School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 4: Conservation Laws for Particles
Due Friday June 25, 2021

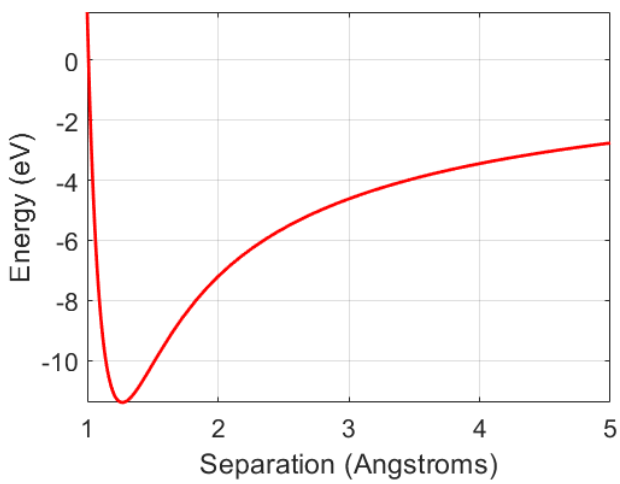
1. The 'Fumi-Tosi' potential is used in molecular dynamics simulation to model ionic bonds in materials (particularly to model behavior at temperatures above the melting point). This paper is an example (there seems to be a typo in the sign of D in the paper). The potential gives the potential energy of a the bond between a K+ and Cl- atom that are separated by a distance r as

$$U = -\frac{e^2}{4\pi\varepsilon_0 r} + B\exp\left(-\alpha r\right) - \frac{C}{r^6} + \frac{D}{r^8}$$

Values for the constants are listed in the table

e	1.6022×10^{-19} Coulombs
ε_0	8.8542×10^{-12} Farads/m
B	8.331×10^{-20} J
C	48×10^{-25} J $(\text{nm})^6$
D	73×10^{-27} J $(\text{nm})^8$
α	2.96 nm^{-1}

1.1 Plot the energy as a function of r (use units of eV for the energy, and Angstroms (1 Angstrom is 10^{-10}m) for the distance r. $1<r<5$ Angstroms gives a clear plot). The unit conversions for the first term are tricky - note that the table gives the charge of an electron e and the permittivity of free space ε_0 in SI units, so if r is in meters, the first term in the formula for U is in Joules. (Just hand in your plot; there is no need to submit MATLAB code)



[2 POINTS]

1.2 Plot a graph showing the force as a function of separation r. (use units of eV/Ansgstrom[1] for the force, and angstroms for the distance r. $1.2<r<5$ Angstroms gives a clear plot) (Just hand in your plot; there is no need to submit MATLAB code)

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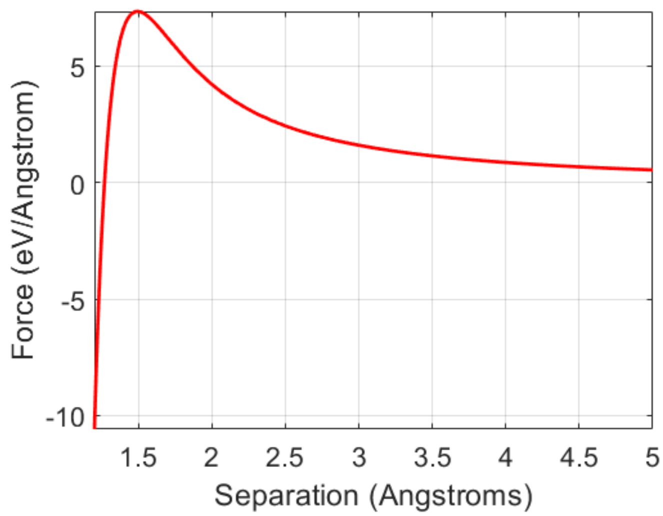
[1] This seems a weird force unit to those of us who don't usually think about individual atoms, but it is quite common in the nanoscience field.



[2 POINTS]

1.3 Find the equilibrium length of the bond between a K+ and Cl- ion (the separation when *F=0*) (in Angstroms)

We can solve for $F=0$ in MATLAB with the result $r_{eq}=1.269\,Angstroms$

[1 POINT]

1.4 Find the stiffness of the bond (at the equilibrium separation) in N/m

The stiffness is $k=\left.\frac{dF}{dr}\right|_{r=r_{eq}}=1638N/m$

[2 POINTS]

1.5 Find the (static) force required to break the bond, in eV/A. If you are using a MATLAB live script to do the calculation you will need to use the 'vpasolve' function to find the value of *r* that maximizes the force of attachment. See Sect 6.1 of the MATLAB tutorial for an example.

We need to maximize *F* – the Live script gives

$$F_{break}=7.34\quad eV/A$$

[2 POINTS]

2. The goal of this problem is to select a power plant and battery for an electrically powered light sport aircraft with the following specifications:

- Total mass 500 kg
- Cruise speed 50 m/s
- Takeoff speed 30 m/s
- Lift:drag ratio 10
- Max climb rate 5 m/s
- Range 500 km
- Reserve: 30 mins in level cruise

(you can take the gravitational acceleration *g=10* to keep the arithmetic simple)

Using the information provided, please calculate:

2.1 The magnitude of the drag force acting on the aircraft during level flight at cruise speed

The magnitude of the drag force is *mg/10* = 500N

[1 POINT]

2.2 The engine power required for level flight at cruise speed

The power-force formula gives $P = \mathbf{F} \cdot \mathbf{v} = 500 \times 50 = 25000W$

[1 POINT]

2.3 The battery capacity required to achieve the necessary range, plus 30 mins reserve (at cruise speed). Please give your answer in kWhr (kilowatt-hours)

The time to travel 500km at 50m/s is 10000s = 25/9 hrs

Adding a reserve of 1/2hr gives a total flight time of 59/18 hrs

The battery capacity required is therefore 25x59/18 kW hr = 1475/18 kWhr = 81.94 kW hr.

[2 POINTS]

2.4 The engine power required to achieve the necessary climb rate (assume the same drag as in level cruise)

The energy equation gives $P_{motor} + P_{gravity} + P_{drag} = \frac{d}{dt}(\frac{1}{2}mv^2) = 0$ (because the speed is constant)

$P_{gravity} = -mg\mathbf{j} \cdot v_y \mathbf{j} = -500 \times 10 \times 5 = -25kW$. Therefore $P_{motor} = (25+25) = 50kW$

[2 POINTS]

2.5 Assume that the magnitude of the drag force on the aircraft, F_D is proportional to the square of its speed *V*, i.e. $F_D = cV^2$ where *c* is a constant. Suppose that during takeoff the engine produces a constant power P_E . Show that the aircraft's acceleration is related to its speed by

$$\frac{dV}{dt} = V\frac{dV}{dx} = \frac{P_E - cV^3}{mV}$$

Hence, find the minimum runway length required for the aircraft to reach takeoff speed.

We have that

$$P_E + P_{drag} = P_E - cV^2V = \frac{d}{dt}(\frac{1}{2}mV^2) = mV\frac{dV}{dt}$$

$$\Rightarrow \frac{P_E - cV^3}{mV} = \frac{dV}{dt}$$

Also the chain rule gives

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = V\frac{dV}{dx}$$

Then

$$\int_0^x dx = \int_0^{V_T} \frac{mV^2}{P_E - cV^3} dV \Rightarrow x = -\frac{m}{3c}\log\left(\frac{P_E - cV_T^3}{P_E}\right)$$

(here log denotes a natural log) We can find *c* from part 2.1 using calculated drag at cruise speed:

$$c = \frac{500}{50^2} = 0.2$$

The engine power follows from 2.4. So

$$x = -\frac{500}{3\times 0.2}\log\left(\frac{50000 - 0.2\times 30^3}{50000}\right) = 95.24m$$

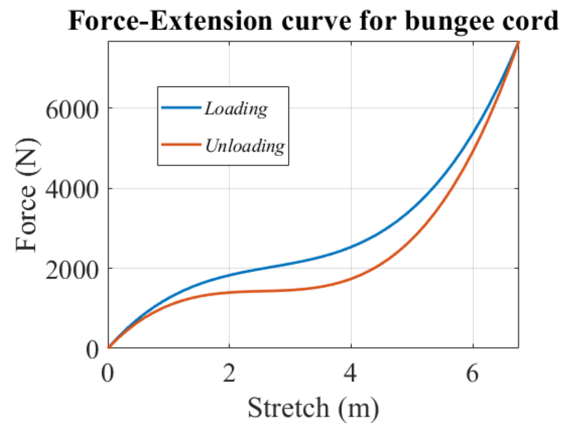
[4 POINTS]

3. This publication describes a design for a catapult system to launch a UAV. The catapult is powered by a bungee cord with a force-extension curve during loading (i.e. the cord is increasing in length) given by

$$F_L = 67.701x^3 - 546.449x^2 + 1735x$$

Where F is in Newtons, and $0<x<5.5$ is in m. During unloading (i.e. the cord starts stretched, and then is allowed to relax) the cord follows a different curve:

$$F_U = 81.032x^3 - 617.95x^2 + 1609x$$



3.1 Calculate the work done on the cord as it is stretched from $x=0$ to x=6.71m

Using the formula (and MATLAB to do the integral)

$$W = \int_0^{6.71} F_L dx = 18.34kJ$$

[1 POINT]

3.2 Calculate the work done by the cord during unloading from x=6.71m to $x=0$.

$$W = -\int_{6.71}^{0} F_U dx = 15.06kJ$$

[1 POINT]

3.3 The UAV has a mass of 175kg. It is launched by placing it on a carriage with mass of 20 kg. The carriage runs up a ramp with slope of 15 degrees. The carriage and UAV are powered by the tension in 20 bungee cords, connected in parallel (so their energy sums), which are stretched by 6.71m prior to launch. Estimate the launch speed of the UAV.

We take the system to be the earth, the carriage, the UAV, and the bungee cord.

The initial state will be with the cord stretched, the carriage and UAV stationary at the bottom of the ramp

The final state will be the cord relaxed to $x=0$, and the carriage and UAV at the top of the ramp.

Therefore

$$T_0 = 0$$
$$U_0 = 20 \times 15.06 \times 10^3$$
$$T_1 = \frac{1}{2}(m_{UAV} + m_{carriage})V^2 = \frac{1}{2} \times 195V^2$$
$$U_1 = (m_{UAV} + m_{carriage})g\Delta x \sin(15^0) = 195 \times 9.81 \times 6.71 \sin(15^0)$$
$$T_0 + U_0 = T_1 + U_1$$
$$\Rightarrow V = \sqrt{\frac{20 \times 15.06 \times 10^3 - 195 \times 9.81 \times 6.71 \sin(15^0)}{195/2}} = 55 m/s$$

Note that we use the recoverable part of the energy (the area under the unloading curve) for the PE of the bungee chords.

[3 POINTS]

3.4 What is the energy efficiency of the catapult (the ratio of the kinetic energy of the UAV just after it is launched to the work done in stretching the bungee cords).

The energy efficiency is

$$\frac{\frac{1}{2} m_{UAV} V^2}{20 \times W_{Loading}} = 72.8\%$$

[2 POINTS]

4. In this publication, a force-plate is used to measure the forces acting on a dog's feet during a standing jump. The results suggest that the (time dependent) horizontal and vertical forces can be approximated by the formula

$$F(t) = \begin{cases} F_0 + F_1 \sin\left(\dfrac{\pi t}{T}\right) & 0 < t < T \\ 0 & t > T \end{cases}$$

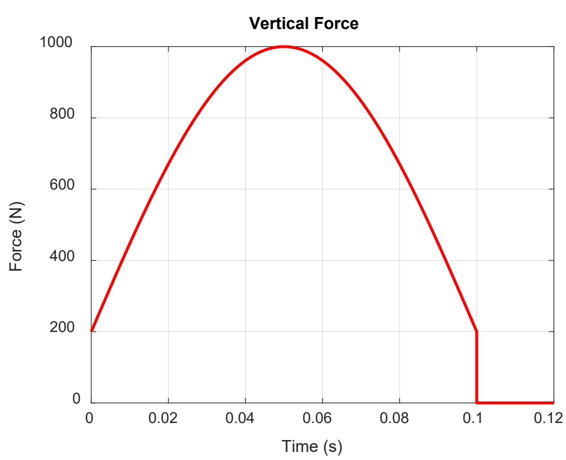
Where t is time, F_0, F_1, T are constants (the horizontal and vertical forces have different values of F_0, F_1, and the same value of T)

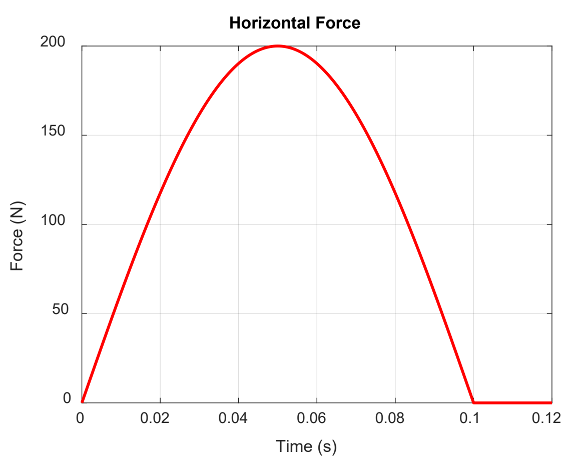
4.1 Find a formula for the impulse exerted by the force *F(t)* during the time interval 0<*t*<*T*.

By definition $\Im = \int_0^T F(t)dt = F_0 T - \left[\dfrac{F_1 T}{\pi}\cos\dfrac{\pi t}{T}\right]_0^T = F_0 T + \dfrac{2F_1 T}{\pi}$

[2 POINTS]

4.2 Use the graphs provided below to estimate the value of *T*, and the values of F_0, F_1 for the horizontal and vertical forces





For the horizontal force $F_0 = 0,\ F_1 = 200N$ (from the max value and the fact that the curve goes through 0 at *t=0*)

For the vertical force $F_0 = 200N,\ F_1 = 800N$

For both *T=0.1s*

[3 POINTS]

4.3 Using data from the graphs and the results of 4.1, calculate the velocity vector of the dog just after it leaves the ground (you can use the figures in 4.2 to estimate the dog's weight). Assume that the dog is at rest at time $t=0$ (i.e. before the impulse).

During the jump, the dog is subjected to the force of gravity and the horizontal and vertical forces acting on its feet.

The gravitational impulse is $\mathfrak{I}_g = -mgT\mathbf{j}$. We can find the dog's weight from the normal force acting at time t=0: mg=200N

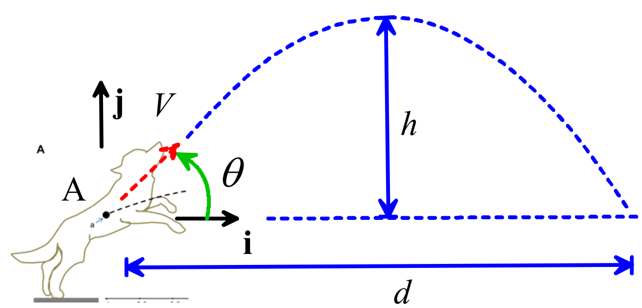
From 4.1, the impulse from the reaction force is $\frac{40}{\pi}\mathbf{i} + \left(20 + \frac{160}{\pi}\right)\mathbf{j}$

Using the impulse-momentum formula and taking m=20kg

$$m\mathbf{v} = \mathfrak{I} \Rightarrow \mathbf{v} = \frac{1}{m}\left(\frac{40}{\pi}\mathbf{i} + \frac{160}{\pi}\mathbf{j}\right) = \left(\frac{2}{\pi}\mathbf{i} + \frac{8}{\pi}\mathbf{j}\right) \text{ m/s}$$

[3 POINTS]

4.4 Calculate the height h and length d of the dog's jump.



We can calculate the height using energy conservation and/or the trajectory equations. We know the horizontal component of velocity is constant, and the vertical component of velocity is zero at the top of the trajectory

$$T_0 + U_0 = T_1 + U_1 \Rightarrow \frac{1}{2}m|\mathbf{v}_0|^2 = \frac{1}{2}m|\mathbf{v}_1|^2 + mgh$$

$$\Rightarrow \left(\frac{2}{\pi}\right)^2 + \left(\frac{8}{\pi}\right)^2 = \left(\frac{2}{\pi}\right)^2 + 2gh$$

$$\Rightarrow h = \frac{1}{2g}\left(\frac{8}{\pi}\right)^2 = \frac{32}{\pi^2 g} \approx 32.4cm$$

The time to reach the top of the trajectory is $t = v_{y0}/g = \frac{8}{\pi g}$; the dog is airborne for twice this time.

The horizontal distance traveled is then

$$2\frac{v_{y0}}{g}v_{x0} = \frac{16}{\pi g}\frac{2}{\pi} = \frac{32}{\pi^2 g} \approx 32cm$$

[3 POINTS]

5. This report describes a protocol for measuring the coefficient of restitution of a golf club head relative to a baseline titanium plate (the club-head must have a restitution coefficient almost identical to that of the reference plate to be legal). The restitution coefficients are measured by firing a golf-ball at both the plate and then (in a separate experiment) the club-head. The plate and club-head are stationary before impact, and free to move after the impact. The speeds of the golf-ball before (V_{in}) and after (V_{out}) the impact are recorded (Note that these are the speeds, i.e. the magnitude of the velocity – the direction reverses). The report gives the following formula for the restitution coefficient in terms of these variables

$$e = \frac{1}{m_C}\left[\left(\frac{V_{out}}{V_{in}}\right)(m_C + m_b) + m_b\right]$$

where m_C is the mass of the club-head and m_b is the mass of the ball.

Derive this formula (use momentum conservation and the restitution formula for straight-line collisions)

The general 1-D collision formulas are:

(1) Momentum conservation: $m_A v_x^{A0} + m_B v_x^{B0} = m_A v_x^{A1} + m_B v_x^{B1}$

(2) Restitution formula: $v_x^{A1} - v_x^{B1} = -e(v_x^{A0} - v_x^{B0})$

We can identify

$$v_x^{A0} = V_{in} \qquad v_x^{B0} = 0$$
$$v_x^{A1} = -V_{out}$$

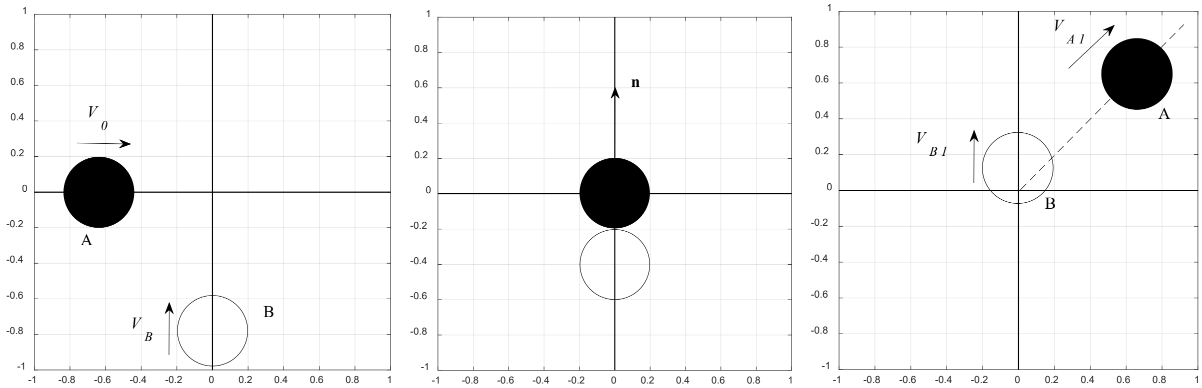
The momentum equation then gives

$$m_b V_{in} = -m_b V_{out} + m_C v_x^{B1} \Rightarrow v_x^{B1} = \frac{m_b}{m_C}(V_{in} + V_{out})$$

Therefore the restitution formula gives

$$-V_{out} - \frac{m_b}{m_C}(V_{in} + V_{out}) = -eV_{in}$$
$$\Rightarrow e = \frac{1}{m_C}\left[(m_C + m_b)\frac{V_{out}}{V_{in}} + m_b\right]$$

[3 POINTS]



6. The figure shows an oblique impact between two frictionless spheres with identical mass. Before the impact sphere A has a velocity $\mathbf{v}^{A0} = V_0\mathbf{i}$ and sphere B moves in the $\mathbf{j}$ direction with a speed to be determined. The collision direction $\mathbf{n}$ is parallel to $\mathbf{j}$. After impact, sphere A moves with a speed to be determined along a line parallel to $\mathbf{i}+\mathbf{j}$

6.1 Explain briefly why sphere B must continue to move parallel to the **j** direction after the collision.

The contact is frictionless, and the collision direction is parallel to **j** so no force acts on sphere B in the **i** direction during the collision. Since there is no force, there is no impulse in the **i** direction, so the **i** component of momentum can't change.

[2 POINTS]

6.2 Explain briefly why the **i** component of sphere A's velocity cannot change during the collision.

Same answer as 6.1!

[2 POINTS]

6.3 Hence, or otherwise, find formulas (in terms of V_0 and the restitution coefficient *e*) for the velocity vectors of spheres A and B after the collision, and the speed of sphere B before the collision.

Since the **i** component of the velocity of A does not change, it follows that the velocity vector of A after impact is

$$\mathbf{v}^{A1} = V_0(\mathbf{i}+\mathbf{j})$$

Since momentum is conserved, we also have that

$$m\mathbf{v}^{B1} + m\mathbf{v}^{A1} = m\mathbf{v}^{B0} + m\mathbf{v}^{A0}$$

$$\Rightarrow m v_y^{B1}\mathbf{j} + mV_0(\mathbf{i}+\mathbf{j}) = m v_y^{B0}\mathbf{j} + mV_0\mathbf{i}$$

In the normal direction the velocities must satisfy the 1D restitution formula

$$v_y^{A1} - v_y^{B1} = -e(v_y^{A0} - v_y^{B0})$$

$$\Rightarrow V_0 - v_y^{B1} = e v_y^{B0}$$

We can solve these equations for v_y^{B1}, v_y^{B0} , with the result

$$v_y^{B1} + V_0 = v_y^{B0}$$
$$V_0 - v_y^{B1} = e v_y^{B0}$$
$$\Rightarrow v_y^{B0} = \frac{2}{1+e} V_0 \qquad v_y^{B1} = \frac{1-e}{1+e} V_0$$

[4 POINTS]