School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 7: Rigid Body Dynamics
Due Friday August 6, 2021

1 A three bladed 10 kW vertical wind-turbine has total mass 150kg (50kg per blade), rotor diameter 5.5m and rotor height 6m.

1.1 The rotor spins at an angular speed of 260 rpm and generates power at a rate of 10kW. What is the torque exerted by the wind on the rotor?

The torque-power formula is $P = Q\omega$ so $Q = P/\omega = 10000/(2\pi.260/60) = 367Nm$

[2 POINTS]

1.2 Suppose that the rotor is spun up from rest by a constant torque with magnitude calculated in problem 1.1, with the generator disconnected. How long will it take the rotor to reach 260 rpm? (you can treat the rotors as slender rods with their axis vertical).

The mass moment of inertia of the rod about its own axis is zero, so the total mass moment of inertia of the rotor is (from the parallel axis theorem) $I_{zz} = 3m(D/2)^2 = 150(5.5/2)^2 = 1134kgm^2$

The angular acceleration of the rotor follows as $\alpha = Q/I_{zz} = 0.323rad/s^2$

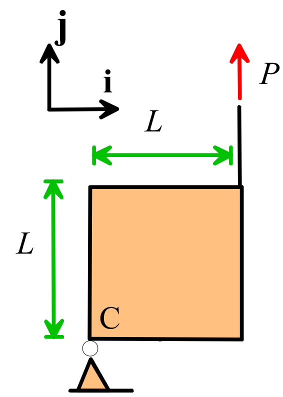
We can calculate the time to reach 260rpm using the constant acceleration formulas (same as straight line motion formulas, except angular acceleration replaces acceleration, angular velocity replaces velocity)

$$\tau = \omega/\alpha = (2\pi.260/60)/0.323 = 84.3s$$

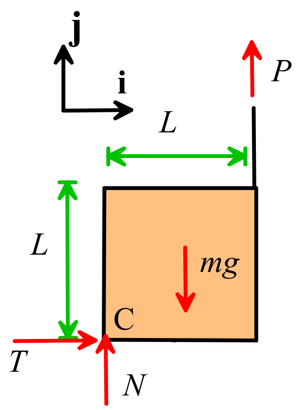
For discussion – we could also do the calculation with power-KE – i.e. total work done on the turbine is $\int_0^t Pdt$ which must equal its change in KE. This works, but since we were told the torque is constant, we can't assume the power is 10kW. Instead, we have to use $P = Q\omega$, where (since the torque is constant) $\omega = \alpha t = Qt/I_{zz}$. This approach ends up being a bit more involved.

[3 POINTS]

2. The point of this problem is to illustrate the choices you can make when you apply the moment-angular momentum equation to calculate the acceleration and angular acceleration of a rigid body subjected to forces. The figure shows a cube with side length L and mass m that is being tipped over by force P applied to one corner. Assume no slip at the contact point C.



2.1 Draw a free body diagram showing the forces acting on the cube



[3 POINTS]

(OK to draw the friction force T acting left instead, since there is no slip. It is *not* OK to label the friction force as μN . No slip means that $|T| < \mu N$; T could have any value between $\pm \mu N$)

2.2 Write down the rigid body kinematics equation that relates the angular acceleration α_z and linear acceleration a_{Gx} of the center of mass of the cube

The rigid body kinematics equation gives

$$\begin{aligned}
&\mathbf{a}_G - \mathbf{a}_C = \alpha_z \mathbf{k} \times (\mathbf{r}_G - \mathbf{r}_C) - \omega^2 (\mathbf{r}_G - \mathbf{r}_C) \\
&= \alpha_z \mathbf{k} \times \frac{L}{2}(\mathbf{i}+\mathbf{j}) - \omega^2 \frac{L}{2}(\mathbf{i}+\mathbf{j}) \\
&\Rightarrow \mathbf{a}_G = \frac{L}{2}\alpha_z(-\mathbf{i}+\mathbf{j}) - \omega^2 \frac{L}{2}(\mathbf{i}+\mathbf{j})
\end{aligned}$$

[1 POINT]

2.3 Write down **F**=m**a** for the cube

F=m**a** gives $T\mathbf{i} + (N + P - mg)\mathbf{j} = m\frac{L}{2}\alpha_z(-\mathbf{i}+\mathbf{j}) - m\omega^2 \frac{L}{2}(\mathbf{i}+\mathbf{j})$

[2 POINTS]

2.4 Write down the equation that relates the total moment acting on the spool to the time derivative of its angular momentum. Take moments (and angular momentum) about the center of mass. Hence, solve the equations in 3.3. and 3.4 to calculate the angular acceleration of the spool and the acceleration of its COM a_{Gx}

Applying the moment-angular momentum equation about the COM gives

$$\left\{T\frac{L}{2}+(P-N)\frac{L}{2}\right\}\mathbf{k}=\mathbf{0}\times m\mathbf{a}_G+\frac{1}{6}mL^2\alpha_z\mathbf{k}$$

To find α_z we need to eliminate T and N from 2.2, 2.3 and 2.4;

$$T=-m\frac{L}{2}\alpha-m\omega^2\frac{L}{2}$$

$$N=-P+mg+m\frac{L}{2}\alpha-m\omega^2\frac{L}{2}$$

Hence

$$\left\{\left(-m\frac{L}{2}\alpha-m\omega^2\frac{L}{2}\right)\frac{L}{2}+P\frac{L}{2}-\left(-P+mg+m\frac{L}{2}\alpha-m\omega^2\frac{L}{2}\right)\frac{L}{2}\right\}\mathbf{k}$$

$$=\mathbf{0}\times m\mathbf{a}_G+\frac{1}{6}mL^2\alpha_z\mathbf{k}$$

$$\Rightarrow PL-mg\frac{L}{2}=\left(\frac{1}{6}mL^2+m\frac{L^2}{2}\right)\alpha_z$$

$$\Rightarrow\alpha_z=\frac{3}{4Lm}(2P-mg)$$

The acceleration of the COM follows as

$$\mathbf{a}_G=\frac{3}{8m}(2P-mg)(-\mathbf{i}+\mathbf{j})-\omega^2\frac{L}{2}(\mathbf{i}+\mathbf{j})$$

(The ω^2 term can be neglected if the cube is assumed to be instantaneously stationary)

[3 POINTS]

2.5 Repeat 2.4, but this time apply the moment – d**h**/dt relation by taking moments about the contact point *C*. Notice that (just like when you do statics) you can simplify the algebra by choosing to take moments about a convenient point – it doesn't change the answer, but can make your life easier.

The moment – mass moment of inertia formula about C gives

$$\sum \mathbf{r}\times\mathbf{F} = \mathbf{r}_G \times m\mathbf{a}_G + I_{Gzz}\alpha_z\mathbf{k}$$

$$\left\{PL - mg\frac{L}{2}\right\}\mathbf{k} = \frac{L}{2}(\mathbf{i}+\mathbf{j})\times m\left(\frac{L}{2}\alpha_z(-\mathbf{i}+\mathbf{j}) - \omega^2\frac{L}{2}(\mathbf{i}+\mathbf{j})\right) + I_{Gzz}\alpha_z\mathbf{k}$$

$$\Rightarrow PL - mg\frac{L}{2} = m\frac{L^2}{2}\alpha_z + \frac{1}{6}mL^2\alpha_z$$

$$\Rightarrow \alpha_z = \frac{3}{4Lm}(2P - mg)$$

[2 POINTS]

2.6 Repeat 2.4, but this time use special version of the moment – d**h**/dt relation for bodies that rotate about a fixed point.

The parallel axis theorem gives

$$I_C = m\left(\frac{L}{\sqrt{2}}\right)^2 + \frac{1}{6}mL^2 = \frac{2}{3}mL^2$$

The moment-d**h**/dt relation gives

$$\left\{PL - mg\frac{L}{2}\right\}\mathbf{k} = \frac{2}{3}mL^2\alpha_z$$

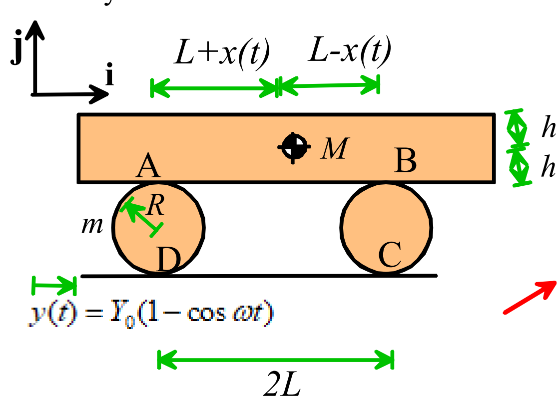
$$\Rightarrow \alpha_z = \frac{3}{4Lm}(2P - mg)$$

[2 POINTS]

3. The figure shows a platform with mass M supported on two cylindrical rollers with radius R and mass m. At time $t=0$ the system is at rest with the COM of the platform midway between the rollers. The base then starts to move with a harmonic displacement

$$y(t) = Y_0(1 - \cos\omega t)$$

The goal of this problem is to calculate the motion $x(t)$ of the platform. To do this, we first need to calculate the acceleration of the platform, and then integrate it.

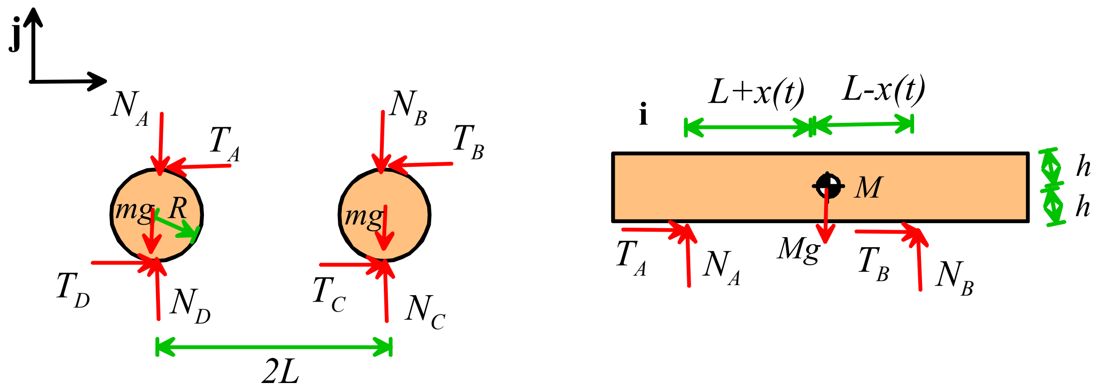


3.1 Find a formula for the horizontal acceleration of the base a_{base}, in terms of Y_0, ω, t

$$a = \frac{d^2y}{dt^2} = Y_0\omega^2\cos\omega t$$

[1 POINT]

3.2 Draw free body diagrams showing forces acting on the rollers, and forces acting on the platform. Include gravity. Assume no slip at the contacts.



[3 POINTS]

3.3 Write down the equations of translational and rotational motion for the rollers (i.e. $\mathbf{F} = m\mathbf{a}_G^{roller}$, $\sum \mathbf{r} \times \mathbf{F} = m\mathbf{r} \times \mathbf{a}_G^{roller} + I_{Gzz}\alpha\mathbf{k}$), in terms of (unknown) reaction forces. You can assume both rollers have the same acceleration.

$$(T_D - T_A)\mathbf{i} + (N_D - N_A - mg)\mathbf{j} = ma_x^{roller}\mathbf{i} \qquad (T_C - T_B)\mathbf{i} + (N_C - N_B - mg)\mathbf{j} = ma_x^{roller}\mathbf{i}$$

$$(T_D + T_A)R = \frac{1}{2}mR^2\alpha \qquad (T_C + T_B)R = \frac{1}{2}mR^2\alpha$$

[3 POINTS]

3.4 Write down the equation of translational and rotational motion for the platform.

$$(T_A + T_B)\mathbf{i} + (N_A + N_B - Mg)\mathbf{j} = Ma_x^{block}\mathbf{i}$$
$$(T_A + T_B)h + N_B(L - x) - N_A(L + x) = 0$$

[2 POINTS]

3.5 Write down kinematic equations relating the acceleration of the platform, the acceleration of the base, and the angular & linear accelerations of the rollers.

Using the rolling wheel formulas

$$a_x^{roller} = a_{base} - \alpha R$$
$$a_x^{block} = a_x^{roller} - \alpha R$$

[2 POINTS]

3.6 Use the results of 4.3-4.5 to show that the acceleration of the block is related to that of the base by

$$a_x^{block} = -\frac{m}{(4M+3m)} a_x^{base}$$

and hence find a formula for the displacement *x(t)* of the block, in terms of $m, M Y_0, \omega, t$

The horizontal components of **F**=m**a** together with the angular momentum equation for the rollers gives

$$\begin{aligned}
&(T_D - T_A) = m a_x^{roller} \qquad && (T_C - T_B) = m a_x^{roller} \\
&(T_D + T_A)R = \frac{1}{2} m R^2 \alpha \qquad && (T_C + T_B)R = \frac{1}{2} m R^2 \alpha \\
&\Rightarrow T_A = \frac{1}{4} m R\alpha - \frac{1}{2} m a_x^{roller} \qquad && T_B = \frac{1}{4} m R\alpha - \frac{1}{2} m a_x^{roller} \\
&\Rightarrow T_A + T_B = \frac{1}{2} m R\alpha - m a_x^{roller} &&
\end{aligned}$$

The horizontal component of **F**=m**a** for the platform gives $T_A + T_B = M a_x^{block}$, so

$$M a_x^{block} = \frac{1}{2} m R\alpha - m a_x^{roller}$$

Using the first kinematics equation

$$M a_x^{block} = \frac{1}{2} m R\alpha - m a_x^{base} + m\alpha R = \frac{3}{2} m R\alpha - m a_x^{base}$$

$$\left.\begin{aligned} a_x^{roller} &= a_{base} - \alpha R \\ a_x^{block} &= a_x^{roller} - \alpha R \end{aligned}\right\} \Rightarrow a_x^{block} = a_x^{base} - 2\alpha R \Rightarrow \alpha = (a_x^{base} - a_x^{block})/(2R)$$

and finally

$$\begin{aligned}
&M a_x^{block} = \frac{3}{4} m (a_x^{base} - a_x^{block}) - m a_x^{base} \\
&\Rightarrow \left(M + \frac{3m}{4} \right) a_x^{block} = -\frac{m}{4} a_x^{base} \\
&\Rightarrow a_x^{block} = -\frac{m}{(4M+3m)} a_x^{base}
\end{aligned}$$

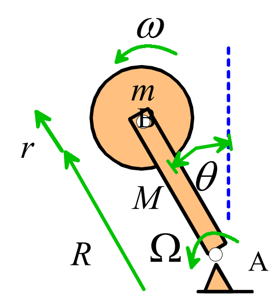
Since the block and base have the same initial conditions, the integrals of their accelerations have the same form, and therefore

$$x(t) = -\frac{m}{(4M+3m)} Y_0 (1-\cos\omega t)$$

Notice that the block moves in the opposite direction to the base!!

[3 POINTS]

4. The figure shows an experiment that is often used to test control systems, and is sometimes used to stabilize walking robots. The bar AB has mass M and length R and rotates freely about A. It is stabilized in an inverted position by a reaction wheel with mass m and radius r at B. The goal of this problem is to design a simple 'Proportional-Derivative' (P-D) controller for the device.



4.1 Suppose that the bar AB rotates with angular speed $\Omega = d\theta / dt$ and the reaction wheel rotates with angular speed ω ***relative to AB*** (i.e. the total angular speed of the reaction wheel relative to a non-rotating frame is $\Omega + \omega$). Find a formula for the total angular momentum of the system about A.

For AB we can use the formula for a solid that rotates about a fixed point

$$\mathbf{h}_{AB} = I_0 \Omega \mathbf{k} = \left(\frac{1}{12} MR^2 + M\left(\frac{R}{2}\right)^2 \right) \Omega \mathbf{k} = \frac{1}{3} MR^2 \Omega$$

For the reaction wheel we have to use the general formula

$$\mathbf{h}_W = \mathbf{r}_G \times m\mathbf{v}_G + I_G \omega \mathbf{k} = mR^2 \Omega \mathbf{k} + \frac{1}{2} mr^2 (\omega + \Omega) \mathbf{k}$$

(we evaluated $\mathbf{r}_G \times m\mathbf{v}_G$ by noting that the center of the reaction wheel is in circular motion about A, and therefore has speed $R\Omega$. The velocity is perpendicular to AB, so $\mathbf{r}_G \times m\mathbf{v}_G = mR^2 \Omega$)

The total angular momentum is therefore

$$\mathbf{h} = \left\{ \left(\frac{1}{3} MR^2 + mR^2 \right) \Omega + \frac{1}{2} mr^2 (\Omega + \omega) \right\} \mathbf{k}$$

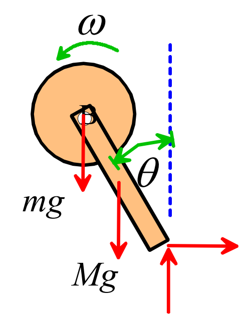
[3 POINTS]

4.2 Hence, use a free body diagram and the moment-angular momentum relation about A to show that θ satisfies the equation of motion

$$\left(\frac{1}{3}MR^2+\frac{1}{2}m(r^2+2R^2)\right)\frac{d^2\theta}{dt^2}-(\frac{MR}{2}+mR)g\sin\theta+\frac{1}{2}mr^2\frac{d\omega}{dt}=0$$

A FBD is shown in the picture. Applying moment-angular momentum about A gives

$$\left(Mg\frac{R}{2}\sin\theta+mgR\sin\theta\right)\mathbf{k}=\frac{d}{dt}\left\{\left(\frac{1}{3}MR^2+mR^2\right)\Omega+\frac{1}{2}mr^2(\Omega+\omega)\right\}\mathbf{k}$$



This can be easily rearranged into the form required.

[3 POINTS]

4.3 The EOM suggests that we could stabilize the pendulum with a simple feedback controller that spins the wheel with an angular acceleration

$$\frac{d\omega}{dt}=k_D\frac{d\theta}{dt}+k_P\theta$$

where k_P, k_D are two constants (called the 'proportional' and 'derivative' gain of the controller.

For the special case M=0 (just to keep the algebra simple), show (by linearizing the EOM and using the solutions to differential equations for vibrating systems)

(i) The pendulum will be stabilized for any proportional gain satisfying $k_P > 2Rg/r^2$

(ii) For a critically damped response the derivative gain must satisfy

$$k_D=2\sqrt{(1+2R^2/r^2)\left(k_P-2Rg/r^2\right)}$$

Set *M=0* in the EOM and substitute the controller equation

$$\frac{1}{2}m(r^2+2R^2)\frac{d^2\theta}{dt^2}-mRg\sin\theta+\frac{1}{2}mr^2\left(k_D\frac{d\theta}{dt}+k_P\theta\right)=0$$

Use the small angle approximation for θ and rearrange into standard form

$$\frac{1}{2}(r^2+2R^2)\frac{d^2\theta}{dt^2}+\frac{1}{2}r^2k_D\frac{d\theta}{dt}+\left(\frac{1}{2}r^2k_P-Rg\right)\theta=0$$

$$\Rightarrow\frac{(r^2+2R^2)}{\left(r^2k_P-2Rg\right)}\frac{d^2\theta}{dt^2}+\frac{r^2k_D}{\left(r^2k_P-2Rg\right)}\frac{d\theta}{dt}+\theta=0$$

This is a case III vibration equation as long as the coefficient of the angular acceleration is positive (if the coefficient is negative the system is unstable). Therefore $k_P > 2Rg/r^2$ for stability.

Note that for a stable system the natural frequency and damping factor satisfy

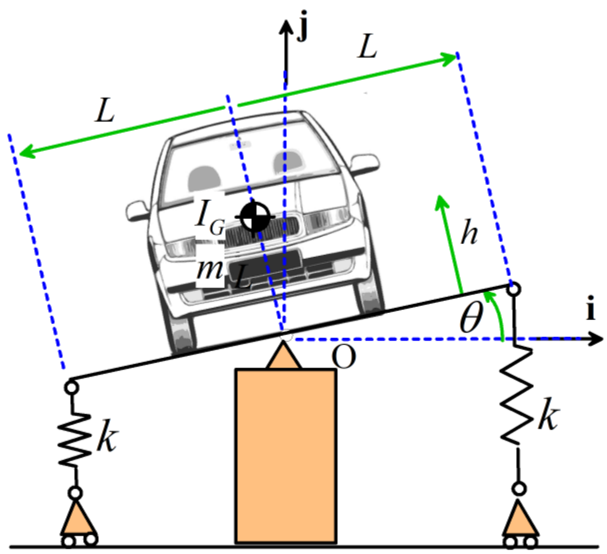
$$\omega_n=\sqrt{\frac{\left(r^2k_P-2Rg\right)}{(r^2+2R^2)}}\qquad\frac{2\zeta}{\omega_n}=\frac{r^2k_D}{\left(r^2k_P-2Rg\right)}$$

$$\Rightarrow\zeta=\frac{1}{2}\frac{r^2k_D}{\left(r^2k_P-2Rg\right)}\sqrt{\frac{\left(r^2k_P-2Rg\right)}{(r^2+2R^2)}}=\frac{1}{2}\frac{r^2k_D}{\sqrt{(r^2+2R^2)\left(r^2k_P-2Rg\right)}}$$

For critical damping $\zeta=1$ and therefore

$$k_D=2\frac{\sqrt{(r^2+2R^2)\left(r^2k_P-2Rg\right)}}{r^2}=2\sqrt{(1+2R^2/r^2)\left(k_P-2Rg/r^2\right)}$$

[3 POINTS]



5. The figure shows a simplified idealization of a commercial instrument for measuring the inertial properties of large objects. It operates by measuring the amplitude and frequency of small oscillations of a platform, together with the horizontal and vertical reaction forces at the pivot.

The goal of this problem is to find a formula for the distance of the COM from the pivot and the mass moment of inertia of the object, in terms of these quantities.

5.1 Use the energy method to derive an equation of motion for the system, and hence find a formula for the natural frequency of vibration of the system, in terms of k, L, h, I_G, m

The mass moment of inertia of the vehicle about the pivot is $I_O = I_G + mh^2$

The kinetic energy is $T = \frac{1}{2} I_O \left(\frac{d\theta}{dt} \right)^2$

The potential energy is $U = mgh\cos\theta + \frac{1}{2}k(d + L\cos\theta - L_0)^2 + \frac{1}{2}k(d - L\cos\theta - L_0)^2$ where d is a constant and L_0 is the unstretched length of the springs.

We know $T+U$ is constant, and therefore

$$\frac{d}{dt}(T+U) = I_0 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \{-mgh\sin\theta - k(d + L\cos\theta - L_0)L\sin\theta + k(d - L\cos\theta - L_0)L\sin\theta\}\frac{d\theta}{dt} = 0$$

$$\Rightarrow I_0 \frac{d^2\theta}{dt^2} + kL^2 \sin 2\theta - mgh\sin\theta = 0$$

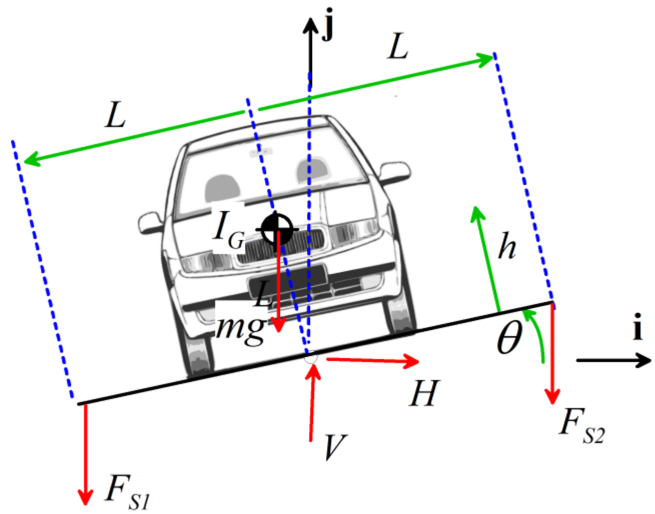
Linearize with the usual small angle approximations

$$\frac{\left(I_G + mh^2\right)}{2kL^2 - mgh}\frac{d^2\theta}{dt^2} + \theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{\left(2kL^2 - mgh\right)}{\left(I_G + mh^2\right)}}$$

[3 POINTS]

5.2 Draw a free body diagram showing the forces acting on the platform/vehicle together



[2 POINTS]

5.3 Assume that the platform vibrates at its natural frequency with a small amplitude $\theta = \theta_0 \sin \omega t$. Show that the horizontal reaction force at O has the form

$$H(t) \approx H_0 \sin \omega_n t$$

and find a formula for H_0 in terms of $m, h, \theta_0, \omega_n, t$

The position vector of the COM is

$$\mathbf{r}_G = -h\sin\theta\mathbf{i} + h\cos\theta\mathbf{j}$$

The velocity and acceleration vectors follow as

$$\mathbf{v}_G = -\left(h\cos\theta\mathbf{i} + h\sin\theta\mathbf{j}\right)\frac{d\theta}{dt}$$

$$\mathbf{a}_G = \left(h\sin\theta\mathbf{i} - h\cos\theta\mathbf{j}\right)\left(\frac{d\theta}{dt}\right)^2 - \left(h\cos\theta\mathbf{i} + h\sin\theta\mathbf{j}\right)\frac{d^2\theta}{dt^2}$$

F=m**a** in the horizontal direction gives

$$H = mh\sin\theta\left(\frac{d\theta}{dt}\right)^2 - hm\cos\theta\frac{d^2\theta}{dt^2} \approx mh\omega_n^2\theta_0\sin\omega_n t$$

[3 POINTS]

5.4 Finally, show that I_G and h can be calculated from the following formulas

$$h = \frac{H_0}{m\omega_n^2\theta_0} \qquad I_G = \frac{2kL^2}{\omega_n^2} - \frac{H_0 g}{\omega_n^4\theta_0}\left(1 + \frac{H_0}{\theta_0 mg}\right)$$

We have that

$$H_0 = mh\omega_n^2\theta_0$$

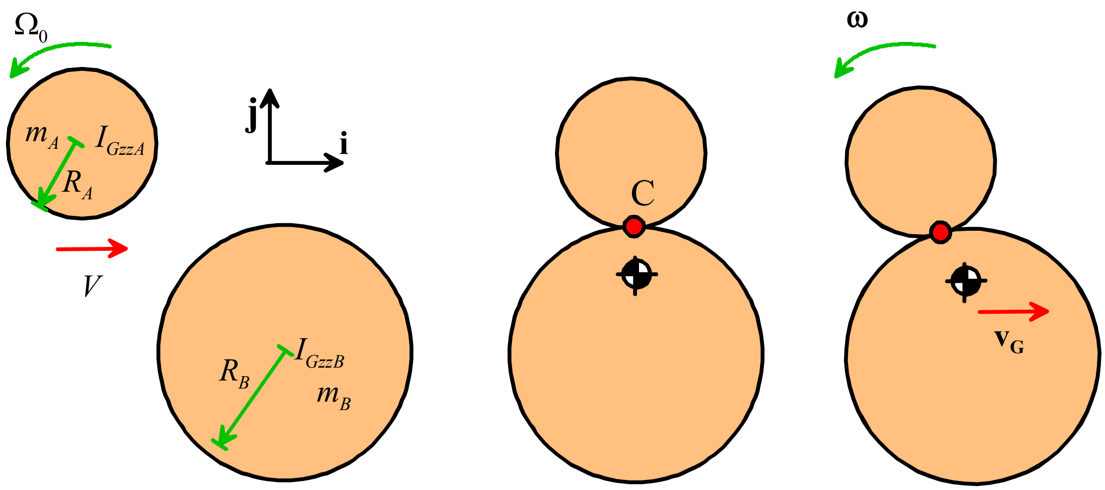
$$\Rightarrow h = \frac{H_0}{m\omega_n^2\theta_0}$$

$$\omega_n^2 = \frac{\left(2kL^2 - mgh\right)}{\left(I_G + mh^2\right)}$$

$$\Rightarrow I_G = \frac{1}{\omega_n^2}\left(2kL^2 - mgh\right) - mh^2 = \frac{1}{\omega_n^2}\left(2kL^2 - mg\frac{H_0}{m\omega_n^2\theta_0}\right) - m\left(\frac{H_0}{m\omega_n^2\theta_0}\right)^2$$

$$= \frac{2kL^2}{\omega_n^2} - \frac{H_0 g}{\omega_n^4\theta_0}\left(1 + \frac{H_0}{\theta_0 mg}\right)$$

[3 POINTS]



6. Several publications describe candidate approaches to capturing a tumbling spacecraft. This example from Stanford sets up a small-scale experiment to test strategies on an air-table. The figure shows a tumbling satellite (A) that has mass m_A and mass moment of inertia I_{GzzA} captured by a larger spacecraft (B) that has mass m_B and mass moment of inertia I_{GzzB} . At time $t=0$ B is stationary, while A moves in the **i** direction with speed V and spins with angular velocity $\Omega_0 \mathbf{k}$. The capture is similar to a plastic collision: the two spacecraft remain in contact after they touch, and no relative rotation of the two occurs. After the capture the center of mass of combined craft moves with velocity vector $\mathbf{v}_G$ and angular velocity $\boldsymbol{\omega}$, to be determined.

6.1 Write down the total linear momentum of the system before the capture.

$$\mathbf{p}_0 = m_A V \mathbf{i}$$

[1 POINT]

6.2 Find the position vector of the center of mass of the system at the instant of capture (take the origin at point C)

$$\mathbf{r}_G = \frac{1}{M}\sum m_i \mathbf{r}_i = \frac{m_A R_A - m_B R_B}{m_A + m_B}\mathbf{j}$$

[1 POINT]

6.2 Choose a point about which to calculate the initial angular momentum (there are an infinite number of choices – anything is fine), and determine the initial angular momentum about the point you chose.

There are many choices:

(i) Point C: $\mathbf{h}_C = \left(-m_A R_A V + I_{GzzA}\Omega_0\right)\mathbf{k}$

(ii) The center of satellite A $\mathbf{h}_A = I_{GzzA}\Omega_0\mathbf{k}$

(iii) The center of satellite B $\mathbf{h}_B = \left(-m_A(R_A + R_B)V + I_{GzzA}\Omega_0\right)\mathbf{k}$

(iv) The COM of the assembly

$$\mathbf{h}_G = -m_A\left(R_A - \frac{m_A R_A - m_B R_B}{m_A + m_B}\right)V + I_{GzzA}\Omega_0\mathbf{k} = \left(-\frac{m_A m_B}{m_A + m_B}\left(R_A + R_B\right)V + I_{GzzA}\Omega_0\right)\mathbf{k}$$

[2 POINTS]

6.3 Explain why both linear and angular momentum of a system consisting of the two satellites is conserved.

There is no external force acting on the system so the external impulse is zero.

[1 POINT]

6.4 Hence, show that $\mathbf{v}_G$ and $\boldsymbol{\omega}$ are given by

$$\mathbf{v}_G = \frac{m_A}{m_A + m_B}V\mathbf{i}$$

$$\boldsymbol{\omega} = \frac{I_{GzzA}(m_A + m_B)\Omega_0 - m_A m_B (R_A + R_B)V}{(m_A + m_B)(I_{GzzA} + I_{GzzB}) + m_A m_B\left(R_A + R_B\right)^2}\mathbf{k}$$

The linear momentum of the system after the capture can be expressed in terms of $\mathbf{v}_G$:

$\mathbf{p}_1 = (m_A + m_B)\mathbf{v}_G$

Momentum conservation gives

$$\mathbf{p}_0 = \mathbf{p}_1 \Rightarrow \mathbf{v}_G = \frac{m_A}{m_A + m_B}V\mathbf{i}$$

Angular momentum is conserved about all points – we can write down the total angular momentum of the system about the same point that was chosen in 6.2 just after the capture in terms of $\mathbf{v}_G$ and $\boldsymbol{\omega}$ and set it equal to the initial angular momentum.

For example, if we choose point C, we note that the velocities of the centers of the two satellite can be calculated using the rigid body velocity formula, eg for A $\mathbf{v}_A - \mathbf{v}_C = \omega\mathbf{k}\times(\mathbf{r}_A - \mathbf{r}_C)$

$$
\begin{aligned}
\mathbf{h}_{C1} &= R_A \mathbf{j} \times m_A(\mathbf{v}_C - \omega R_A \mathbf{i}) + I_{GzzA}\omega\mathbf{k} + (-R_B\mathbf{j}) \times m_B(\mathbf{v}_C + \omega R_B \mathbf{i}) + I_{GzzB}\omega\mathbf{k} \\
&= (m_A R_A - m_B R_B)(\mathbf{j} \times \mathbf{v}_C) + \left(m_A R_A^2 + I_{GzzA} + m_B R_B^2 + I_{GzzA}\right)\omega\mathbf{k}
\end{aligned}
$$

We also know that

$$
\begin{aligned}
&\mathbf{v}_C - \mathbf{v}_G = \omega\mathbf{k} \times (\mathbf{r}_C - \mathbf{r}_G) \\
&\Rightarrow \mathbf{v}_C = \frac{m_A}{m_A + m_B} V\mathbf{i} + \omega\mathbf{k} \times \frac{m_B R_B - m_A R_A}{m_A + m_B}\mathbf{j} \\
&= \frac{m_A}{m_A + m_B} V\mathbf{i} + \omega\left(\frac{m_A R_A - m_B R_B}{m_A + m_B}\right)\mathbf{i}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\mathbf{h}_{C1} &= R_A \mathbf{j} \times m_A(\mathbf{v}_C + \omega R_A \mathbf{i}) + I_{GzzA}\omega\mathbf{k} + (-R_B\mathbf{j}) \times m_B(\mathbf{v}_C - \omega R_B \mathbf{i}) + I_{GzzB}\omega\mathbf{k} \\
&= -(m_A R_A - m_B R_B)\left[\frac{m_A}{m_A + m_B} V + \omega\left(\frac{m_A R_A - m_B R_B}{m_A + m_B}\right)\right]\mathbf{k} + \left(m_A R_A^2 + I_{GzzA} + m_B R_B^2 + I_{GzzB}\right)\omega\mathbf{k}
\end{aligned}
$$

whence

$$
\begin{aligned}
&\left(-m_A R_A V + I_{GzzA}\Omega_0\right)\mathbf{k} = -(m_A R_A - m_B R_B)\left[\frac{m_A}{m_A + m_B} V + \omega\left(\frac{m_A R_A - m_B R_B}{m_A + m_B}\right)\right]\mathbf{k} + \left(m_A R_A^2 + I_{GzzA} + m_B R_B^2 + I_{GzzB}\right)\omega\mathbf{k} \\
&\Rightarrow -\frac{m_A m_B (R_A + R_B)}{m_A + m_B} V + I_{GzzA}\Omega_0 = \left(m_A R_A^2 + I_{GzzA} + m_B R_B^2 + I_{GzzB} - \frac{\left(m_A R_A - m_B R_B\right)^2}{m_A + m_B}\right)\omega \\
&\Rightarrow -\frac{m_A m_B (R_A + R_B)}{m_A + m_B} V + I_{GzzA}\Omega_0 = \left(I_{GzzA} + I_{GzzB} + \frac{m_A m_B \left(R_A + R_B\right)^2}{m_A + m_B}\right)\omega \\
&\Rightarrow \omega = \frac{I_{GzzA}(m_A + m_B)\Omega_0 - m_A m_B (R_A + R_B) V}{(m_A + m_B)(I_{GzzA} + I_{GzzB}) + m_A m_B \left(R_A + R_B\right)^2}
\end{aligned}
$$

As a second approach, we can apply angular momentum conservation about the COM. In this case we can find the combined mass moment of inertia of the entire connected assembly about the COM using the parallel axis theorem:

$$
\begin{aligned}
I_{Gzz} &= I_{GzzA} + I_{GzzB} + m_A\left(R_A + \frac{m_B R_B - m_A R_A}{m_A + m_B}\right)^2 + m_B\left(R_B - \frac{m_B R_B - m_A R_A}{m_A + m_B}\right)^2 \\
&= I_{GzzA} + I_{GzzB} + \frac{m_A m_B \left(R_A + R_B\right)^2}{m_A + m_B}
\end{aligned}
$$

Then we get directly

$$
\begin{aligned}
&\left(-\frac{m_A m_B}{m_A + m_B}\left(R_A + R_B\right) V + I_{GzzA}\Omega_0\right)\mathbf{k} = \left\{I_{GzzA} + I_{GzzB} + \frac{m_A m_B \left(R_A + R_B\right)^2}{m_A + m_B}\right\}\omega\mathbf{k} \\
&\Rightarrow \omega = \frac{I_{GzzA}(m_A + m_B)\Omega_0 - m_A m_B (R_A + R_B) V}{(m_A + m_B)(I_{GzzA} + I_{GzzB}) + m_A m_B \left(R_A + R_B\right)^2}
\end{aligned}
$$

[4 POINTS]