_0 + \left(V_0 - U_0\right)\exp\left(-\frac{9}{2}\frac{\eta t}{\rho R^2}\right)$$

$$\int_0^x dx = \int_0^t\left\{U_0 + \left(V_0 - U_0\right)\exp\left(-\frac{9}{2}\frac{\eta t}{\rho R^2}\right)\right\}dt$$

$$\Rightarrow x = U_0 t + \frac{2\rho R^2}{9\eta}\left(V_0 - U_0\right)\left[1 - \exp\left(-\frac{9}{2}\frac{\eta t}{\rho R^2}\right)\right]$$

$$\frac{dv_y}{dt} = -g - \frac{9}{2}\frac{\eta}{\rho R^2} v_y \Rightarrow \int_0^{v_y} \frac{dv_y}{\left(2\rho g R^2/(9\eta) + v_y\right)} = \int_0^t -\frac{9}{2}\frac{\eta}{\rho R^2} dt$$

$$\Rightarrow \log \frac{2\rho g R^2/(9\eta) + v_y}{2\rho g R^2/(9\eta)} = -\frac{9}{2}\frac{\eta}{\rho R^2} t$$

$$\Rightarrow v_y = -\frac{2\rho g R^2}{9\eta}\left(1 - \exp\left(-\frac{9}{2}\frac{\eta}{\rho R^2} t\right)\right)$$

$$\Rightarrow \int_H^y dy = \int_0^t -\frac{2\rho g R^2}{9\eta}\left(1 - \exp\left(-\frac{9}{2}\frac{\eta}{\rho R^2} t\right)\right) dt$$

$$y = H - \frac{2\rho g R^2}{9\eta} t + \left(\frac{2\rho R^2}{9\eta}\right)^2 g \left(1 - \exp\left(-\frac{9}{2}\frac{\eta}{\rho R^2} t\right)\right)$$

[4 POINTS]

4.3 Plot a graph showing the horizontal distance traveled by a droplet as a function of its radius, using the range $0.025 < R < 0.3mm$. (There will be 3 curves on your graph, one for each value of U_0

Use the following values for parameters (references: [1] [2])

$H = 2m$

$U_0 = -1m/s \quad , \quad 0m/s, \quad 1m/s$

$\eta = 18.1\mu Ns/m^2$

$\rho = 1000kg/m^3$

$V_0 = 4m/s$

Which travel furthest – small or large droplets?

Note: Doing the plot in MATLAB is a bit tricky. It's easiest to create a vector of *R* values, then (using a loop), for each value of *R* in the vector, calculate the horizontal distance *x* (and store the results in another vector). Then you can use plot(R,x) to display the graph. For each value of *R*, you will need to calculate the time for the droplet to fall to the ground, by solving the equation *y=0* for *t*. If you do the calculation using a 'Live Script' you can use the 'vpasolve' function to do this, eg using

```
tval = vpasolve(z==0,t,20);
```

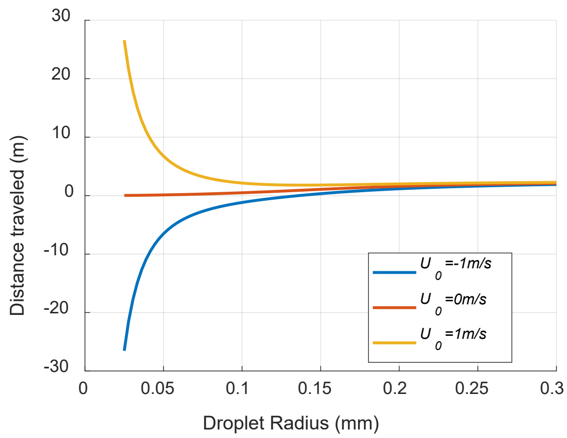
Here, the variable *z* is a symbolic formula for the height of the droplet as a function of time *t*. It's important to give 'vpasolve' a large value for the initial guess for *t* , otherwise it will return a negative time. Alternatively, if you do the calculation in an 'm' file, you can use fzero, (see the MATLAB tutorial for more details) eg

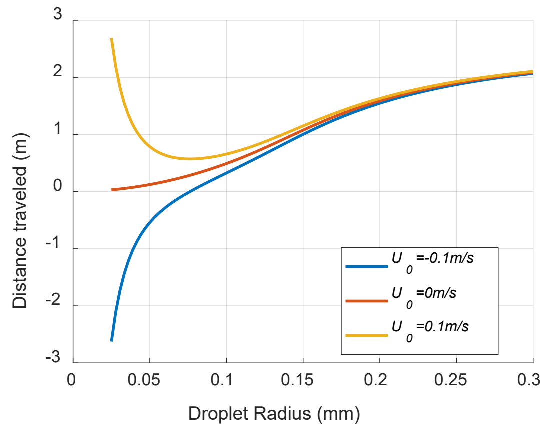
```
t = fzero(@(t) height(t,H,R,rho,eta,g),[0.05,30])
```

Here, 'height' is a function that calculates the height of the droplet. Again, it's important to specify a sensible range 0.05<t<30 for the region to search for the solution *t*, otherwise fzero will return an unphysical number (or fail altogether).

See matlab script or Live script for solution. The plots for various values of U_0 are shown (either version should get credit – the problem asked for -1, 0, +1, but some people might have run the -0.1, 0. 0.1

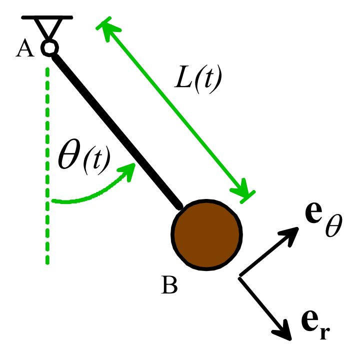
version because that's what was in the solutions. A Live script or matlab upload was not required for this problem, only the plot, but some people might have submitted something. Don't deduct points if they didn't…





[4 POINTS]

5. This publication describes various proposed methods for damping the swinging motion of a payload suspended from a drone. One proposal is to vary the length of the cable at twice the natural frequency of the pendulum (as this problem will show, this will work, but needs to be done rather carefully).



The figure shows the problem to be solved. The length of the pendulum is

$$L = L_0 + \Delta L \sin(2\Omega t) \qquad \Omega = \sqrt{g / L_0}$$

At time t=0, $\theta = 0$ and the pendulum is set swinging by giving the mass an initial horizontal velocity V_0

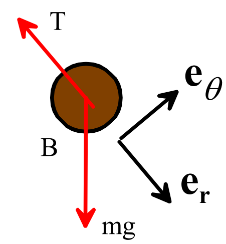
5.1 Find the acceleration vector of the mass at B, in terms of the angle θ and its time derivatives, L_0, ΔL and Ω. Express your answer using the polar coordinate basis vectors shown in the figure.

$$\mathbf{a} = \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right)\mathbf{e}_r + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \right)\mathbf{e}_\theta$$

$$\mathbf{a} = \left(-4\Delta L\Omega^2 \sin 2\Omega t - (L_0 + \Delta L \sin 2\Omega t)\left(\frac{d\theta}{dt}\right)^2 \right)\mathbf{e}_r + \left(4\Delta L\Omega \cos 2\Omega t \frac{d\theta}{dt} + (L_0 + \Delta L \sin 2\Omega t)\frac{d^2\theta}{dt^2} \right)\mathbf{e}_\theta$$

[3 POINTS]

5.2 Draw a free body diagram showing the force acting on the mass at B.



[3 POINTS]

5.3 Using Newton's law, show that the equation of motion for the angle θ is

$$\left(L_0 + \Delta L \sin 2\Omega t\right)\frac{d^2\theta}{dt^2} + 4\Delta L\Omega \frac{d\theta}{dt}\cos 2\Omega t + g\sin\theta = 0$$

Newton's law is

$$(mg\cos\theta - T)\mathbf{e}_r - mg\sin\theta\mathbf{e}_\theta =$$

$$m\left(-4\Delta L\Omega^2 \sin 2\Omega t - \left(L_0 + \Delta L\sin 2\Omega t\right)\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + m\left(4\Delta L\Omega\cos 2\Omega t\frac{d\theta}{dt} + \left(L_0 + \Delta L\sin 2\Omega t\right)\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta$$

The $\mathbf{e}_\theta$ component of this equation gives the answer stated.

[2 POINTS]

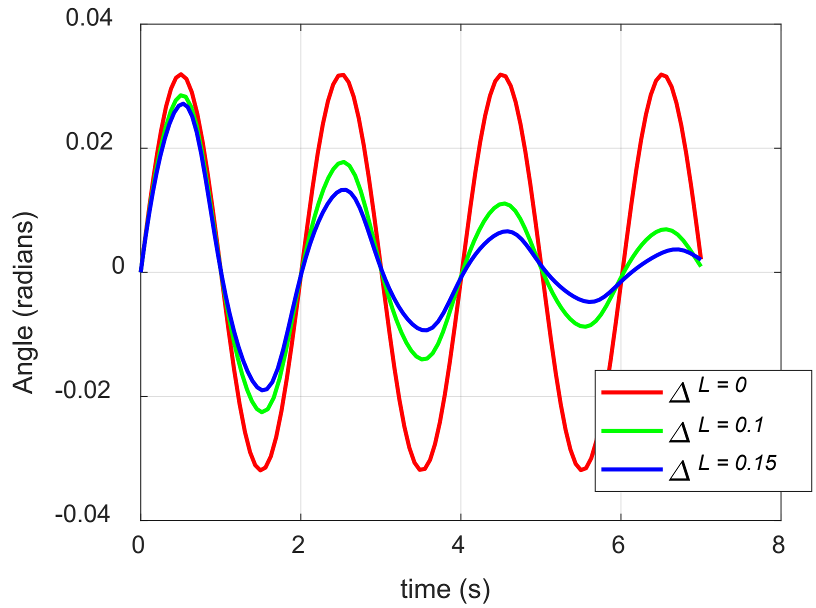
5.4 Rearrange the equation of motion into a form that MATLAB can solve.

We turn the 2nd order ODE into two first order ODEs in the usual way

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ \left(-4\Delta L\Omega\omega\cos 2\Omega t - g\sin\theta\right)/\left(L_0 + \Delta L\sin 2\Omega t\right)\end{bmatrix}$$

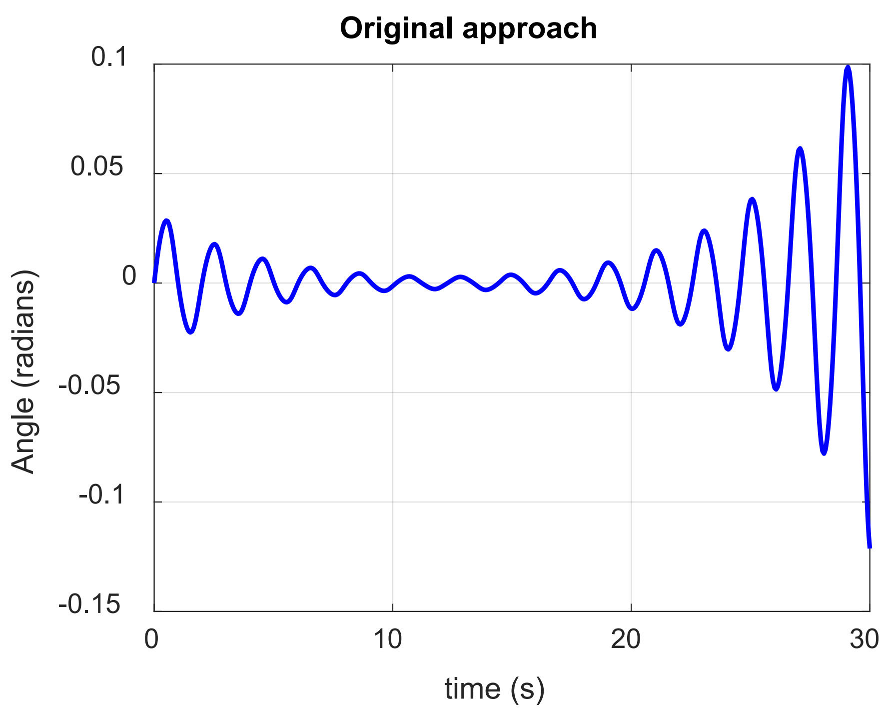
[1 POINT]

5.5 Write a MATLAB script that will solve the equations derived in part 5.4. Plot graphs showing the variation of the angle θ with time, for a time interval $0<t<7$ sec, initial conditions $\theta = 0$ $\quad d\theta/dt = 0.1$ and $L_0 = 1m$, $\Omega = \sqrt{g/L_0}$ for $\Delta L = 0, \Delta L = 0.1, \Delta L = 0.15$. These plots should show that, in principle, changing the pendulum length will successfully damp vibrations (at least for a short time).



[4 POINTS]

5.6 Repeat the simulation with initial conditions $\theta = 0$ $d\theta / dt = 0.1$ and $L_0 = 1m$, $\Delta L = 0.1$, but this time run the simulation for 0<t<30 sec. Your prediction should show that this approach has a big problem!

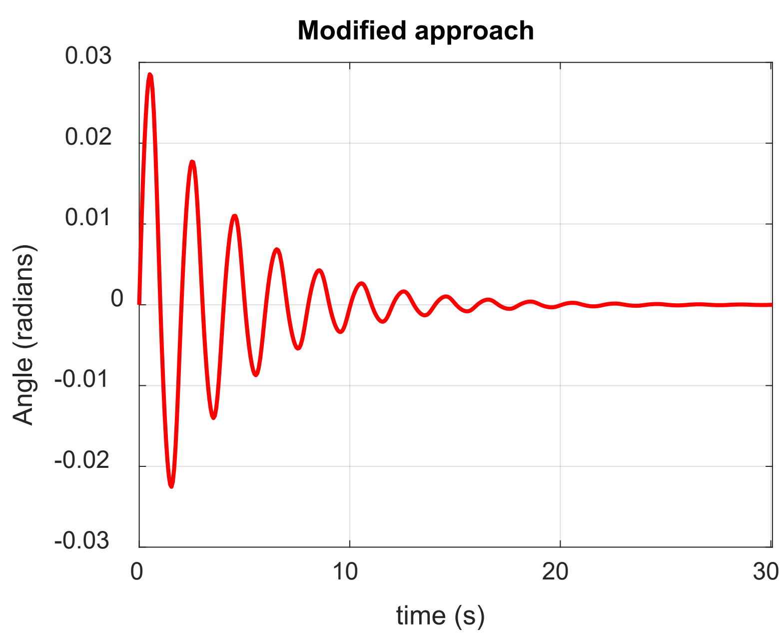


[4 POINTS]

5.7 The authors of this paper suggest that the problem can be fixed by making the length of the pendulum equal to $\left(L_0+\Delta L\sin\Omega(t-t_0)\right)$ where t_0 is the most recent time that $\theta=0$. To test this, modify your MATLAB code for parts 5.5 and 5.6 to add an 'event' function called 'stop_pendulum' that will terminate the calculation at the instant that $\theta=0$. Then add the lines of code shown below to your 'main' matlab function (you will need to study the code a bit to understand how this implements the modified design, and may need to modify the line with 'ode45' on it to make it consistent with the rest of your solution)

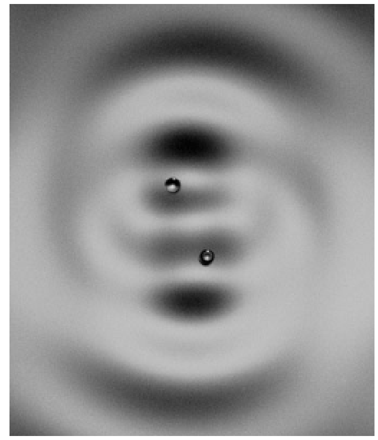
```
options = odeset('Events',@(t,w) stop_pendulum(t,w));
sols = [];
times = [];
for cycle=1:30
  [time_cycle,sol_cycle] = ode45(@(t,w) pendulum(t,w,g,L0,dL),...
                                   [0,15],initial_w,options);
  initial_w = [0,sol_cycle(end,2)];
  if (isempty(times))
      times = time_cycle;
  else
     times = [times;times(end)+time_cycle];
  end
  sols = [sols;sol_cycle];
end

figure
plot(times,sols(:,1))
```



[4 POINTS]

6. If a fluid droplet hits the surface of a pool, it can bounce back off. If the surface of the pool is in continuous wave motion, the droplets can continue bouncing forever. Under the right circumstances the droplet will bounce from one wave trough to the neighboring one, and consequently drifts over the pool's surface. If two or more droplets bounce near each other, they interact, because each droplet radiates waves that influences its neighbors. Arrays of droplets can self-organize into fascinating patterns. The goal of this problem is to predict the trajectories of two such interacting fluid droplets.



Protiere *et al* give the following equations of motion governing the horizontal position and average in-plane velocity[1] of two identical interacting droplets with mass m bouncing in-phase on a fluid surface

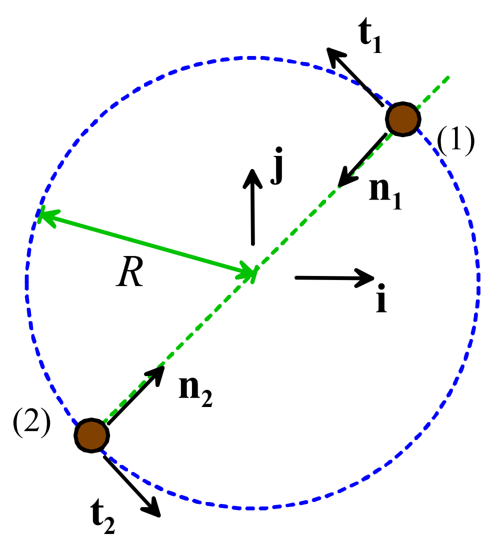
$$m\frac{d^2\mathbf{r}_1}{dt^2} = F_0 \sin(V_1/U_0)\frac{\mathbf{v}_1}{V_1} - \eta\mathbf{v}_1 + L_0 F_0 \frac{\mathbf{r}_1 - \mathbf{r}_2}{d^2}\sin(2\pi d/L_0)$$

$$m\frac{d^2\mathbf{r}_2}{dt^2} = F_0 \sin(V_2/U_0)\frac{\mathbf{v}_2}{V_2} - \eta\mathbf{v}_2 + L_0 F_0 \frac{\mathbf{r}_2 - \mathbf{r}_1}{d^2}\sin(2\pi d/L_0)$$

where $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$ are the position vectors of the two particles; $d = |\mathbf{r}_2 - \mathbf{r}_1|$ is the distance between them, $\mathbf{v}_1 = d\mathbf{r}_1/dt$ $\mathbf{v}_2 = d\mathbf{r}_2/dt$ are their velocities, $V_1 = |\mathbf{v}_1|, V_2 = |\mathbf{v}_2|$ are their speeds, and F_0, U_0, η, L_0 are constants (U_0 can be controlled by changing the frequency of the waves on the fluid surface, η is the air viscosity, and L_0 is the wavelength of the surface waves).

6.1 Show that steady circular motion (with the two particles at diametrically opposed points on the circle) is one possible trajectory that satisfies the equations of motion. To do this:

(i) Assume that both particles move around a circle with radius R with the same speed V. Write down their position, velocity and acceleration vectors (in terms of R,V) in normal-tangential coordinates, using basis vectors $\{\mathbf{n}_1, \mathbf{t}_1\}$, $\{\mathbf{n}_2, \mathbf{t}_2\}$ for particles (1) and (2), respectively.



We can use the usual circular motion formulas

$$\mathbf{r}_1 = -R\mathbf{n}_1 = R\mathbf{n}_2 \qquad \mathbf{r}_2 = -R\mathbf{n}_2 = R\mathbf{n}_1$$

$$\mathbf{v}_1 = V\mathbf{t}_1 \qquad \mathbf{v}_2 = V\mathbf{t}_2$$

$$\mathbf{a}_1 = \frac{V^2}{R}\mathbf{n}_1 \qquad \mathbf{a}_2 = \frac{V^2}{R}\mathbf{n}_2$$

[2 POINTS]

[1] The bouncing motion is averaged out

(ii) Show that both equations of motion are satisfied by a speed V and radius R that satisfy

$$\sin\frac{V}{U_0} = V\frac{\eta}{F_0} \qquad \frac{4\pi R}{L_0} = \sin^{-1}\left(-\frac{2mV^2}{L_0F_0}\right)$$

These equations make four useful predictions: (a) steady motion is possible only if $U_0 < F_0/\eta$ (otherwise the first equation only has the trivial solution $V=0$), (b) The droplets drift speed is in the range $0 < V < F_0/\eta$, (c) If a particular speed V is desired, it is straightforward to calculate the necessary value of U_0, and (d) the droplets may follow circular paths with a large number of possible discrete radii. Our goal in the next sections of this problem will be to check the stability of these paths.

Substituting the formulas in (i) into the EOM and noting that $d=2R$ gives

$$m\frac{V^2}{R}\mathbf{n}_1 = \left\{F_0\sin(V/U_0) - \eta V\right\}\mathbf{t}_1 - \frac{L_0F_0}{2R}\mathbf{n}_1\sin(4\pi R/L_0)$$
$$m\frac{V^2}{R}\mathbf{n}_2 = \left\{F_0\sin(V/U_0) - \eta V\right\}\mathbf{t}_2 - \frac{L_0F_0}{2R}\mathbf{n}_2\sin(4\pi R/L_0)$$

The $\mathbf{t}$ components of these vector equations give the formula for V directly.

[2 POINTS]

6.2 Write down differential equations for the unknowns $\mathbf{w} = [x_1, y_1, x_2, y_2, v_{x1}, v_{y1}, v_{x2}, v_{y2}]$ in a form that can be coded in the MATLAB 'ode45' differential equation solver.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \end{bmatrix} = \begin{bmatrix} v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ (F_0/m)\sin(V_1/U_0)v_{x1}/V_1 - (\eta/m)v_{x1} + (L_0F_0/(md^2))(x_1 - x_2)\sin(kd) \\ (F_0/m)\sin(V_1/U_0)v_{y1}/V_1 - (\eta/m)v_{y1} + L_0F_0/(md^2))(y_1 - y_2)\sin(kd) \\ (F_0/m)\sin(V_2/U_0)v_{x2}/V_2 - (\eta/m)v_{x2} + (L_0F_0/(md^2))(x_2 - x_1)\sin(kd) \\ (F_0/m)\sin(V_2/U_0)v_{y2}/V_2 - (\eta/m)v_{y2} + L_0F_0/(md^2))(y_2 - y_1)\sin(kd) \end{bmatrix}$$

[2 POINTS]

6.3 Write a MATLAB script that uses ‘ode45’ to integrate the equations of motion in 6.2. Use your code to calculate and plot the trajectory for the following values of parameters: $m = 10^{-6} kg$, $F_0 = 10^{-6} N$, $\eta = 10^{-6} Ns/m$, $L_0 = 5mm$ and choose U_0 to give V=2.5mm/s. To start the droplets in a circular orbit, chose their initial velocities to be $\mathbf{v}_1 = -V\mathbf{i}$ $\quad \mathbf{v}_2 = V\mathbf{i}$, and run simulations for the following initial positions:

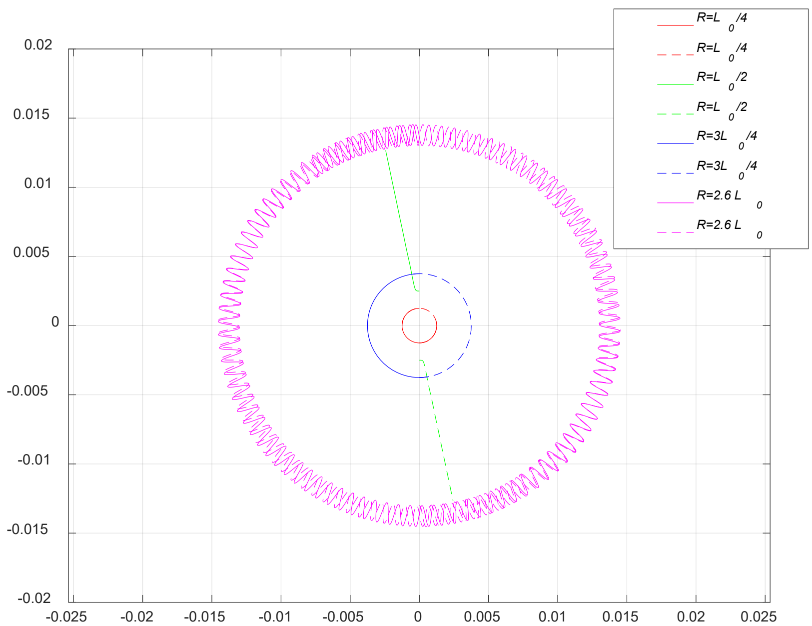
(a) $\mathbf{r}_1 = (L_0/4)\mathbf{j} \quad \mathbf{r}_2 = -(L_0/4)\mathbf{j}$ (this should give a stable circular orbit) Run this for 2.5 sec

(b) $\mathbf{r}_1 = (L_0/2)\mathbf{j} \quad \mathbf{r}_2 = -(L_0/2)\mathbf{j}$ (this will give an unstable orbit) Run this for 0.5 sec

(c) $\mathbf{r}_1 = (3L_0/4)\mathbf{j} \quad \mathbf{r}_2 = -(3L_0/4)\mathbf{j}$ (another stable orbit) Run this for 5 sec

(d) $\mathbf{r}_1 = (2.6L_0)\mathbf{j} \quad \mathbf{r}_2 = -(2.6L_0)\mathbf{j}$ (close to, but not exactly on the stable orbit) Run this for 30 sec

You can put all the trajectories on the same plot.



[4 POINTS]