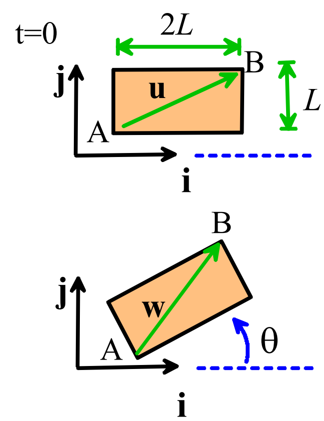
**School of Engineering**
**Brown University**

# EN40: Dynamics and Vibrations

## Homework 6: Rigid Body Kinematics, Inertial properties of rigid bodies
## Due Friday July 30, 2021

**1.** A vector **u** connecting two points A and B in a rigid body changes to a new vector **w** after the body is rotated through an angle $\theta$ about the **k** axis. The matrix describing the rotation is




$$\mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Show that **w** and **u** have the same length (this verifies that **R** is a rigid rotation)

The brute-force way to show this would be to calculate the components of **w** for an arbitrary **u**

$$\begin{bmatrix}w_x\\ w_y\\ w_z\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}u_x\\ u_y\\ u_z\end{bmatrix}=\begin{bmatrix}u_x\cos\theta-u_y\sin\theta\\ u_x\sin\theta+u_y\cos\theta\\ u_z\end{bmatrix}$$

and then calculate the length of **w**

$$\left|\mathbf{w}\right|=\sqrt{\left(u_x\cos\theta-u_y\sin\theta\right)^2+\left(u_x\sin\theta+u_y\cos\theta\right)+u_z^2}$$

$$=\sqrt{u_x^2(\cos^2\theta+\sin^2\theta)+u_y^2(\sin^2\theta+\cos^2\theta)+u_z^2}=\sqrt{u_x^2+u_y^2+u_z^2}=\left|\mathbf{u}\right|$$

If you want to show off your linear algebra muscles (Brain cells? Neurons?) you can also do

$$\mathbf{w}\cdot\mathbf{w}=\left(\mathbf{Ru}\right)\cdot\left(\mathbf{Ru}\right)=\mathbf{u}\cdot\left(\mathbf{R}^T\mathbf{R}\right)\mathbf{u}$$

$$\mathbf{R}^T\mathbf{R}=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

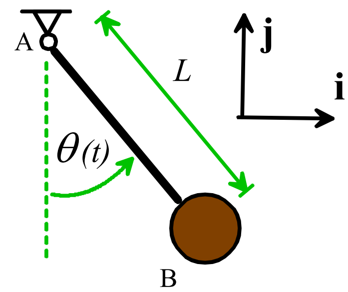
$$=\begin{bmatrix}\sin^2\theta+\cos^2\theta & 0 & 0\\ 0 & \sin^2\theta+\cos^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\mathbf{I}$$

$$\Rightarrow\mathbf{u}\cdot\left(\mathbf{R}^T\mathbf{R}\right)\mathbf{u}=\mathbf{u}\cdot\mathbf{Iu}=\mathbf{u}\cdot\mathbf{u}$$

$$\Rightarrow\left|\mathbf{w}\right|^2=\left|\mathbf{u}\right|$$

**[3 POINTS]**

**2.** A pendulum with length $L$ swings in the $\{\mathbf{i},\mathbf{j}\}$ plane. It is at rest at time $t=0$ and its shaft subtends a small angle $\theta_0$ to the vertical. Write down formulas for the angular velocity vector and angular acceleration vector of the shaft of the pendulum, as a function of $L$, $\theta$, $t$ and $g$. (You don't need to analyze the motion of the pendulum – just use the solution derived in lectures)




We know $\theta(t)$ is harmonic with amplitude $\theta_0$ and angular frequency $\sqrt{\frac{g}{L}}$. Therefore (using the given initial conditions)

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}t\right)$$

$$\boldsymbol{\omega} = -\theta_0 \sqrt{\frac{g}{L}} \sin\left(\sqrt{\frac{g}{L}}t\right)\mathbf{k}$$

$$\boldsymbol{\alpha} = -\theta_0 \frac{g}{L} \cos\left(\sqrt{\frac{g}{L}}t\right)\mathbf{k}$$

**[3 POINTS]**

**3.** A time dependent 3D rotation is described by the matrix

$$\mathbf{R} = \begin{bmatrix} \cos^2(\Omega t) & \cos(\Omega t)\sin(\Omega t) & \sin(\Omega t) \\ \cos(\Omega t)\sin(\Omega t) & \sin^2(\Omega t) & -\cos(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \end{bmatrix}$$

where $\Omega$ is a constant.

3.1 Find a formula for the spin tensor (matrix) (MATLAB will make the calculus/matrix product painless)

Here's the MATLAB, and solution. Remember the formula $\mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T$

```
clear all
syms t Omega
RR = [cos(Omega*t)^2,cos(Omega*t)*sin(Omega*t),sin(Omega*t);...
      cos(Omega*t)*sin(Omega*t),sin(Omega*t)^2,-cos(Omega*t);...
      -sin(Omega*t),cos(Omega*t),0]
simplify(diff(RR,t)*transpose(RR))
```

```
ans =
```

$$\begin{pmatrix} 0 & -\Omega & \Omega\cos(\Omega t) \\ \Omega & 0 & \Omega\sin(\Omega t) \\ -\Omega\cos(\Omega t) & -\Omega\sin(\Omega t) & 0 \end{pmatrix}$$

**[2 POINTS]**

3.2 Find a formula for the angular velocity vector

Recall that

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Therefore

$$\boldsymbol{\omega} = -\Omega(\sin\Omega t\mathbf{i} - \cos\Omega t\mathbf{j} - \mathbf{k})$$

**[2 POINTS]**

3.3 Find a formula for the angular acceleration vector

We can $\boldsymbol{\alpha} = \dfrac{d\boldsymbol{\omega}}{dt} = -\Omega^2(\cos\Omega t\mathbf{i} + \sin\Omega t\mathbf{j})$

**[1 POINT]**

**4.** Let {**i**,**j**,**k**} be a Cartesian basis. A rigid body is subjected to a sequence of two rotations:
(a) A 90 degree clockwise rotation about the **i** axis
(b) A 90 degree clockwise rotation about the **j** axis.

An intelligent life-form without knowledge of linear algebra or rigid body dynamics (eg a humanities major) suggests that it should be possible to return the body to its original orientation by:
(c) A 90 degree counterclockwise rotation about the **i** axis, followed by
(d) A 90 degree counterclockwise rotation about the **j** axis.

Does this work? If not, find the axis and angle of the (single) rotation that will return the body to its initial orientation after step (d).

The general formula for the rotation matrix corresponding to a rotation through $\theta$ about an axis parallel to **n** is

$$\mathbf{R}=\begin{bmatrix}\cos\theta+(1-\cos\theta)n_x^2 & (1-\cos\theta)n_xn_y-\sin\theta n_z & (1-\cos\theta)n_xn_z+\sin\theta n_y\\(1-\cos\theta)n_xn_y+\sin\theta n_z & \cos\theta+(1-\cos\theta)n_y^2 & (1-\cos\theta)n_yn_z-\sin\theta n_x\\(1-\cos\theta)n_xn_z-\sin\theta n_y & (1-\cos\theta)n_yn_z+\sin\theta n_x & \cos\theta+(1-\cos\theta)n_z^2\end{bmatrix}$$

A 90 degree clockwise rotation (which makes $\theta=-\pi/2$ )with axis **i**, is $\mathbf{R}^{(1)}=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$

A 90 degree rotation with axis **j**, which gives $\mathbf{R}^{(2)}=\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}$

A 90 degree counterclockwise rotation about **i** is $\mathbf{R}^{(3)}=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$

And finally a 90 degree counterclockwise rotation about **j** is $\mathbf{R}^{(2)}=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}$

The sequence of four rotations is

$$\mathbf{R}=\mathbf{R}^{(4)}\mathbf{R}^{(3)}\mathbf{R}^{(2)}\mathbf{R}^{(1)}$$

Its straightforward to do this multiplication by hand but MATLAB is quicker and easier

```
R1 = [1,0,0;0,0,1;0,-1,0]
R2 = [0,0,-1;0,1,0;1,0,0]
R3 = [1,0,0;0,0,-1;0,1,0]
R4 = [0,0,1;0,1,0;-1,0,0]
Rfinal = R4*R3*R2*R1
```

```
Rfinal = 3×3
     0     0     1
    -1     0     0
     0    -1     0
```

Since this is not equal to the identity matrix (1s on diagonal, zeros everywhere else) the sequence of rotations has not returned the body to its original orientation.

**[2 POINTS]**

We can use the formula to find the axis and angle that will rotate the body from its initial to its final state

$$1+2\cos\theta = R_{xx} + R_{yy} + R_{zz}$$

$$\mathbf{n} = \frac{1}{2\sin\theta}\left[\left(R_{zy}-R_{yz}\right)\mathbf{i} + \left(R_{xz}-R_{zx}\right)\mathbf{j} + \left(R_{yx}-R_{xy}\right)\mathbf{k}\right]$$

This gives $\cos\theta = -(1/2) \Rightarrow \theta = 120^0 \quad \mathbf{n} = \frac{1}{2\sqrt{3}/2}\left[-\mathbf{i}+\mathbf{j}-\mathbf{k}\right] = \frac{1}{\sqrt{3}}\left[-\mathbf{i}+\mathbf{j}-\mathbf{k}\right]$

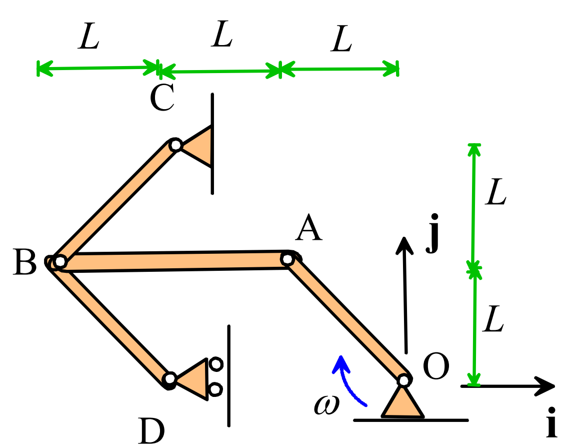
To get back to the initial state, therefore, we just need to change the sign of the angle or the direction of the axis. Thus either

$$\theta = 120, \quad \mathbf{n} = \frac{1}{\sqrt{3}}\left[\mathbf{i}-\mathbf{j}+\mathbf{k}\right]$$

Or $\theta = -120, \quad \mathbf{n} = \frac{1}{\sqrt{3}}\left[-\mathbf{i}+\mathbf{j}-\mathbf{k}\right]$

**[2 POINTS]**

**5.** The figure shows a mechanism used in a hand-operated press. Member OA rotates clockwise with constant angular speed $\omega$ For the configuration shown in the figure:




5.1 calculate the velocity of point D, along with the angular speeds of members AB, BC and BD.

We can use the standard procedure – start at O and work around to point C.

$$\mathbf{v}_A - \mathbf{v}_O = -\omega\mathbf{k}\times(-L\mathbf{i}+L\mathbf{j}) = \omega L(\mathbf{j}+\mathbf{i})$$
$$\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{k}\times(-2L\mathbf{i}) = -2\omega_{AB}L\mathbf{j}$$
$$\mathbf{v}_C - \mathbf{v}_B = \omega_{BC}\mathbf{k}\times(L\mathbf{i}+L\mathbf{j}) = \omega_{BC}L(\mathbf{j}-\mathbf{i})$$
$$\Rightarrow \mathbf{v}_C - \mathbf{v}_O = (\omega-\omega_{BC})L\mathbf{i} + (\omega-2\omega_{AB}+\omega_{BC})L\mathbf{j} = \mathbf{0}$$

Hence using the **i,j** components give two equations

$$(\omega-\omega_{BC}) = 0 \qquad (\omega-2\omega_{AB}+\omega_{BC}) = 0 \quad \Rightarrow \omega_{BC} = \omega \quad \omega_{AB} = \omega$$

Now do the calculation from O to D

$$\mathbf{v}_A - \mathbf{v}_O = -\omega\mathbf{k}\times(-L\mathbf{i}+L\mathbf{j}) = \omega L(\mathbf{j}+\mathbf{i})$$
$$\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{k}\times(-2L\mathbf{i}) = -2\omega_{AB}L\mathbf{j}$$
$$\mathbf{v}_D - \mathbf{v}_B = \omega_{BC}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) = \omega_{BD}L(\mathbf{j}+\mathbf{i})$$
$$\Rightarrow \mathbf{v}_D - \mathbf{v}_O = (\omega+\omega_{BD})L\mathbf{i} + (\omega - 2\omega_{AB} + \omega_{BD})L\mathbf{j} = V_D\mathbf{j}$$

This once again gives two equations

$$(\omega+\omega_{BD}) = 0 \qquad (\omega - 2\omega_{AB} + \omega_{BD})L = V_D$$
$$\Rightarrow \omega_{BD} = -\omega \qquad V_D = -2L\omega$$

**[3 POINTS]**

5.2 calculate the acceleration of point D, along with the angular accelerations of members AB, BC and BD.

We can repeat the process for accelerations

$$\mathbf{a}_A - \mathbf{a}_O = \alpha_{AO}\mathbf{k}\times(-L\mathbf{i}+L\mathbf{j}) - \omega^2(-L\mathbf{i}+L\mathbf{j}) = \omega^2(L\mathbf{i}-L\mathbf{j})$$
$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{AB}\mathbf{k}\times(-2L\mathbf{i}) - \omega_{AB}^2(-2L\mathbf{i}) = -\alpha_{AB}L\mathbf{j} - \omega_{AB}^2(-2L\mathbf{i})$$
$$\mathbf{a}_C - \mathbf{a}_B = \alpha_{BC}\mathbf{k}\times(L\mathbf{i}+L\mathbf{j}) - \omega_{AB}^2(L\mathbf{i}+L\mathbf{j}) = \alpha_{BC}L(\mathbf{j}-\mathbf{i}) - \omega_{BC}^2L(\mathbf{i}+\mathbf{j})$$
$$\Rightarrow \mathbf{a}_C - \mathbf{a}_O = (-\alpha_{BC} + \omega^2 + 2\omega_{AB}^2 - \omega_{BC}^2)L\mathbf{i} + (-\alpha_{AB} + \alpha_{BC} - \omega^2 - \omega_{BC}^2)L\mathbf{j} = \mathbf{0}$$

Hence using the **i,j** components give two equations

$$(-\alpha_{BC} + \omega^2 + 2\omega_{AB}^2 - \omega_{BC}^2) = 0 \qquad (-\alpha_{AB} + \alpha_{BC} - \omega^2 - \omega_{BC}^2) = 0 \Rightarrow \alpha_{BC} = 2\omega^2 \quad \alpha_{AB} = 0$$

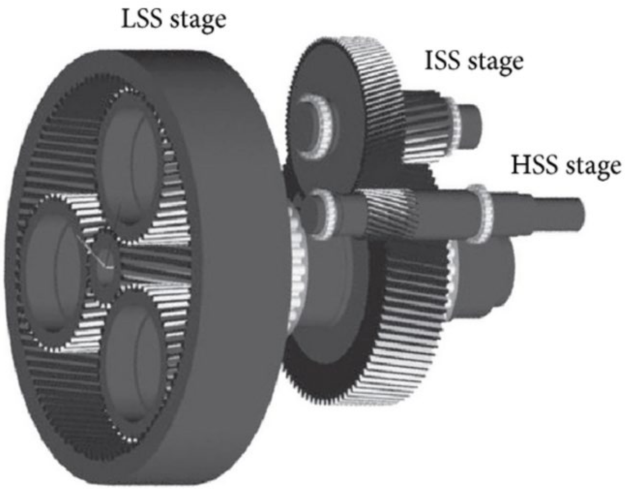
Now do the calculation from O to D

$$\mathbf{a}_A - \mathbf{a}_O = \alpha_{AO}\mathbf{k}\times(-L\mathbf{i}+L\mathbf{j}) - \omega^2(-L\mathbf{i}+L\mathbf{j}) = \omega^2(L\mathbf{i}-L\mathbf{j})$$
$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{AB}\mathbf{k}\times(-2L\mathbf{i}) - \omega_{AB}^2(-2L\mathbf{i}) = -\alpha_{AB}L\mathbf{j} - \omega_{AB}^2(-2L\mathbf{i})$$
$$\mathbf{a}_D - \mathbf{a}_B = \alpha_{BD}\mathbf{k}\times(L\mathbf{i}-L\mathbf{j}) - \omega_{BD}^2(L\mathbf{i}-L\mathbf{j}) = \alpha_{BD}L(\mathbf{j}+\mathbf{i}) - \omega_{BD}^2L(\mathbf{i}-\mathbf{j})$$
$$\Rightarrow \mathbf{a}_D - \mathbf{a}_O = (\alpha_{BD} + \omega^2 + 2\omega_{AB}^2 - \omega_{BD}^2)L\mathbf{i} + (-\alpha_{AB} + \alpha_{BD} - \omega^2 + \omega_{BD}^2)L\mathbf{j} = a_{Dy}\mathbf{j}$$

$$(\alpha_{BD} + \omega^2 + 2\omega_{AB}^2 - \omega_{BD}^2) = 0 \qquad (-\alpha_{AB} + \alpha_{BD} - \omega^2 + \omega_{BD}^2)L = a_{Dy}$$
$$\Rightarrow \alpha_{BD} = -2\omega^2 \qquad a_{Dy} = -2L\omega^2$$

**[3 POINTS]**

**6.** The gearbox in a wind turbine is intended to allow the turbine (which rotates at a slow angular speed) and the generator (which runs at a fast angular speed – about 100 times faster than the turbine) to rotate at their most efficient speeds. The gearbox typically has two or 3 stages of gearing. The first stage is nearly always an epicyclic gearbox. The input to the epicyclic is connected to the turbine; the output must rotate as fast as possible.




There are 3 possible ways to connect the epicyclic: (i) run with the sun stationary and the planet carrier or ring connected to the turbine; (ii) the planet carrier stationary, and the sun or ring connected to the turbine; and (iii) the ring stationary, and the planet carrier or sun connected to the turbine.

Which option will give the biggest ratio between the speed of the output shaft and the speed of the turbine?

The formulas for the angular speeds of ring, planet carrier and sun are

$$\frac{\omega_{zR} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\frac{N_S}{N_R} \qquad N_R = N_S + 2N_P$$

This can be rearranged to

$$\omega_{zR} + \omega_{zS}\frac{N_S}{N_R} = \omega_{zPC}\left(1 + \frac{N_S}{N_R}\right)$$

So the three options are

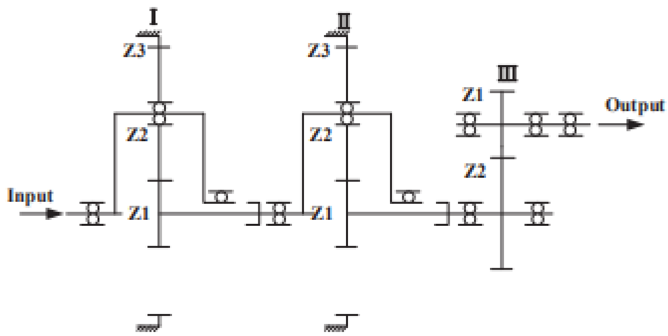
(i) Fix the planet carrier, connect ring to turbine and take output from sun. The ratio of sun speed to ring speed is $\omega_{zS} / \omega_{zR} = -N_R / N_S$

(ii) Fix the sun, connect the planet carrier to the turbine, and take the output from the ring. The ration of ring speed to planet carrier speed is $\omega_{zR} / \omega_{zPC} = (1 + N_S / N_R)$

(iii) Fix the ring, connect the planet carrier to the turbine, and take output from the sun. The ratio of planet carrier speed to sun speed is $\omega_{zS} / \omega_{zPC} = (1 + N_R / N_S)$

We know that $N_R > N_S$ , so option (iii) gives the greatest increase in speed.

**[3 POINTS]**




| Stage | Number of planet gears | Number of teeth | | |
|---|---|---|---|---|
| | | $z_1$ | $z_2$ | $z_3$ |
| I | 3 | 22 | 36 | 95 |
| II | 3 | 22 | 39 | 101 |
| III | \ | 29 | 98 | \ |

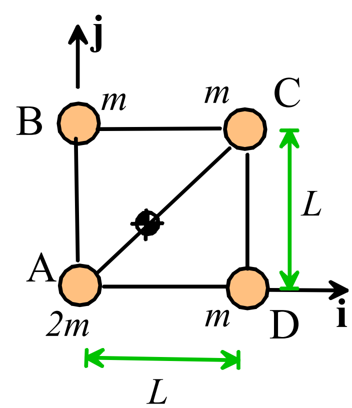
**7.** This publication analyzes vibrations in a 3 stage wind-turbine gearbox. A schematic diagram of the transmission is shown in the figure. The first two stages are epicyclics, operating with the ring gear fixed, input to the planet carrier, and output from the sun. The final stage is a regular gear pair. The table lists the number of teeth on the various gears in the system (for example, z1 is the number of teeth on the sun, z3 is the number of teeth on the ring in the two epicyclic gears. Note that the gears are helical so don't quite satisfy the usual relation $N_R = N_S + 2N_P$ ). Calculate the gear ratio of the gearbox (i.e. the ratio of the output speed to the input speed).

From the previous problem the epicyclics increase the input by a factor $\omega_{zS} / \omega_{zPC} = (1 + N_R / N_S)$. The last stage is a standard gear pair with gear ratio $\omega_B / \omega_A = N_A / N_B$ . The combined gear ratio is therefore

$$\omega_{out} / \omega_{in} = -\left(1 + \frac{95}{22}\right)\left(1 + \frac{101}{22}\right)\left(\frac{98}{29}\right) = -100.5$$

**[2 POINTS]**

**8.** The figure shows four particles connected by rigid massless links. The particle at A has mass $2m$; those at B, C and D have mass $m$. The assembly rotates at constant angular speed $\omega$ about an axis parallel to **k** passing through the center of mass. The point of this problem is to demonstrate that the rigid body formula for the kinetic energy of the system gives the same answer as calculating the kinetic energy of each mass separately, and summing them. The rigid body formulas for angular momentum and kinetic energy are just fast ways of summing the total angular momentum and KE of a system of particles.




8.1 Calculate the position of the center of mass of the assembly

$$\mathbf{r}_G = \frac{1}{M}\sum_i m_i \mathbf{r}_i = \frac{1}{5m}\left(mL\mathbf{j} + mL(\mathbf{i}+\mathbf{j}) + mL\mathbf{i}\right) = \frac{2}{5}L(\mathbf{i}+\mathbf{j})$$

**[1 POINT]**

7.2 Calculate the 2D mass moment of inertia of the system about the center of mass

$$I_{Gzz} = \sum_i m_i \left(d_{xi}^2 + d_{yi}^2\right)$$

where $\mathbf{d}_i = d_{xi}\mathbf{i} + d_{yi}\mathbf{j} = \mathbf{r}_i - \mathbf{r}_G$ is the position vector of the *i*th particle with respect to the center of mass.

Use the formula

$$I_{Gzz} = \sum_i m_i \left( d_{xi}^2 + d_{yi}^2 \right)$$

$$= 2m\left[\left(\frac{2L}{5}\right)^2 + \left(\frac{2L}{5}\right)^2\right] + m\left[\left(\frac{2L}{5}\right)^2 + \left(\frac{3L}{5}\right)^2\right] + m\left[\left(\frac{3L}{5}\right)^2 + \left(\frac{3L}{5}\right)^2\right] + m\left[\left(\frac{3L}{5}\right)^2 + \left(\frac{2L}{5}\right)^2\right]$$

$$= \frac{12}{5} mL^2$$

**[1 POINT]**

8.3 Suppose that the assembly rotates about its center of mass with angular velocity $\omega\mathbf{k}$ (the center of mass is stationary). What are the speeds of the particles A,B and C?

We can use the circular motion formula. The distances of A, B and C from the COM are

$$d_A = \sqrt{\left(\frac{2L}{5}\right)^2 + \left(\frac{2L}{5}\right)^2} = \frac{2\sqrt{2}}{5} L$$

$$d_B = d_D = \sqrt{\left(\frac{2L}{5}\right)^2 + \left(\frac{3L}{5}\right)^2} = \frac{\sqrt{13}}{5} L$$

$$d_C = \sqrt{\left(\frac{3L}{5}\right)^2 + \left(\frac{3L}{5}\right)^2} = \frac{3\sqrt{2}}{5} L$$

The speeds are

$$V_A = \frac{2\sqrt{2}}{4}\omega L \qquad V_B = V_D = \frac{\sqrt{13}}{5}\omega L \qquad V_C = \frac{3\sqrt{2}}{5}\omega L$$

**[3 POINTS]**

8.4 Calculate the total kinetic energy of the system (a) using your answer to 8.2; and (b) using your answer to 8.3. (The point of this problem is to demonstrate that the rigid body formula $(1/2)I\omega^2$ is just a quick way of summing the kinetic energies of the 7 masses. For the simple 2D system here it is quite simple to prove the equivalence for any arrangement of masses. For 3D the derivation is more complicated, but the idea is the same.)

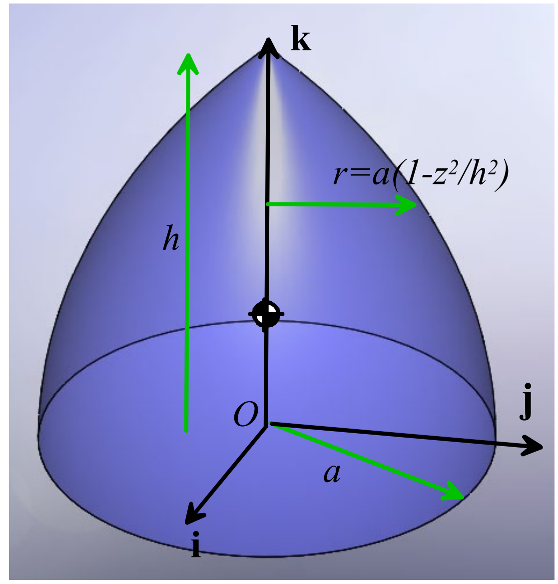
We can use the rigid body formula: $KE = \frac{1}{2} I_G \omega^2 = \frac{6}{5} mL^2\omega^2$

Or we can sum the KEs of the four masses

$$KE = \frac{1}{2} 2mV_A^2 + mV_B^2 + \frac{1}{2} mV_C^2 = m\left(\frac{2\sqrt{2}}{5}\omega L\right)^2 + m\left(\frac{\sqrt{13}}{5}\omega L\right)^2 + \frac{1}{2} m\left(\frac{3\sqrt{2}}{5}\omega L\right)^2 = \frac{30}{25} mL^2\omega^2 = \frac{6}{5} mL^2\omega^2$$

**[2 POINTS]**

**9** The figure shows a proposed design for a rocket nose-cone (or if you prefer, a Hershey's Kiss). It is a solid of revolution with base radius *a,* height *h*, and profile $r = a(1 - z^2 / h^2)$ . It has uniform mass density $\rho$ Using a Matlab 'Live Script', calculate




91 The total mass *M* (you will need to do the relevant integrals using cylindrical-polar coordinates)

$$M = \int_0^h \int_0^{2\pi} \int_0^{a(1-z^2/h^2)} \rho R dR d\theta dz = 8\pi a^2 h \rho / 15$$

You can do the integrals in different orders, of course, but the limits may change.

(see below for Live Script)

**[1 POINT]**

8.2 The position vector of the center of mass (with respect to the origin shown in the figure)

$$\mathbf{r}_{COM} = \frac{1}{M} \int_0^h \int_0^{2\pi} \int_0^{a(1-z^2/h^2)} \rho (R\cos\theta \mathbf{i} + R\sin\theta \mathbf{j} + z\mathbf{k}) R dR d\theta dz = (5h/16)\mathbf{k}$$ (see below for Live Script)

**[2 POINTS]**

8.3 The inertia tensor (matrix) about the center of mass, in the basis shown

$$\mathbf{I}_{COM} = \int_0^h \int_0^{2\pi} \int_0^{a(1-z^2/h^2)} \rho \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix} R dR d\theta dz$$

$$d_x = 0 \qquad d_y = 0 \qquad d_z = z - (5h/16)$$

(See live script for the solution – it's messy)

**[3 POINTS]**

8.4 Using the parallel axis theorem, calculate the mass moment of inertia about the origin O.

The general formula is

$$\mathbf{I}_O = \mathbf{I}_G + M \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix}$$

**[2 POINTS]**

Please upload your 'Live script' solution to Canvas.

**Graders – only the Live Script upload is required….**

```
syms a h rho r theta z M real
x = r*cos(theta); y=r*sin(theta);
pos = [x,y,z];
mass = int(int(int(rho*r,r,[0,a*(1-z^2/h^2)]),theta,[0,2*pi]),z,[0,h])
rG = simplify(int(int(int(pos*rho*r,r,[0,a*(1-z^2/h^2)]),theta,[0,2*pi]),z,[0,h])/mass)
d = pos-rG;
integrand = (dot(d,d)*eye(3) - d'*d)*rho*r;
Ig = M*simplify(int(int(int(integrand,r,[0,a*(1-z^2/h^2)]),theta,[0,2*pi]),z,[0,h])/mass)
integrand = (dot(pos,pos)*eye(3) - pos'*pos)*rho*r;
Io = M*simplify(int(int(int(integrand,r,[0,a*(1-z^2/h^2)]),theta,[0,2*pi]),z,[0,h])/mass)
d = -rG;
IoParralelAxis = simplify(Ig + M*(dot(d,d)*eye(3) - d'*d))
```

mass =

$$\frac{8\pi a^2 h \rho}{15}$$

rG =

$$\begin{pmatrix} 0 & 0 & \frac{5h}{16} \end{pmatrix}$$

Ig =

$$\begin{pmatrix} M\left(\frac{4a^2}{21}+\frac{81h^2}{1792}\right) & 0 & 0 \\ 0 & M\left(\frac{4a^2}{21}+\frac{81h^2}{1792}\right) & 0 \\ 0 & 0 & \frac{8Ma^2}{21} \end{pmatrix}$$

Io =

$$\begin{pmatrix} M\left(\frac{4a^2}{21}+\frac{h^2}{7}\right) & 0 & 0 \\ 0 & M\left(\frac{4a^2}{21}+\frac{h^2}{7}\right) & 0 \\ 0 & 0 & \frac{8Ma^2}{21} \end{pmatrix}$$

IoParralelAxis =

$$\begin{pmatrix} \frac{M(4a^2+3h^2)}{21} & 0 & 0 \\ 0 & \frac{M(4a^2+3h^2)}{21} & 0 \\ 0 & 0 & \frac{8Ma^2}{21} \end{pmatrix}$$