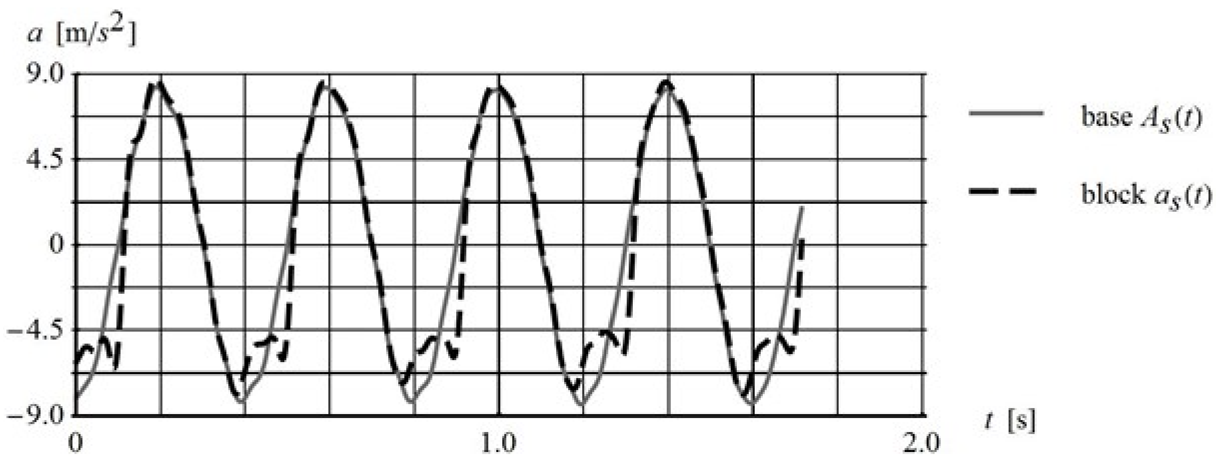
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday July 16, 2021



1. The solid line labeled 'base' on the figure (from this publication) shows a measurement from an accelerometer attached to a vibrating inclined ramp (the experiment was designed to show that earthquakes can cause sand and earth-piles to collapse)

1.1 The amplitude of the acceleration

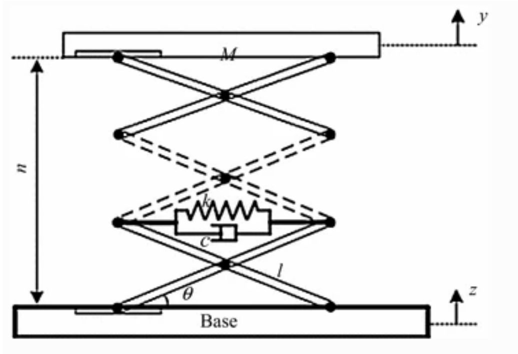
1.2 The period of the vibration

1.3 The frequency (in Hertz) and angular frequency (in rad/s)

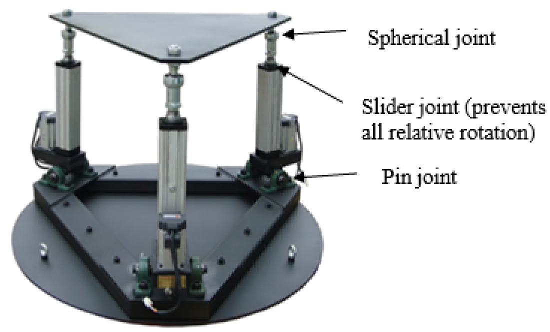
1.4 The amplitude of the velocity

1.5 The amplitude of the displacement

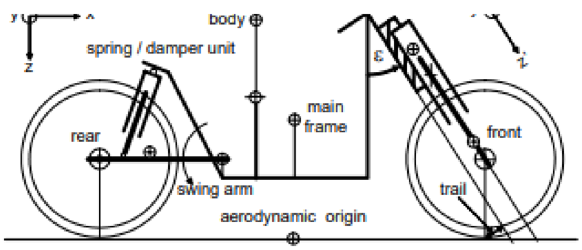
2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures (you may need to consult the publications to understand the system)



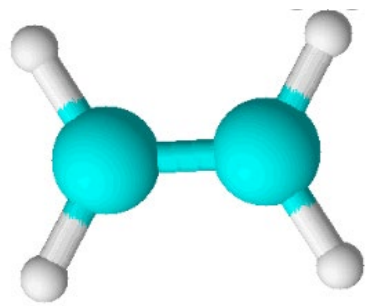
(a) 2D model of an energy harvesting system (assume the base is fixed)



(b) Motion simulation platform



(c) 2D idealization of a motorcycle (ignore steering, and treat the rider and frame as a single rigid body)



(d) Ethylene molecule (you can see the vibration modes here)

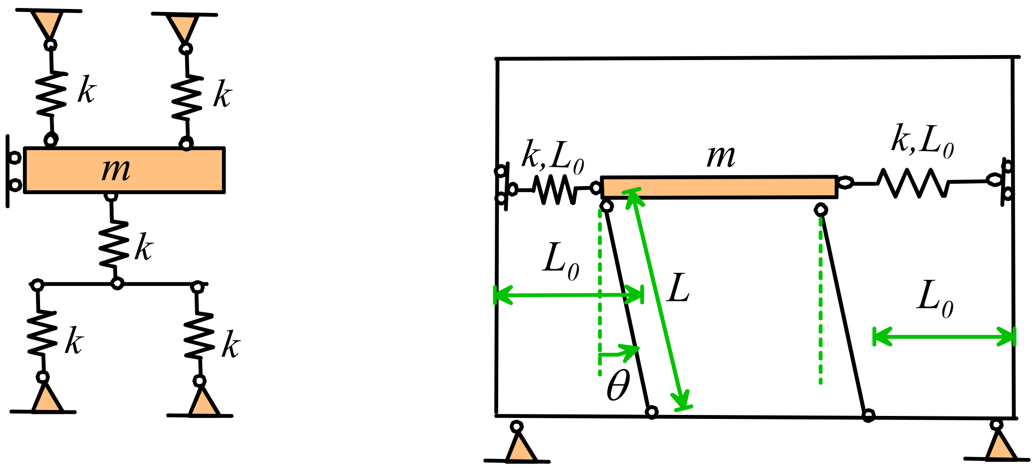
3. Solve the following differential equations (please solve them by hand, using the tabulated solutions to differential equation – you can check the answers with matlab if you like)

3.1 $\frac{d^2y}{dt^2} + 81y = 9 \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

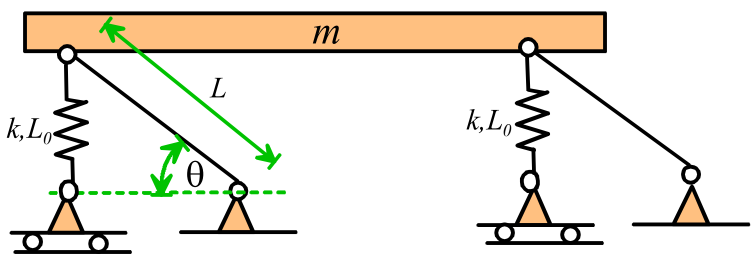
3.2 $\frac{1}{4}\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = 0 \qquad y = 1 \qquad \frac{dy}{dt} = 1 \qquad t = 0$

3.3 $\frac{d^2y}{dt^2} + 16y = \sin 5t \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

4. Find formulas for the natural frequency of vibration for the systems shown in the figure



5. The figure shows a proposed design for a vibrating conveyor (it's a bit simpler than a real configuration – which usually has inclined springs - to make it easier to analyze!). It vibrates at a frequency equal to its natural frequency. The goal of this problem is to find a formula for its natural frequency. When the system is at rest, the angle θ is 45 degrees.



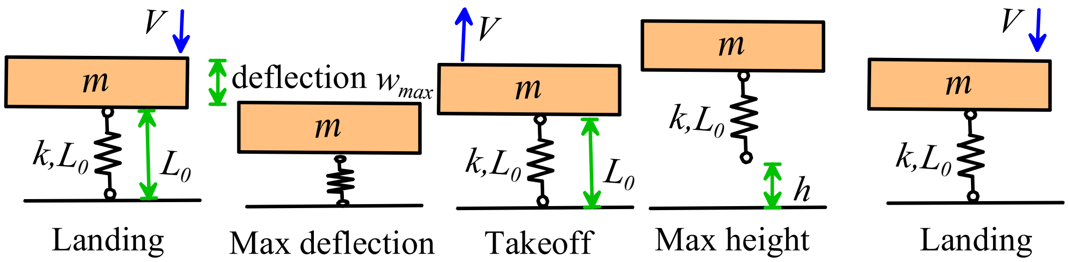
5.1 Use the energy method to show that θ satisfies the equation of motion

$$mL^2 \frac{d^2\theta}{dt^2} + kL(L\sin 2\theta - 2L_0 \cos\theta) = -mgL\cos\theta$$

5.2 Recall that θ is 45 when the system is at rest. Use the EOM to show that this requires an unstretched spring length

$$L_0 = \frac{L}{\sqrt{2}} + \frac{mg}{2k}$$

5.3 If the vibration amplitude is small then $\theta \approx \frac{\pi}{4} + \delta\theta$ where $\delta\theta \ll 1$. Linearize the equation of motion for small $\delta\theta$ and hence find a formula for the natural frequency (use the solution to 5.2 to eliminate L_0)



6. This famous publication shows that many aspects of the motion of a human hopping up and down in place can be predicted by idealizing the person jumping as a spring-mass system, as shown in the figure. This problem will repeat some of the authors calculations.

6.1 Suppose the person jumps to a height h above the ground. Find formulas for (1) the time that the person is airborne; and (2) the person's speed just before hitting the ground, in terms of g and h (this is not a vibration problem, of course – use straight line motion, eg).

6.2 Find a formula for the maximum deflection of the spring (which represents the person's legs bending) after the person hits the ground in terms of m,g,k and h (you could do this with energy. Include gravity, of course). Use the answer to show that the maximum force in the spring is

$$F_{\max} = kw_{\max} = mg\left(1+\sqrt{1+2kh/(mg)}\right)$$

6.3 Write down the equation of motion for the downward deflection of the mass w during the phase of motion while the spring is in contact with the ground (include gravity – you can get the equation using energy, Newton's law, or just write down the answer if you know it).

6.4 Write down the initial conditions for the equation of motion (i.e. the value of w and dw/dt at the instant the spring just touches the ground) in terms of L_0, g, h .

6.5 Hence, use the tabulated solutions to show that the solution for w as a function of time is

$$w = \frac{mg}{k} + \frac{mg}{k}\sqrt{1+\frac{2kh}{mg}}\sin\left(\sqrt{\frac{k}{m}}t+\phi\right) \qquad \phi = -\sin^{-1}\frac{1}{\sqrt{1+\dfrac{2kh}{mg}}}$$

(take $t=0$ to be the instant when the spring just touches the ground).

6.6 Use your solution to the previous problem to show that the person is in contact with the ground for a time

$$t_c = \sqrt{\frac{m}{k}}\left(\pi + 2\sin^{-1}\left\{1/\sqrt{1+\frac{2kh}{mg}}\right\}\right)$$

(you will need to use your solution to 6.5, and then find the roots of $\sin(\omega_n t+\phi) = \sin\phi$. You can figure this out by hand using a sketch of the sin function, or MATLAB will solve it for you).

6.7 If you jump up and down, you can control (roughly speaking) the maximum force in the spring $F_{\max}$ (by controlling the forces in your muscles) and the maximum spring deflection $w_{\max}$ (by choosing how much to bend your knees). Note that this means $k = F_{\max} / w_{\max}$ Show that, in terms of these variables:

(i) The height of your jump is given by

$$h = w_{\max}\left(\frac{F_{\max}}{2mg} - 1\right)$$

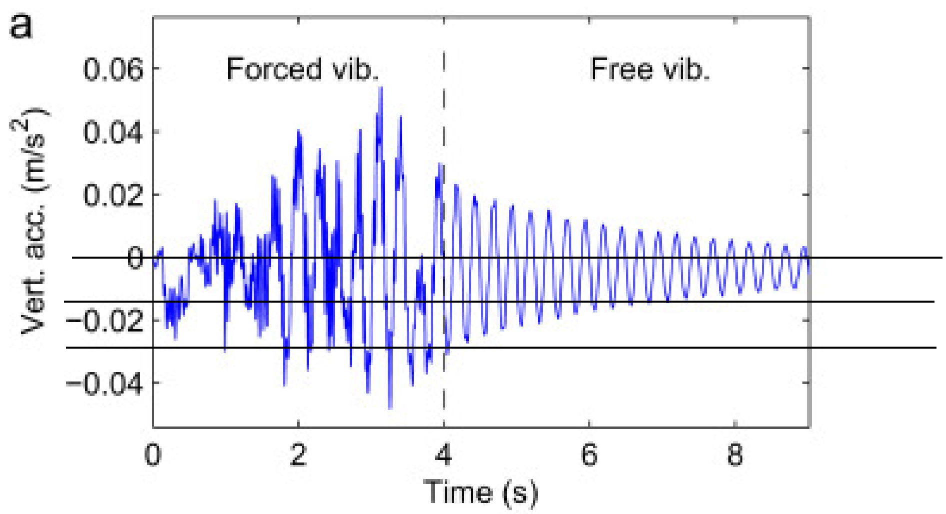
(use the solution to 6.2 to find *h* and eliminate *k* using $k = F_{\max} / w_{\max}$)

(ii) The time for one complete jump is given by

$$t_{jump} = 2\sqrt{\frac{w_{\max}}{g}\left(\frac{F_{\max}}{mg} - 2\right)} + \sqrt{\frac{mw_{\max}}{F_{\max}}}\left(\pi + 2\sin^{-1}\frac{1}{\sqrt{1 + \frac{F_{\max}}{mg}\left(\frac{F_{\max}}{mg} - 2\right)}}\right)$$

(use the solutions to 6.1 and 6.6 to find the total time for a jump, and eliminate *h* using (i), and *k* using $k = F_{\max} / w_{\max}$)

6.8 The publication suggests that (for an average person) the maximum effective spring force that leg muscles can develop is about 3000N, and the maximum leg displacement that the muscles are able to sustain is $w_{\max} \approx 0.5m$. Estimate the maximum jump height that you can achieve, and the resulting number of jumps per second. You could try to confirm your prediction experimentally, if you are curious, or just need the exercise.

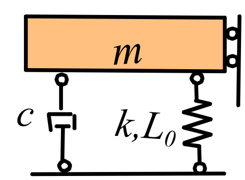


7. The figure (from this publication) shows the measured damped vibration response of a railway bridge.

7.1 Calculate the period and log decrement for the signal

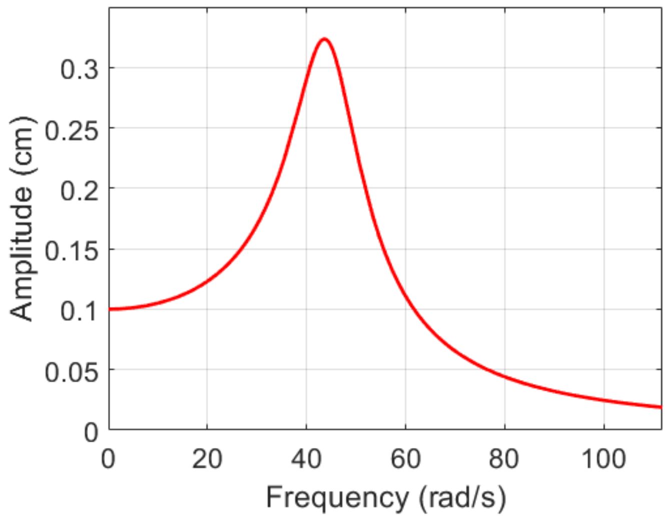
7.2 Hence determine the undamped natural frequency ω_n and damping coefficient ζ for the system

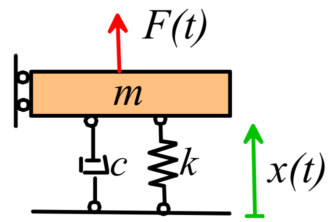
7.3 The effective mass of the bridge can be estimated from the numbers given in the table in the paper as 3.9×10^5kg. Estimate its effective stiffness k and dashpot coefficient c.



7.4 What value of c would be required to make the bridge critically damped?

7.5 If the system were critically damped, and at time $t=0$ is stationary and has an acceleration of 0.02 m/s^2 how long would it take for the acceleration to decay to 0.01m/s^2?





8. The spring-mass system shown in the figure is subjected to a harmonic force with amplitude 100N. The figure shows the measured steady-state amplitude of vibration of the mass as a function of the frequency of the force.

8.1 What (approximately) is the natural frequency of vibration of the system ω_n ?

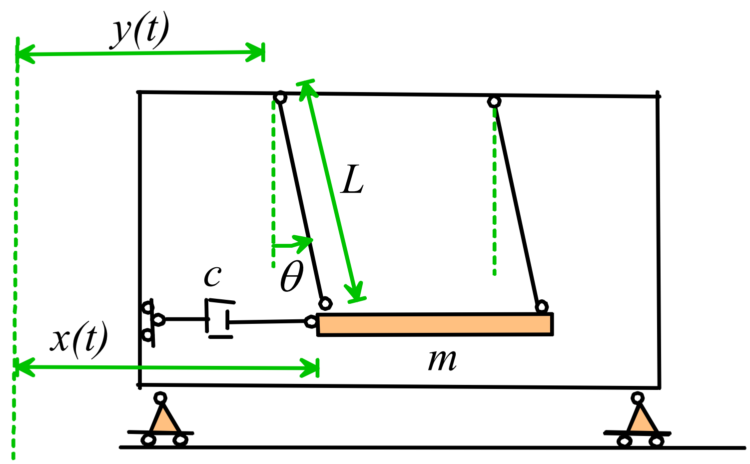
8.2 What is the damping factor ζ ?

8.3 What is the stiffness of the spring *k*,, the mass *m* and the dashpot coefficient *c*?

9 The spring-mass system described in the previous problem is at rest at time $t=0$, and has no force acting on it. At time $t>0$, a harmonic force with amplitude 100N and frequency 90 rad/s starts to act on the mass. **Neglect gravity**.

Plot a graph showing the displacement of the mass as a function of time, for $0<t<1$s (you will need to use the solutions to the differential equations for vibrating systems, and include the transient response. Be careful to get the phase right!). You only need to submit your plot and a brief explanation of your calculation – there is no need to submit MATLAB code

10 The figure shows a design for a vibration isolation system (see eg this design for a vibration isolation system for an atomic force microscope as practical example). The outer case vibrates horizontally with a harmonic displacement $y(t) = Y_0 \sin \omega t$. The goal of the design is to minimize the horizontal displacement of the platform $x(t)$.



10.1 Show that the vertical acceleration of the platform is related the angle θ by

$$a = L\cos\theta\left(\frac{d\theta}{dt}\right)^2 + L\sin\theta\frac{d^2\theta}{dt^2}$$

10.2 Draw a free body diagram showing the forces acting on the platform.

10.3 Note that problem 10.1 shows that if $\theta << 1$ the vertical acceleration of the platform is much smaller than its horizontal acceleration, and can be neglected. Show (with Newton's law and your solution to 10.2) that with this approximation, and assuming $\cos\theta \approx 1$, the equation of motion relating x to y is (approximately)

$$\frac{L}{g}\frac{d^2x}{dt^2} + \frac{Lc}{mg}\frac{dx}{dt} + x = \frac{Lc}{mg}\frac{dy}{dt} + y$$

Hence find formulas for the constants ω_n, ζ, K in the standard form for the equation.

10.4 Following the publication, the system is to be designed with the following specifications:
(1) The mass of the platform is 146kg, and carries a payload of 40kg
(2) The resonant frequency of the system is 0.5Hz
(3) The damping ratio of the system is $\zeta = 0.58$

Calculate the values of L and c that will meet this specification.

10.5 The lab in which the system is to be installed has a vibration with *acceleration* amplitude 8 m/s^2 at 11 Hz. What is the expected amplitude of the *displacement* of the platform of the vibration isolator?