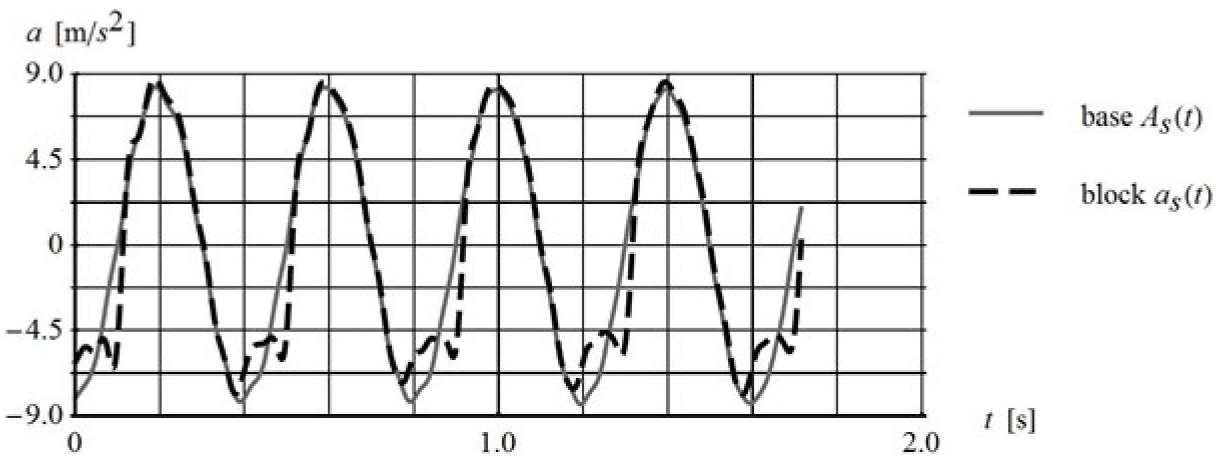
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday July 16, 2021



1. The solid line labeled 'base' on the figure (from this publication) shows a measurement from an accelerometer attached to a vibrating inclined ramp (the experiment was designed to show that earthquakes can cause sand and earth-piles to collapse)

1.1 The amplitude of the acceleration

From the graph, $A_0 \approx 8.5 m/s^2$

[1 POINT]

1.2 The period of the vibration

4 cycles takes 1.6 secs so T=0.4s.

[1 POINT]

1.3 The frequency (in Hertz) and angular frequency (in rad/s)

The frequency is 1/T=2.5Hz, or $\frac{2\pi}{0.4} = 5\pi rad/s$ rad/s

[1 POINT]

1.4 The amplitude of the velocity

The simple harmonic motion formulas give $A_0 = \omega V_0 \Rightarrow V_0 = 8.5/(5\pi) = 0.541 m/s$

[1 POINT]

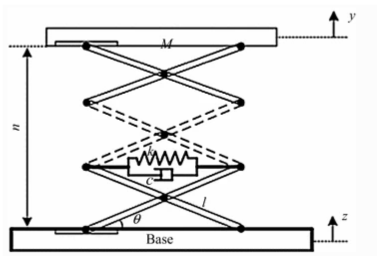
1.5 The amplitude of the displacement

The simple harmonic motion formulas give

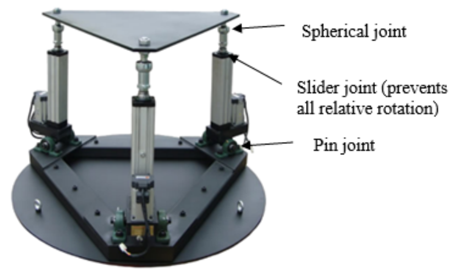
$V_0 = \omega X_0 \Rightarrow X_0 = 0.541/(5\pi) = 0.0344m$

[1 POINT]

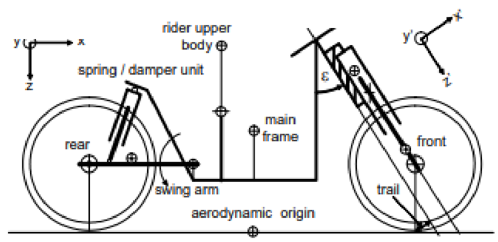
2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures (you may need to consult the publications to understand the system)



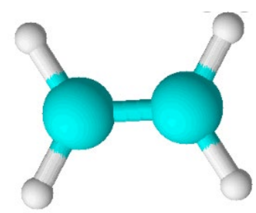
(a)2D model of an energy harvesting system (assume the base is fixed)



(b) Motion simulation platform



(c) 2D idealization of a motorcycle (ignore steering, and treat the rider and frame as a single rigid body)



(d) Ethylene molecule (you can see the vibration modes here)

For (a): the mass can only move vertically, so clearly there is only 1 DOF. But we can derive this with the formula too. There are 7 rigid bodies (the mass, plus the 6 links), 9 pin joints (2 constraints each) and two slider joints (1 constraint each). So the formula gives #DOF = 3r + 2p-c = 21 – 18 – 2 = 1. There are no rigid body modes since the base is fixed, so 1 vibration mode.

For (b) there are again 7 rigid bodies (the platform, plus two for each leg); there are 3 pin joints (5 constraints each), 3 spherical joints (3 constraints each) and 3 prismatic joints (5 constraints each – 3 rotations and relative motion in two directions are prevented). The formula gives #DOF = 6r + 3p –c = 6x7 – 3*5 – 3*3 – 3*5 = 3*(14-5-3-5) = 3 . There are no rigid body modes since the base is fixed, so 3 vibration modes.

For (c) there are 6 rigid bodies (the rear wheel, the swing arm attached to the rear wheel, the spring/damper unit, the main frame/rider, the front forks, and the front wheel). There are 4 pin joints (2 constraints each), two contacts (2 constraints each), two slider joints (2 constraints each – prevents rotation, and relative motion in one direction). So the formula gives #DOF = 3r + 2p-c = 18-8-4-4 = 2 DOF. There is one rigid body mode, since the bike can roll steadily in the horizontal direction, so 1 vibration mode. As an aside, the figure in the publication is actually incorrect – there should be another pin joint between the swing arm and the spring-damper unit, otherwise the rear suspension is over-constrained and can't move… If we use the figure, the only DOFs are rolling left/right, and rocking about the rear axle. If there was another pin joint the DOFs would be rolling, vertical motion of the body, and rocking of the body (with the wheels moving during body motion as dictated by the constraints)

For (d) 6 particles, no constraints so 18 DOF. There are 6 rigid body modes, so 12 vibration modes.

[8 POINTS]

3. Solve the following differential equations (please solve them by hand, using the tabulated solutions to differential equation – you can check the answers with matlab if you like)

3.1 $\frac{d^2 y}{dt^2} + 81y = 9 \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

3.2 $\frac{1}{4}\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + 4y = 0 \qquad y = 1 \qquad \frac{dy}{dt} = 1 \qquad t = 0$

3.3 $\frac{d^2 y}{dt^2} + 16y = \sin 5t \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

3.1

Rearrange in standard form

$$\frac{1}{81}\frac{d^2 y}{dt^2} + y = \frac{1}{9}$$

This is a Case I equation - compare with the standard form to see that $\omega_n = 9 \qquad C = 1/9$

The solution is

$$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

We are given $x_0 = 0 \quad v_0 = 0$ so

$$y(t) = \frac{1}{9}(1 - \cos 9t)$$

[3 POINTS]

3.2

Rearrange in standard form

$$\frac{1}{16}\frac{d^2 y}{dt^2} + \frac{2}{4}\frac{dy}{dt} + y = 0 \qquad y = 1 \qquad \frac{dy}{dt} = 1 \qquad t = 0$$

This is a Case III equation - compare with the standard form to see that $\omega_n = 4, \ \zeta = 1 \ C = 0$

The solution is

$$x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n(x_0 - C)\right]t\right\}\exp(-\omega_n t)$$

We are given $x_0 = 1 \quad v_0 = 1$ so

$$x(t) = \left\{1 + \left[1 + 4\right]t\right\}\exp(-4t) = \left\{1 + 5t\right\}\exp(-4t)$$

[3 POINTS]

3.3 $\frac{d^2 y}{dt^2} + 16y = \sin 5t \qquad y = 0 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

We can rearrange this as a Case 4 equation

$$\frac{1}{4^2}\frac{d^2 y}{dt^2} + y = \frac{1}{16}\sin 5t$$

It appears that $\omega = 5,\ KF_0 = 1/16,\ \omega_n = 4,\ \zeta = 0\ \ C = 0$.

The steady-state solution follows as

$$x_p(t) = X_0 \sin(\omega t + \phi) \qquad X_0 = KF_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \quad (-\pi < \phi < 0)$$

The homogeneous solution is

$$x_h(t) = \exp(-\varsigma\omega_n t)\left\{x_0^h \cos\omega_d t + \frac{v_0^h + \varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

where $\omega_d = \omega_n\sqrt{1-\varsigma^2}$ and

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin\phi$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0\omega\cos\phi$$

Substituting numbers gives

$$M(\omega/\omega_n, \zeta) = \frac{1}{\left\{(1-25/16)^2\right\}^{1/2}} = \frac{16}{9} \qquad \phi = \tan^{-1}(0) \quad = 0$$

$$X_0 = \frac{1}{9}$$

$$x_0^h = 0 \qquad v_0^h = -5/9$$

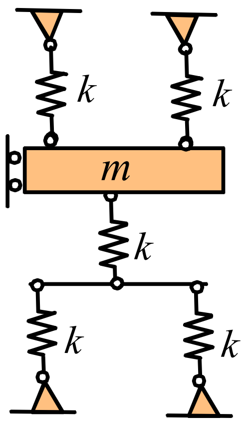
The total solution is therefore

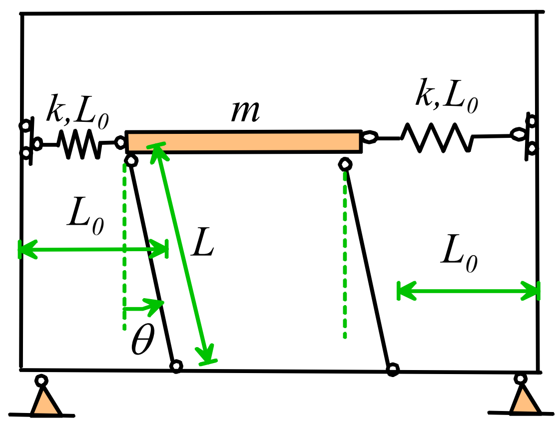
$$y(t) = \frac{1}{9}\sin 5t - \frac{5}{36}\sin 4t$$

We can check that this is correct by substituting it into the differential equation, and by substituting *t=0* into *y* and *dy/dt* and checking that initial conditions are satisfied.

[3 POINTS]

4. Find formulas for the natural frequency of vibration for the systems shown in the figure





For the first system, we can replace the springs with an equivalent single spring. On the bottom we have two springs in parallel, which together are in series with a single spring. The effective stiffness of this combination is

$$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{2k} \Rightarrow k_{eff} = 2k/3$$

This equivalent spring is in parallel with the pair of springs on top, so the effective stiffness of the entire assembly is 8*k/3*. The formula for natural frequency gives $\omega = \sqrt{8k/(3m)}$

[2 POINTS]

We can get an EOM for the second system using the energy method. The platform is in circular motion, so its speed (from the circular motion formula) is

$$v = L\left(\frac{d\theta}{dt}\right)$$

and therefore the KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$

The PE includes gravity and the energy of the springs. Geometry shows that the spring lengths are $L_0 + L\sin\theta,\ \ L_0 - L\sin\theta$ so

$$U = mgL\cos\theta + \frac{1}{2}k\left(L_0 + L\sin\theta - L_0\right)^2 + + \frac{1}{2}k\left(L_0 - L\sin\theta - L_0\right)^2$$

$$T+U=const \Rightarrow \frac{d}{dt}(T+U)=0$$

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right)-mgL\sin\theta\frac{d\theta}{dt}+kL^2\sin 2\theta\frac{d\theta}{dt}=0$$

$$mL^2\left(\frac{d^2\theta}{dt^2}\right)-mgL\sin\theta+kL^2\sin 2\theta=0$$

(here we used the formula $2\sin\theta\cos\theta=\sin 2\theta$ to make finding the small angle approximation easier but its fine to leave this term as just $2\sin\theta\cos\theta$)

To linearize the equation just set $\sin\theta\approx\theta,\ \ \sin 2\theta\approx 2\theta$, which gives

$$mL^2\left(\frac{d^2\theta}{dt^2}\right)+\left(2kL^2-mgL\right)\theta=0$$

$$\Rightarrow \frac{mL^2}{\left(2kL^2-mgL\right)}\left(\frac{d^2\theta}{dt^2}\right)+\theta=0$$

and compare to the standard case I EOM to see that

$$\omega_n=\sqrt{\frac{2kL^2-mgL}{mL^2}}$$

Remarks:

(1) This arrangement is sometimes used to design a vibration isolation system. You can tune the spring stiffness to make the natural frequency as low as you like.

(2) It's worth thinking about what happens if $mgL>2kL^2$. If that happens our formula predicts that the natural frequency is complex – what does that mean, exactly? We can figure this out in two different ways. One idea is to recognize that our equation of motion

$$mL^2\left(\frac{d^2\theta}{dt^2}\right)+\left(2kL^2-mgL\right)\theta=0$$

has a problem in this limit, because the coefficient of θ is negative. So it's not a 'Case I' equation. But we can turn it into a 'Case II' equation if we want

$$mL^2\left(\frac{d^2\theta}{dt^2}\right)-\left(mgL-2kL^2\right)\theta=0$$

$$\frac{mL^2}{\left(mgL-2kL^2\right)}\left(\frac{d^2\theta}{dt^2}\right)-\theta=0$$

Which has the form

$$\frac{1}{\alpha^2}\left(\frac{d^2\theta}{dt^2}\right)-\theta=0$$

The solution this equation has the form $\theta=Ae^{\alpha t}+Be^{-\alpha t}$, where A, B are constants (they depend on the initial conditions). So the math tells us that this system won't vibrate, but instead θ will increase exponentially. This looks a bit weird too, because that means the platform would zip around in a circle at progressively increasing speed, but of course our EOM is approximate, and only accurate when θ is small. So what we are really learning is that the system is unstable, and will collapse, if the springs are too soft. That makes sense.

There's another way to look at the math if you happen to be familiar with trig functions of complex numbers. Math will actually let us work with complex valued natural frequencies – if we have an equation of the form

$$\frac{1}{\omega_n^2}\left(\frac{d^2\theta}{dt^2}\right)+\theta=0$$

where $\omega_n = i\alpha$ is a complex number, we can use the 'Case I' solution $\theta = A\sin i\alpha t + B\cos i\alpha t$. Euler's formula for complex numbers tells us that trig functions of complex numbers are actually exponentials – for example

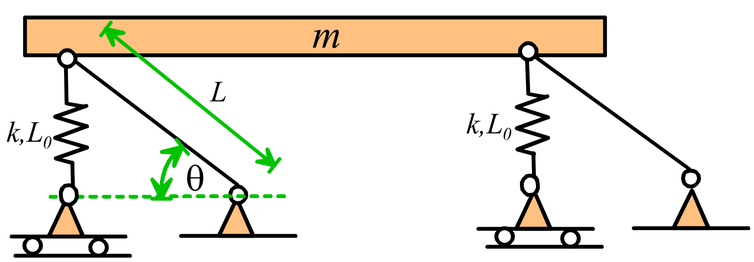
$$e^{i\beta} = \cos\beta + i\sin\beta \Rightarrow \cos\beta = (e^{i\beta t} + e^{-i\beta t})/2$$

$$\cos i\alpha t = (e^{-\alpha t} + e^{\alpha t})/2$$

So again, the math is telling us that the motion is not harmonic vibrations any more, but exponential growth, i.e. collapse.

[3 POINTS]

5. The figure shows a proposed design for a vibrating conveyor (it's a bit simpler than a real configuration – which usually has inclined springs - to make it easier to analyze!). It vibrates at a frequency equal to its natural frequency. The goal of this problem is to find a formula for its natural frequency, and hence to determine the spring stiffness *k* and unstretched spring length L_0 needed. When the system is at rest, the angle θ is 45 degrees.



5.1 Use the energy method to show that *y* satisfies the equation of motion

$$mL^2\frac{d^2\theta}{dt^2} + kL(L\sin 2\theta - 2L_0\cos\theta) = -mgL\cos\theta$$

The kinetic energy of the system is $T = \frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2$ (to see this, note that (i) the platform has to remain level; (ii) All points on the platform have the same speed; (iii) The points where the platform is attached to the inclined bars clearly are in circular motion about the pivot. We can get the speed using the circular motion formula).

The potential energy is $U = k\left(L\sin\theta - L_0\right)^2 + mgL\sin\theta$

Using the energy method we get

$$\frac{d}{dt}\left(T+U\right) = mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + 2k(L\sin\theta - L_0)L\cos\theta\frac{d\theta}{dt} + mgL\cos\theta\frac{d\theta}{dt} = 0$$

$$\Rightarrow mL^2\frac{d^2\theta}{dt^2} + 2kL(L\sin\theta\cos\theta - L_0\cos\theta) = -mgL\cos\theta$$

$$\Rightarrow mL^2\frac{d^2\theta}{dt^2} + kL(L\sin 2\theta - 2L_0\cos\theta) = -mgL\cos\theta$$

[3 POINTS]

5.2 Recall that θ is 45 when the system is at rest. Use the EOM to show that this requires an unstretched spring length

$$L_0 = \frac{L}{\sqrt{2}} + \frac{mg}{2k}$$

If the system is at rest the angular acceleration is zero, so the equation of motion gives

$$kL(L\sin 90 - 2L_0\cos 45) = -mgL\cos 45$$

$$\Rightarrow L_0 = \frac{L}{\sqrt{2}} + \frac{mg}{2k}$$

[2 POINTS]

5.3 If the vibration amplitude is small then $\theta \approx \frac{\pi}{4} + \delta\theta$ where $\delta\theta \ll 1$. Linearize the equation of motion for small $\delta\theta$ and hence find a formula for the natural frequency (use the solution to 5.2 to eliminate L_0)

$$mL^2\frac{d^2\delta\theta}{dt^2} + kL\left(L\sin\left(\frac{\pi}{2}+2\delta\theta\right) - 2\left\{\frac{L}{\sqrt{2}} + \frac{mg}{2k}\right\}\cos\left(\frac{\pi}{4}+\delta\theta\right)\right) = -mgL\cos\left(\frac{\pi}{4}+\delta\theta\right)$$

$$\Rightarrow mL^2\frac{d^2\delta\theta}{dt^2} + kL^2\delta\theta \approx 0$$

$$\Rightarrow \frac{m}{k}\frac{d^2\delta\theta}{dt^2} + \delta\theta \approx 0$$

$$\omega_n = \sqrt{k/m}$$

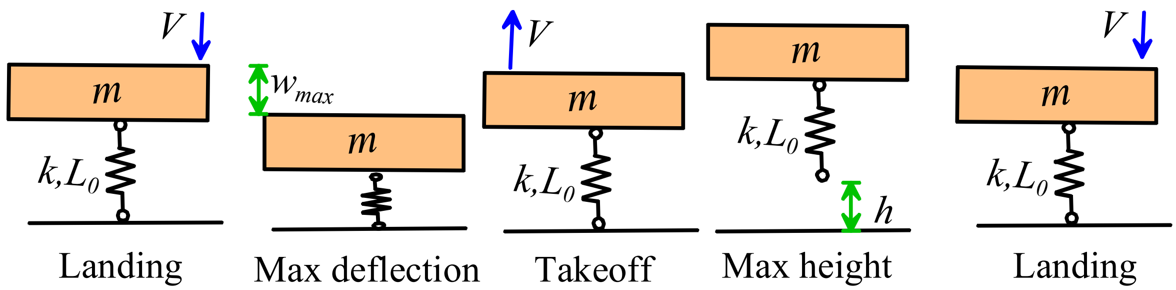
MATLAB can help with the Taylor series, eg

```
syms x m g k L
f = k*L*(L*sin(pi/2+x)-2*(L/sqrt(2)+m*g/(2*k))*cos(pi/4+x)) + m*g*L*cos(pi/4+x)
simplify(series(f,x,'Order',1))
```

But if you prefer you can also just expand all the terms with the double angle formulas

In retrospect the answer is obvious – we have two springs in parallel, which have effective stiffness $2k$. Since the mass moves at a 45 degree angle to the vertical, its speed in the horizontal direction and vertical directions are equal – it therefore has kinetic energy mv_y^2 - i.e. twice the KE of a mass moving only vertically. It therefore has an apparent mass equal to $2m$. So the natural frequency is $\omega_n = \sqrt{2k/(2m)} = \sqrt{k/m}$

[3 POINTS]



6. This famous publication shows that many aspects of the motion of a human hopping up and down in place can be predicted by idealizing the person jumping as a spring-mass system, as shown in the figure. This problem will repeat some of the authors calculations.

6.1 Suppose the person jumps to a height h above the ground. Find formulas for (1) the time that the person is airborne; and (2) the person's speed just before hitting the ground, in terms of g and h.

We can solve the second problem with energy:

- take the system to be the earth + the jumper; the initial state as the highest point of the jump; the final state as the instant just before landing.
- In the initial state the person has zero velocity so the KE is zero, and the PE is mgh.
- In the final state the person has speed V; the KE is $(1/2)mV^2$ and the PE is zero.
- No external forces means energy is conserved, so $mgh = mV^2/2 \Rightarrow V = \sqrt{2gh}$.

We could get the same answer using straight-line motion constant accel formulas too, of course.

We can get the time airborne using impulse-momentum:

- Take the system to be the jumper;
- he/she is subjected to a gravitational force $-mg\mathbf{j}$ while airborne.
- Take the initial state to be the instant just after take-off and the final state to be the instant just before landing. Note that since the PE is the same at both times, and energy is conserved, the jumper must have the same speed at both instants.
- The initial linear momentum is $mV\mathbf{j} = m\sqrt{2gh}\mathbf{j}$; the final linear momentum is $-mV\mathbf{j} = -m\sqrt{2gh}\mathbf{j}$.
- The gravitational force exerts an impulse $-mgt_a\mathbf{j}$ where t_a is the time airborne.
- Impulse-momentum gives $-mgt_a\mathbf{j} = -m\sqrt{2gh}\mathbf{j} - m\sqrt{2gh}\mathbf{j} \Rightarrow t_a = 2\sqrt{2h/g}$

[2 POINTS]

6.2 Find the maximum deflection of the spring (which represents the person's legs bending) after the person hits the ground (you could do this with energy. Include gravity, of course). Use the answer to find a formula for the maximum force in the spring.

We can do this with energy as well

- Take the system to be earth+jumper
- Take the initial state to be
- In the initial state the person has zero velocity so the KE is zero, and the PE is *mgh*.
- In the final state the pe is $-mgw_{\max} + \frac{1}{2}kw_{\max}^2$ and the KE is again zero.

Hence

$$mgh = -mgw_{\max} + \frac{1}{2}kw_{\max}^2$$
$$\Rightarrow w_{\max} = \frac{mg + \sqrt{(mg)^2 + 2mghk}}{k} = \frac{mg}{k}\left(1 + \sqrt{1 + 2kh/(mg)}\right)$$

The maximum force is $F_{\max} = kw_{\max} = mg\left(1 + \sqrt{1 + 2kh/(mg)}\right)$

[3 POINTS]

6.3 Write down the equation of motion for the downward deflection of the mass w during the phase of motion while the spring is in contact with the ground (include gravity – you can get the equation using energy, Newton's law, or just write down the answer if you know it.

Using the energy method:

- the KE is $T = \frac{1}{2}m\left(\frac{dw}{dy}\right)^2$
- the PE is $U = \frac{1}{2}kw^2 - mgw$
- Energy conservation implies that

$$\frac{d}{dt}\left(T + U\right) = m\frac{dw}{dt}\frac{d^2w}{dt^2} + kw\frac{dw}{dt} - mg\frac{dw}{dt} = 0$$
$$\Rightarrow \frac{m}{k}\frac{d^2w}{dt^2} + w = \frac{mg}{k}$$

[2 POINTS]

6.4 Write down the initial conditions for the equation of motion (i.e. the value of *w* and *dw/dt* at the instant the spring just touches the ground) in terms of L_0, g, h .

The initial conditions are $w = 0 \quad \frac{dw}{dt} = \sqrt{2gh}$

[1 POINT]

6.5 Hence, a formula for w as a function of time (take $t=0$ to be the instant when the spring just touches the ground).

Using the tabulated solution to the 'Case I' equation we get

$$w = \frac{mg}{k} + \sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2gh}{k/m}} \sin\left(\sqrt{\frac{k}{m}}t + \phi\right) = \frac{mg}{k} + \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}} \sin\left(\sqrt{\frac{k}{m}}t + \phi\right)$$

$$\phi = \sin^{-1} \frac{(-mg/k)}{\sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2gh}{k/m}}} = -\sin^{-1} \frac{1}{\sqrt{1 + \frac{2kh}{mg}}}$$

Or alternatively

$$w = \frac{mg}{k} - \frac{mg}{k}\cos\left(\sqrt{\frac{k}{m}}t\right) + \sqrt{\frac{2gh}{k/m}} \sin\left(\sqrt{\frac{k}{m}}t\right)$$

[3 POINTS]

6.5 Use your solution to the previous problem to find a formula for the time that the person is in contact with the ground, in terms of k,m,g,h (you will need to use your solution to 6.5, and then find the roots of $\sin(\omega_n t + \phi) = \sin\phi$. You can figure this out by hand using a sketch of the sin function, but if you like, MATLAB will do this for you too).

The person leaves the ground when $w = 0$ so

$$0 = \frac{mg}{k} + \frac{mg}{k}\sqrt{1 + \frac{2kh}{mg}} \sin\left(\sqrt{\frac{k}{m}}t + \phi\right)$$

$$\Rightarrow \sin\left(\sqrt{\frac{k}{m}}t + \phi\right) = \frac{-1}{\sqrt{1 + \frac{2kh}{mg}}} = \sin(\phi)$$

Where we have used the solution for ϕ from 6.4 to get the last line.

This has roots $t_c = 0, \qquad \sqrt{k/m}\, t_c = \pi - 2\phi \Rightarrow t_c = \pi\sqrt{m/k} + 2\sqrt{m/k}\sin^{-1} \frac{1}{\sqrt{1 + \frac{2kh}{mg}}}$

[3 POINTS]

6.6 If you jump up and down, you can control (roughly speaking) the maximum force in the spring $F_{\max}$ (by controlling the forces in your muscles) and the maximum spring deflection $w_{\max}$ (by choosing how much to bend your knees). Note that this means $k = F_{\max} / w_{\max}$ Show that, in terms of these variables:

(i) The height of your jump is given by

$$h = w_{\max}\left(\frac{F_{\max}}{2mg} - 1\right)$$

(ii) The time for one complete jump is given by

$$t_{jump} = 2\sqrt{\frac{w_{\max}}{g}\left(\frac{F_{\max}}{mg} - 2\right)} + \sqrt{\frac{mw_{\max}}{F_{\max}}}\left(\pi + 2\sin^{-1}\frac{1}{\sqrt{1 + \frac{F_{\max}}{mg}\left(\frac{F_{\max}}{mg} - 2\right)}}\right)$$

Problem 6.2 tells us that

$$F_{\max} = mg\left(1 + \sqrt{1 + 2kh/(mg)}\right)$$

$$\Rightarrow 1 + 2kh/(mg) = \left(\frac{F_{\max}}{mg} - 1\right)^2$$

$$\Rightarrow h = \frac{mg}{2k}\left\{\left(\frac{F_{\max}}{mg} - 1\right)^2 - 1\right\} = \frac{mgw_{\max}}{2F_{\max}}\left\{\left(\frac{F_{\max}}{mg} - 2\right)\frac{F_{\max}}{mg}\right\}$$

$$h = w_{\max}\left(\frac{F_{\max}}{2mg} - 1\right)$$

Problems 6.1 and 6.6 tell us that

$$t_{jump} = 2\sqrt{2h/g} + \pi\sqrt{m/k} + 2\sqrt{m/k}\sin^{-1}\frac{1}{\sqrt{1 + \frac{2kh}{mg}}}$$

[3 POINTS]

6.7 The publication suggests that (for an average person) the maximum effective spring force that leg muscles can develop is about 3000N, and the maximum leg displacement that the muscles are able to sustain is $w_{\max} \approx 0.5m$. Estimate the maximum jump height that you can achieve, and the resulting number of jumps per second

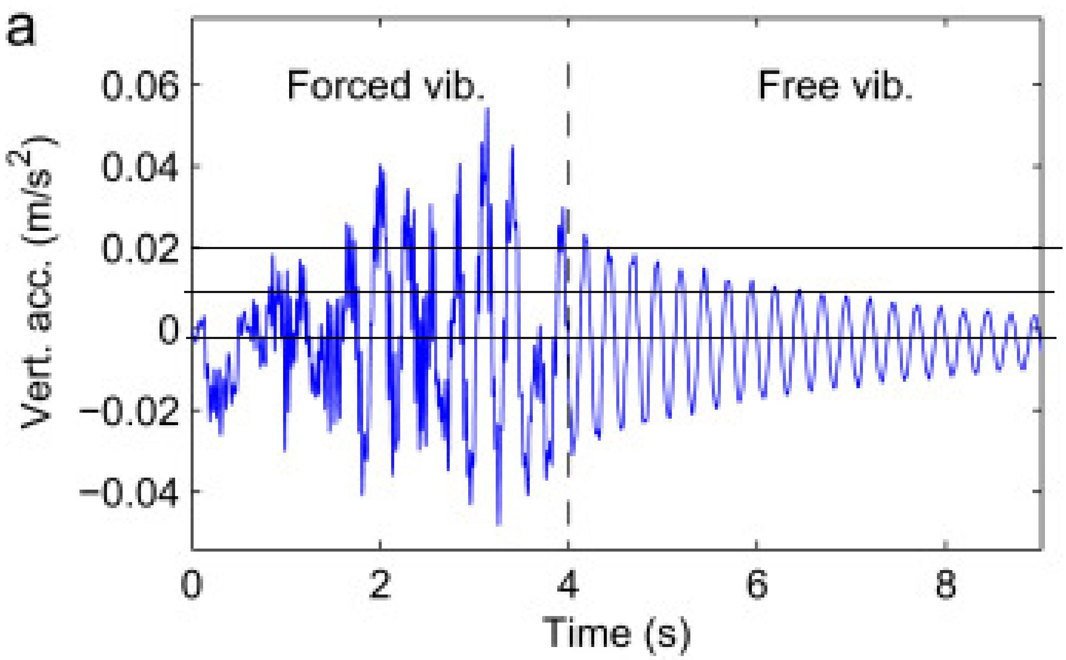
The answer will depend on the person's weight. For a 70 kg person we get

$$h \approx 0.59m$$

$$t_a = 0.69s \qquad t_c = 0.404 \qquad t_{jump} = 1.1 \qquad \Rightarrow N = 1/t_{jump} = 0.91 jumps/s$$

You are in better shape than I am if you are able to achieve this. The force-plate measurements conducted in lecture ?? suggest 2 jumps per second, and a peak force of 2400 N.

[2 POINTS]



7. The figure (from this publication) shows the measured damped vibration response of a railway bridge.

7.1 Calculate the period and log decrement for the signal

There are about 20 cycles in 5 seconds – the period is 5/20 = 0.25s.

Take $x_0 = 0.02$ to be the second peak after 4 sec, and note that the amplitude has decayed to about $x_9 = 0.01$ after 9 cycles. Therefore

$$\delta = \frac{1}{9}\ln(\frac{0.02}{0.01}) = 0.0770$$

[2 POINTS]

7.2 Hence determine the undamped natural frequency ω_n and damping coefficient ζ for the system

We can use the formulas from the lectures/notes

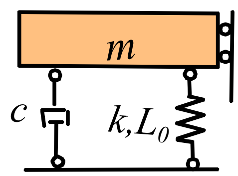
$$\omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T} = 25.13 \;\; rad/s$$

-

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.0123$$

[2 POINTS]

7.3 The effective mass of the bridge can be estimated from the numbers given in the table in the paper as $3.9 \text{x} 10^5$kg. Estimate its effective stiffness *k* and dashpot coefficient *c*.



We know that

$$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2 = 2.46 \times 10^8 \, N/m$$

$$\zeta = \frac{c}{2\sqrt{km}} \Rightarrow c = 2\zeta\sqrt{km} = 2.4 \times 10^5 \, N/m$$

[2 POINTS]

7.4 What value of *c* would be required to make the bridge critically damped?

For critical damping $\zeta = 1 \Rightarrow 2\zeta\sqrt{km} = 1.96 \times 10^7 \, Ns/m$

[1 POINT]

7.5 If the system were critically damped, and at time *t=0* is stationary and has an acceleration of 0.02 m/s^2 how long would it take for the acceleration to decay to 0.01m/s^2?

The solution for the critically damped spring-mass system is

$$x(t) = C + \left\{ (x_0 - C) + \left[v_0 + \omega_n (x_0 - C) \right] t \right\} \exp(-\omega_n t)$$

The acceleration follows as

$$a(t) = \frac{d^2 x}{dt^2} = \left(\omega_n \left\{ (x_0 - C) + \left[v_0 + \omega_n (x_0 - C) \right] t \right\} - 2\left[v_0 + \omega_n (x_0 - C) \right] \right) \omega_n \exp(-\omega_n t)$$

With zero initial velocity this simplifies to

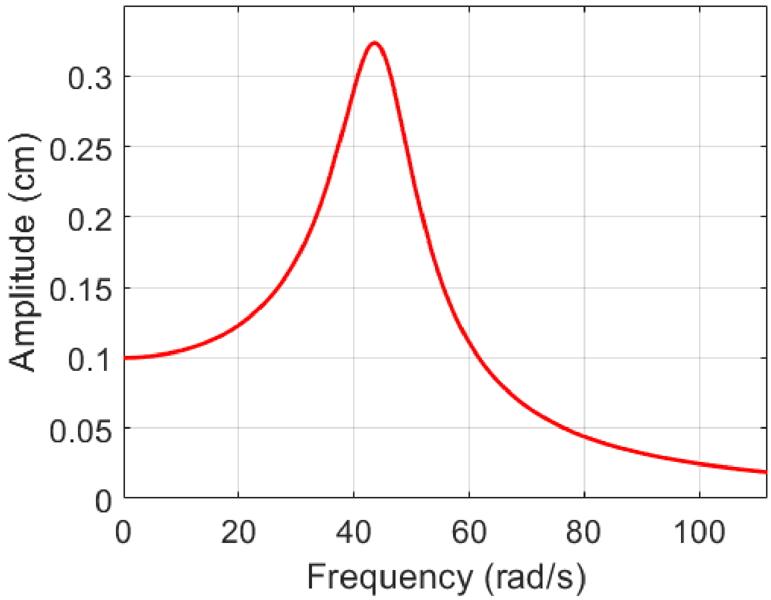
$$a(t) = (C - x_0)\omega_n^2 (1 - t\omega_n) \exp(-\omega_n t)$$

If *a(0)*=0.02 then $\omega_n^2 (C - x_0) = 0.02 \Rightarrow a(t) = 0.02(1 - t\omega_n)\exp(-\omega_n t)$. The time to reach *a*=0.01m/s^2 satisfies

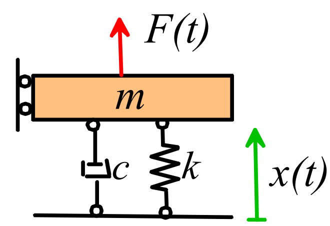
$$0.02(1 - t\omega_n)\exp(-\omega_n t) = 0.01$$

We can substitute $\omega_n = 25.13$ and ask MATLAB to solve for *t*. This gives *t=0.0125s*.

[3 POINTS]



8. The spring-mass system shown in the figure is subjected to a harmonic force with amplitude 100N. The figure shows the measured steady-state amplitude of vibration of the mass as a function of the frequency of the force.



8.1 What (approximately) is the natural frequency of vibration of the system ω_n ?

This is a standard spring-mass system so there is no need to derive the EOM; we can use the solution from the notes. The formula for vibration amplitude is

$$X_0 = KF_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{1}{\left\{ \left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2 \right\}^{1/2}}$$

$$\omega_n = \sqrt{k/m} \qquad \zeta = c/(2\sqrt{km}) \qquad K = 1/k$$

From the discussion in the notes, we know that (as long as the damping is fairly small) the maximum occurs when $\omega \approx \omega_n$, so $\omega_n \approx 45\, rad/s$

[1 POINT]

8.2 8.3 What is the damping factor ζ ?

The maximum value of the magnification *M* occurs roughly when $\omega = \omega_n$ and has value $M = 1/2\zeta$. We have to be careful to calculate *M* correctly. We know that

(1) *M=1* when $\omega = 0$, therefore we can get the value of KF_0 from the displacement amplitude at zero frequency, i.e. $KF_0 = 0.001\,\mathrm{m}$

(2) We can now calculate $M = X_0 / (KF_0)$. At the peak, $M = 0.00325 / 0.001 = 3.25$

Therefore $\zeta = \dfrac{1}{2M_{\max}} = \dfrac{1}{6.5} = 0.15$

[2 POINTS]

8.3 What is the stiffness of the spring *k*,, the mass *m* and the dashpot coefficient *c*?

We know that $KF_0 = 0.001\,\mathrm{m}$ (from the previous problem) and $F_0 = 100$ N, therefore K=10^{-4} m/N, and since $K = 1/k$ we find that k=10^5 N/m.

Then $\omega_n = \sqrt{k/m} \Rightarrow m = k/\omega_n^2 = 49kg$

And finally $\zeta = c/(2\sqrt{km}) \Rightarrow c = 2\zeta\sqrt{km} = 660Ns/m$

[3 POINTS]

9 The spring-mass system described in the previous problem is at rest at time *t=0*, and has no force acting on it. At time *t>0*, a harmonic force with amplitude 100N and frequency 90 rad/s starts to act on the mass. **Neglect gravity**.

Plot a graph showing the displacement of the mass as a function of time, for 0<*t*< 1s (you will need to use the solutions to the differential equations for vibrating systems, and include the transient response. Be careful to get the phase right!). You only need to submit your plot and a brief explanation of your calculation – there is no need to submit MATLAB code

We know the properties of the system from the previous problem:

$\omega_n \approx 45\,rad/s$

$\zeta = 0.15$

$KF_0 = 0.001\,\mathrm{m}$

The tabulated solutions give the solution as

$$x(t) = C + x_h(t) + x_p(t)$$

The value of *C* is zero for the standard spring-mass system with no gravity, and

$$x_p(t) = X_0 \sin(\omega t + \phi) \qquad X_0 = KF_0 M(\omega / \omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{1}{\left\{ \left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2 \right\}^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2} \right) \quad (-\pi < \phi < 0)$$

and (since the system is underdamped) the transient solution is

$$x_h(t) = \exp(-\varsigma\omega_n t) \left\{ x_0^h \cos\omega_d t + \frac{v_0^h + \varsigma\omega_n x_0^h}{\omega_d} \sin\omega_d t \right\}$$

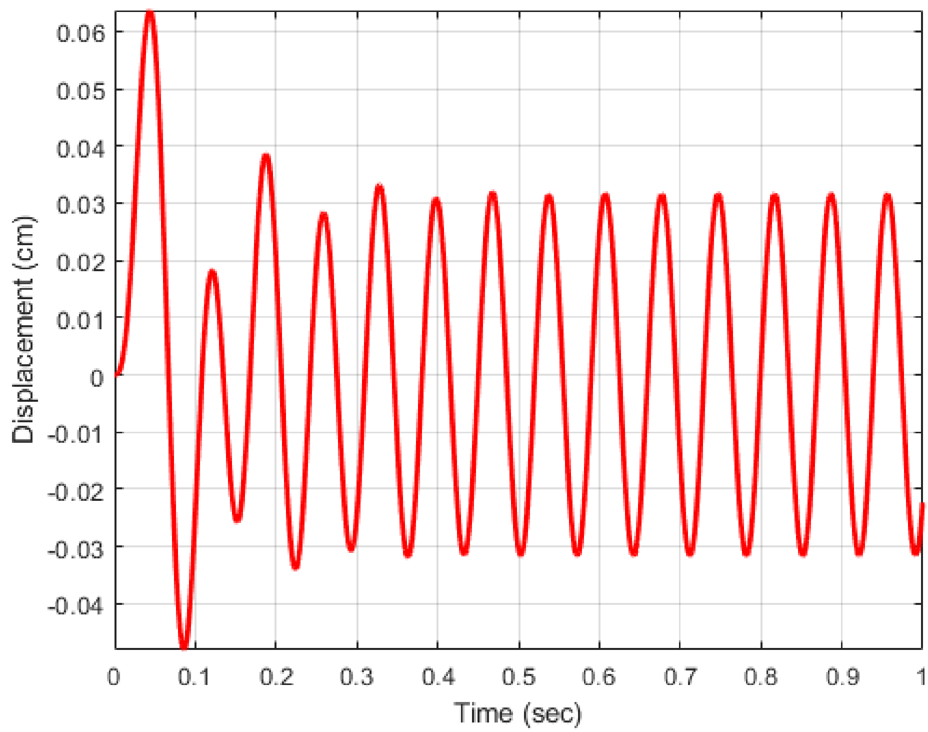
where $\omega_d = \omega_n \sqrt{1-\varsigma^2}$ and

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin\phi$$

$$v_0^h = v_0 - \left. \frac{dx_p}{dt} \right|_{t=0} = v_0 - X_0 \omega \cos\phi$$

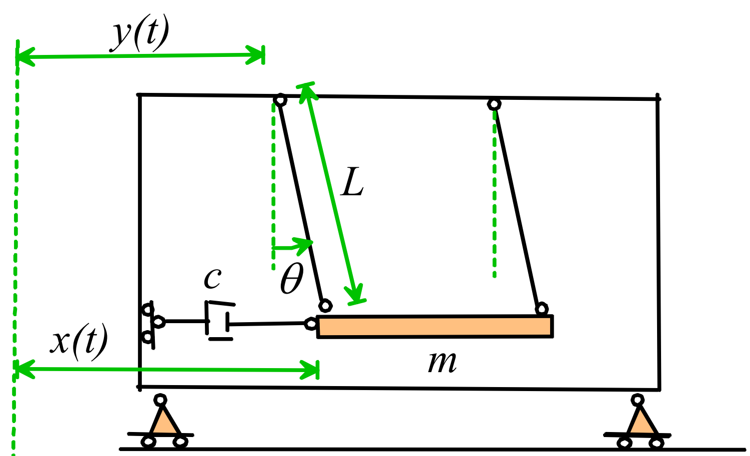
We are given that $x_0 = v_0 = 0$. It's straightforward to code all this stuff in MATLAB:

```
clear all
syms t
wn = 45;
zeta = 0.25;
wd = wn*sqrt(1-zeta^2);
KF0 = 0.1;
w = 90;
M = 1/sqrt( (1-w^2/wn^2)^2 + (2*zeta*w/wn)^2 );
X0 = KF0*M;
phi = atan(-2*zeta*(w/wn)/(1-w^2/wn^2));
if (phi>0)
    phi = phi-pi;
end
xp = X0*sin(w*t+phi);
x0h = -X0*sin(phi);
v0h = -X0*w*cos(phi);
xh = exp(-zeta*wn*t)*(x0h*cos(wd*t) + (v0h+zeta*wn*x0h)/wd*sin(wd*t));
x = xp + xh;
fplot(x,[0,1],'LineWidth',2,'Color',[1,0,0]);
xlabel('Time (sec)')
ylabel('Displacement (cm)')
grid on
```



[3 POINTS]

10 The figure shows a design for a vibration isolation system (see eg this design for a vibration isolation system for an atomic force microscope as practical example). The outer case vibrates horizontally with a harmonic displacement $y(t) = Y_0 \sin \omega t$. The goal of the design is to minimize the horizontal displacement of the platform $x(t)$.



10.1 Show that the vertical acceleration of the platform is related the angle θ by

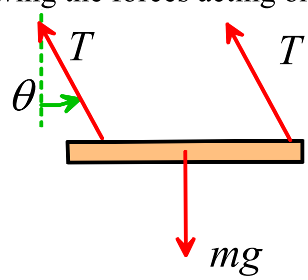
$$a = L\cos\theta\left(\frac{d\theta}{dt}\right)^2 + L\sin\theta\frac{d^2\theta}{dt^2}$$

We can take the origin for the vertical coordinate at one of the pivots. The height of the platform above the pivot is

$$h = -L\cos\theta$$
$$\frac{dh}{dt} = L\sin\theta\frac{d\theta}{dt}$$
$$a = L\cos\theta\left(\frac{d\theta}{dt}\right)^2 + L\sin\theta\frac{d^2\theta}{dt^2}$$

[2 POINTS]

10.2 Draw a free body diagram showing the forces acting on the platform.



[2 POINTS]

10.3 Note that problem 3.1 shows that if $\theta << 1$ the vertical acceleration of the platform is much smaller than its horizontal acceleration, and can be neglected. Show that with this approximation, and assuming $\cos\theta \approx 1$ the equation of motion relating x to y is (approximately)

$$\frac{L}{g}\frac{d^2x}{dt^2} + \frac{Lc}{mg}\frac{dx}{dt} + x = \frac{Lc}{mg}\frac{dy}{dt} + y$$

Hence find formulas for the constants ω_n, ζ, K in the standard form for the equation.

Geometry shows that $\sin\theta = (x-y)/L$.

Newton's law in the horizontal and vertical directions gives

$$m\frac{d^2x}{dt^2} = -2T\sin\theta - c\frac{d(x-y)}{dt}$$
$$ma = 2T\cos\theta - mg \Rightarrow 2T \approx mg$$
$$\Rightarrow m\frac{d^2x}{dt^2} = -mg\frac{x-y}{L} - c\frac{d(x-y)}{dt}$$
$$\frac{L}{g}\frac{d^2x}{dt^2} + \frac{Lc}{mg}\frac{dx}{dt} + x = \frac{Lc}{mg}\frac{dy}{dt} + y$$

Comparing these with the standard form gives

$$\omega_n = \sqrt{\frac{g}{L}} \qquad \frac{2\zeta}{\omega_n} = \frac{Lc}{mg} \Rightarrow \zeta = \frac{Lc}{2mg}\sqrt{\frac{g}{L}} = \frac{c}{2m}\sqrt{\frac{L}{g}}$$

[3 POINTS]

10.4 Following the publication, the system is to be designed with the following specifications:
(1) The mass of the platform is 146kg, and carries a payload of 40kg
(2) The resonant frequency of the system is 0.5Hz
(3) The damping ratio of the system is $\zeta = 0.58$

Calculate the values of L and c that will meet this specification.

Here's the MATLAB calculation

```
m = 186
f = 0.5;
wn = 2*pi*f;
zeta = 0.58;
g = 9.81;
L=g/wn^2
c = zeta*2*m*sqrt(g/L)
```

```
L = 0.9940
c = 677.8300
```

So $L = 1m, \quad c = 678Ns/m$

[2 POINTS]

10.5 The lab in which the system is to be installed has a vibration with *acceleration* amplitude 8 m/s^2 at 11 Hz. What is the expected amplitude of the *displacement* of the platform of the vibration isolator?

The amplitude of the displacement of the lab is $Y_0 = A_0/\omega^2 = 0.0017m$

The vibration solutions give the amplitude of the platform as

$$X_0 = KY_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{\left\{1 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}$$

Substituting numbers (see matlab below) gives $X_0 = 88\mu m$

```
w = 11*2*pi;
A0 = 8;
Y0 = 8/w^2
M = sqrt( 1 + (2*zeta*w/wn)^2 )/sqrt( (1-w^2/wn^2)^2 + (2*zeta*w/wn)^2 );
X0 = Y0*M
```

[2 POINTS]