



School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 2: Kinematics and Dynamics of Particles Due Friday Feb 18, 2022

1. Straight Line Motion. The goal of this problem is to estimate the timing for two coordinated traffic lights (for those with a particular interest in this topic will find the transportation research board [signal timing manual](#) entertaining reading).

Suppose that the lights are 80m apart (the length of a typical NYC city block), and that a queue of 5 cars starts at rest at the first light, separated by a distance of 5m (i.e. the distance between the fronts of 2 cars in the queue). Assume that, after the first light changes



- The first car accelerates with an acceleration $0.2g$ until it reaches 11m/s (25mph), and thereafter travels at constant speed.
- Subsequent vehicles in the queue wait until the car ahead of them travels a distance of 3m, and then begin to move with the same history of acceleration and speed.
- Vehicles will brake to a stop with a deceleration of $0.2g$

1.1 How long does it take a vehicle to accelerate from rest to 11 m/s ?

Using the straight line motion formula $t = v / a = 11 / (0.2 \times 9.81) = 5.6s$

[1 POINT]

1.2 How far does a vehicle travel while accelerating (or decelerating) to 11m/s ?

Another straight line motion formula: $d = v^2 / (2a) = 11^2 / (2 \times 0.2 \times 9.81) = 30.8m$

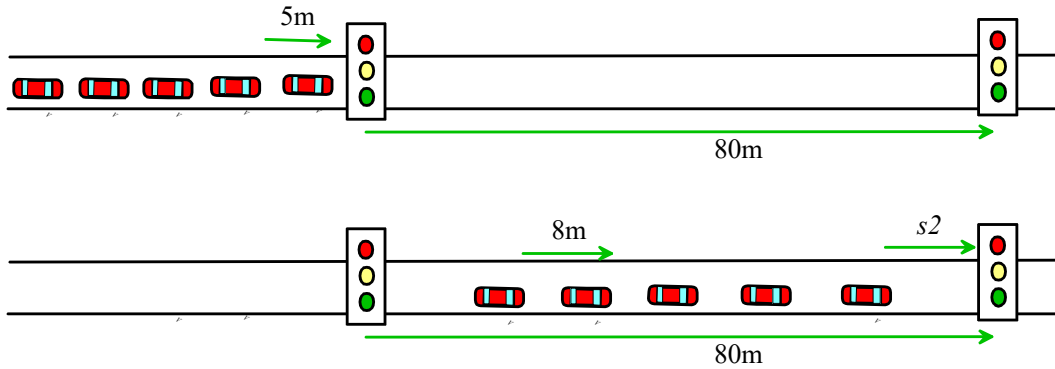
[1 POINT]

1.3 How long does it take for a vehicle to travel 3m while accelerating from rest?

And the 3rd straight line motion formula... $t = \sqrt{2d / a} = \sqrt{2 \times 3 / (0.2 \times 9.81)} = 1.75s$

[1 POINT]

1.4 To ensure that drivers can continue unimpeded through the second light, the second light must begin to change from red to red/amber before the first car reaches the distance found in 1.2 from the light. How long after the first light changes should the second light begin to change?



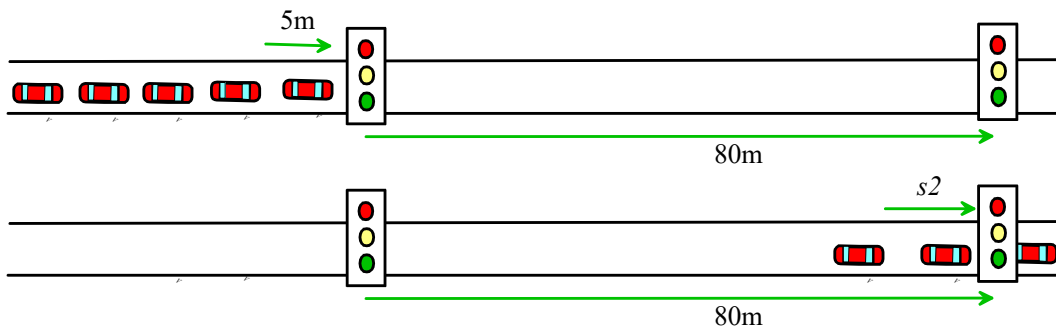
The second light must begin to change before the first car reaches a distance of 30.8m from the second light. It travels 30.8m while accelerating, leaving a distance of 80-61.7m to travel at constant speed.

This takes 1.66s at 11 m/s.

The **total time** is therefore 7.27s

[2 POINTS]

1.4 The second light may begin to change from green to amber once the last car in the queue reaches the distance found in 1.2 from the second light. How long after the first light changes should the second light change from green to amber?



The 5th car in the queue begins to move $4 \times 1.74 = 6.995$ s after the first. It travels a distance $5 \times 4 = 20$ m further than the first car. It must travel this distance at 11m/s, which takes 1.82s.

The second light may therefore change back to amber after a time $6.995 + 1.82 + 7.27 = 16.09$ s after the first light changes.

This calculation gives a cycle time for the light of about 20s – this is shorter than would likely be used in practice – typical cycle times are of order 1 min or so.

[2 POINTS]

2. Calculus review (apologies for inflicting this on you but hopefully it will be helpful. We suggest doing the problems by hand rather than MATLAB; the problems are meant to help you brush up on calculus that sometimes appears on exams. The calculus in this problem appears again in later parts of the course, so try to remember it!): A particle starts at position $x = 0$ with speed $v = V_0$ at time $t = 0$. For each case below, please find formulas for the speed v and position x of the particle as functions of time.

2.1 The acceleration of the particle depends on time $a(t) = -V_0 t \exp(-t)$ (here $\exp(x)$ denotes the exponential function, i.e. $\exp(x) = e^x$ (you might find it helpful to start by integrating $t \exp(-t)$ by parts)

Following the suggestion:

$$\int_0^v t \exp(-t) dt = [-t \exp(-t)]_0^t + \int_0^t \exp(-t) dt = -t \exp(-t) - (\exp(-t) - 1)$$

Then

$$\begin{aligned} a(t) &= \frac{dv}{dt} = V_0 t \exp(-t) \Rightarrow \int_{V_0}^v dv = \int_0^t -V_0 t \exp(-t) dt \\ \Rightarrow v - V_0 &= V_0 (t \exp(-t) + \exp(-t) - 1) \\ \Rightarrow v &= V_0 (t \exp(-t) + \exp(-t)) \end{aligned}$$

Hence

$$\begin{aligned} v &= \frac{dx}{dt} = V_0 (t \exp(-t) + \exp(-t)) \Rightarrow \int_0^x dx = \int_0^t V_0 (t \exp(-t) + \exp(-t)) dt \\ \Rightarrow x &= V_0 (1 - t \exp(-t) - \exp(-t)) + V_0 (1 - \exp(-t)) \\ \Rightarrow x &= V_0 (2 - 2 \exp(-t) - t \exp(-t)) \end{aligned}$$

[3 POINTS]

2.2 The acceleration of the particle depends on its speed as $a(v) = \omega \sqrt{V_0^2 - v^2}$, where ω is a constant.

$$a(v) = \omega \sqrt{V_0^2 - v^2} \Rightarrow \int_{V_0}^v \frac{dv}{\sqrt{V_0^2 - v^2}} = \int_0^t \omega dt$$

We can evaluate the integral with the substitution

$$v = V_0 \sin \theta \Rightarrow dv = (V_0 \cos \theta d\theta)$$

$$\int_{V_0}^v \frac{dv}{\sqrt{V_0^2 - v^2}} = \int_{\sin^{-1}(1)}^{\sin^{-1}(v/V_0)} \frac{\cos \theta d\theta}{\sqrt{1 - \sin^2 \theta}} = \sin^{-1}(v/V_0) - \pi/2$$

$$\text{And hence } v = V_0 \sin(\omega t + \pi/2)$$

Next

$$\begin{aligned} v &= \frac{dx}{dt} = V_0 \sin(\omega t + \pi/2) \Rightarrow \int_0^x dx = \int_0^t V_0 \sin(\omega t + \pi/2) dt \\ \Rightarrow x &= -(V_0 / \omega) \cos(\omega t + \pi/2) = (V_0 / \omega) \sin(\omega t) \end{aligned}$$

[3 POINTS]

2.3 The acceleration of the particle depends on its position as

$$a(x) = (x - V_0)$$

(You can solve this problem by finding v as a function of x first, then find x as a function of t , and finally use your answer to find v as a function of t . There are other ways too, if you know how to solve differential equations).

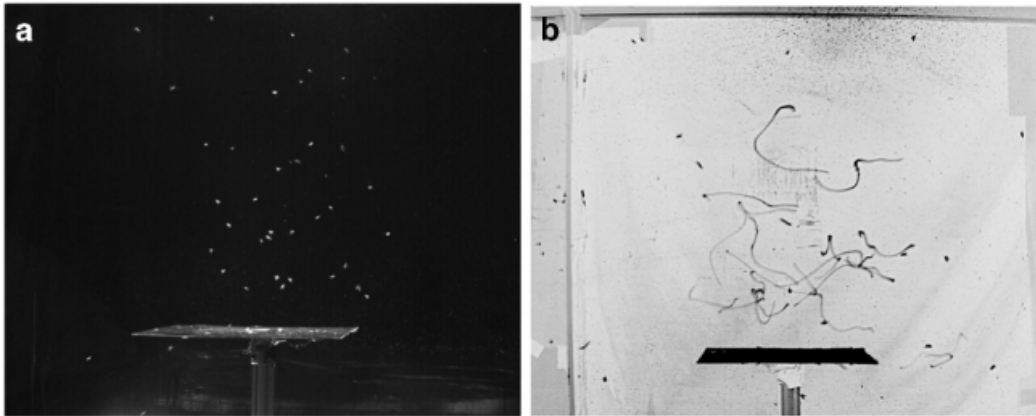
$$\begin{aligned} a = v \frac{dv}{dx} = x - V_0 &\Rightarrow \int_{V_0}^v v dv = \int_0^x (x - V_0) dx \\ \Rightarrow \frac{1}{2}(v^2 - V_0^2) &= \frac{1}{2}x^2 - V_0x \\ \Rightarrow v = \sqrt{x^2 - 2V_0x + V_0^2} &= V_0 - x \end{aligned}$$

(note that we took $v = V_0 - x$ rather than $x - V_0$ to satisfy the initial condition)

Then

$$\begin{aligned} \frac{dx}{dt} = V_0 - x &\Rightarrow \int_0^x \frac{dx}{V_0 - x} = \int_0^t dt \\ \Rightarrow -\log\left(\frac{V_0 - x}{V_0}\right) &= t \\ \Rightarrow x = V_0(1 - \exp(-t)) \\ \Rightarrow v = V_0 \exp(-t) \end{aligned}$$

[3 POINTS]



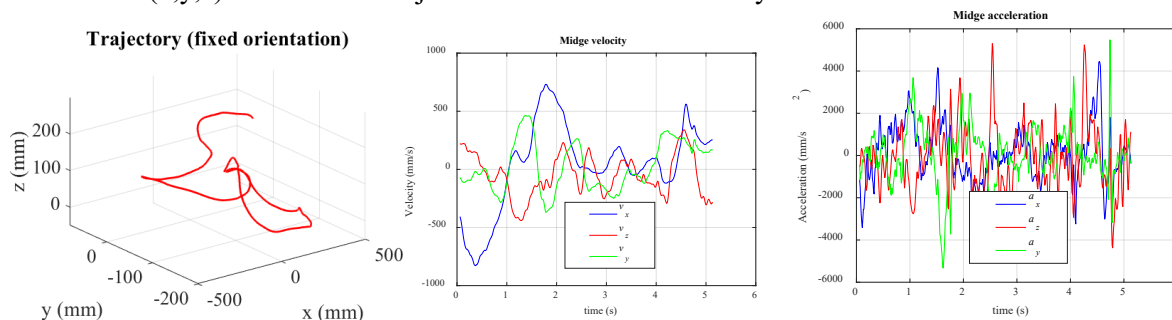
3. Using MATLAB to process position / acceleration / velocity measurements: [This data file](#) (in csv format, downloaded from [this data repository](#)) stores measured trajectories, velocities and accelerations for all 15 insects in a swarm of the non-biting midge *Chironomus Riparius* (the figure shows a snapshot and traces of the trajectories). A description of the data (the meanings of the columns, and the units) can be found in [this article](#). Please (i) include screenshots of the graphs in your pdf upload; and (ii) upload your MATLAB script to CANVAS as a solution to this problem.

3.1

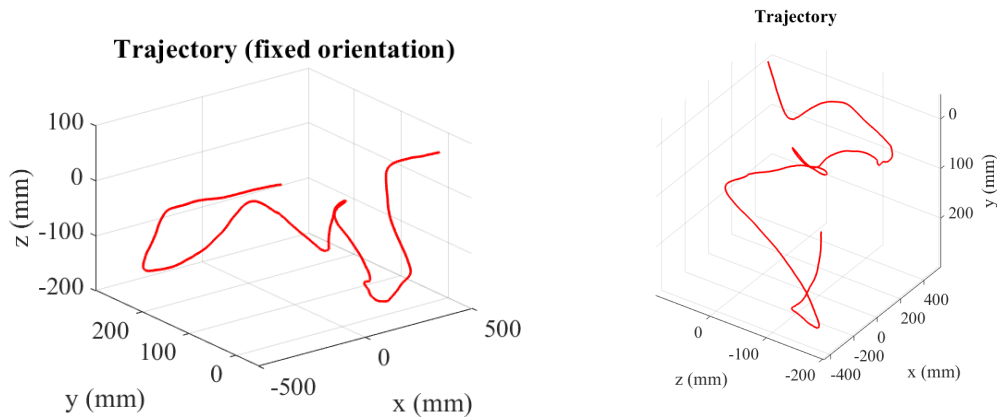
- Read the data into a matrix using the MATLAB 'csvread' command
- Extract the time, position, velocity, acceleration for midge number 4 from the matrix. You can do this by using a loop to check all the rows of the matrix, and using a conditional statement to store the data for the midge with the id (in the first column) of interest.
- Plot the trajectory (on a 3D plot).
- Plot the measured (v_x, v_y, v_z) (3 separate curves, on the same plot) as a function of time
- Plot the measured (a_x, a_y, a_z) (3 separate curves, on the same plot) as a function of time

Plots are shown below. See the MATLAB for solution.

The data description file gives the order of the coords as (x,z,y) and the plots below use this but if this is switched to (x,y,z) that's fine... it just labels the axes differently



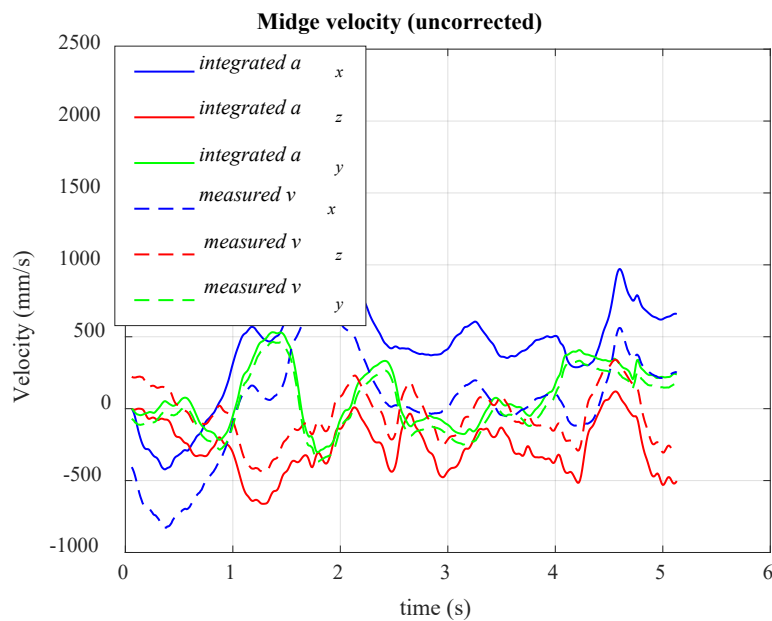
The trajectory plot looks a bit different if y and z are switched. In the plot below y is vertical – the fig on the left is the default MATLAB camera view but you can position the camera to have y vertical in the plot. The other plots are the same – only the legend subscripts change....



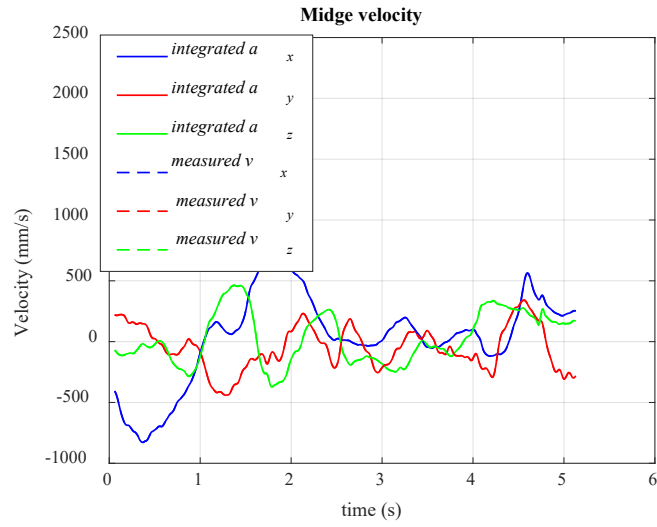
[3 POINTS]

3.2 Add MATLAB code to check the consistency of the data (for midge 4). Calculate the velocity by integrating the acceleration (a_x, a_y, a_z) using the 'cumtrapz' function, and (on the same graph) plot the measured (v_x, v_y, v_z). You will find that the integrated accelerations don't agree with the measured data – explain why, and see if you can find a way to correct the integrated accelerations.

The figure below shows the velocity calculated by integrating the accelerations (without any correction). They don't agree with the data in the file because the 'cumtrapz' function assumes the velocity is zero at time $t=0$.

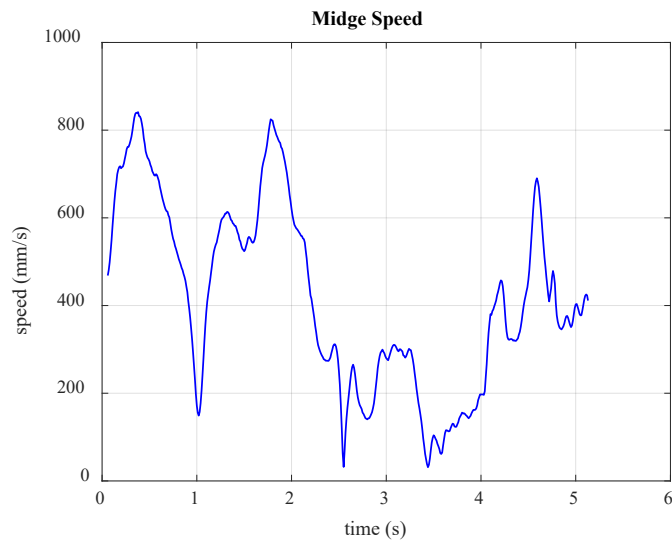


We can correct this by adding the velocity at time $t=0$ to the integrated accelerations. Results are shown below.



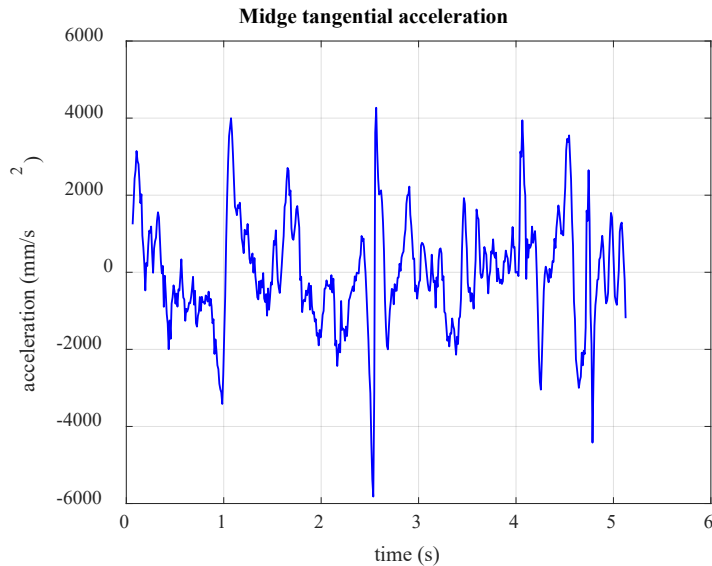
[3 POINTS]

3.3 Calculate (using a loop, eg) and plot the speed of midge number 4.



[2 POINTS]

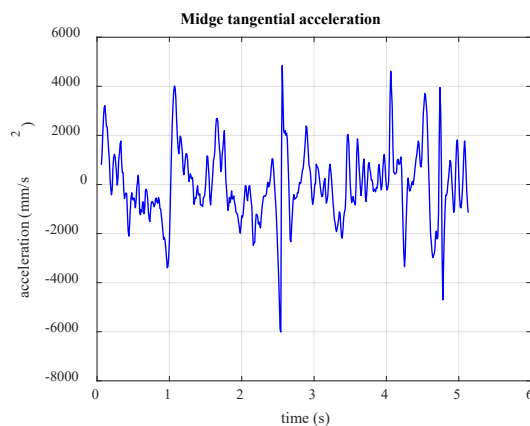
3.4 Calculate the tangential component of acceleration of midge 4 by differentiating the speed (you can find a review of how to differentiate a discrete data set in the [lecture videos](#))



[3 POINTS]

3.5 Suggest a different way to calculate the tangential acceleration directly from the data in the csv file (i.e. without calculating the speed and differentiating it). You don't actually have to do the calculation, unless you are curious to compare it with the result of 3.4....

You can also calculate the tangential acceleration using the formula $a_t = \mathbf{a} \cdot \mathbf{t}$, where \mathbf{t} is a unit vector tangent to the path. Since the direction of the velocity vector is parallel to the tangent, we can calculate \mathbf{t} using $\mathbf{t} = \mathbf{v} / |\mathbf{v}|$. The resulting plot is shown below. It looks slightly different to the results obtained by evaluating the numerical derivative of the speed. This is because the authors of the dataset smoothed the acceleration data in the file.



[2 POINTS]

4. Simple circular motion problem A satellite in geostationary orbit has an altitude 35786km above the earth's surface, and completes an orbit in 24 hours. The earth's radius is 6371km.



Calculate the speed of the satellite and the magnitude of its acceleration.

The distance (arc length) around the circumference of an orbit is $2\pi R = 264.88 \times 10^3 \text{ km}$. The speed is the distance/time, i.e.

$$V = \frac{264.88 \times 10^3}{24 \times 3600} = 3.066 \text{ km/s}$$

The acceleration can be calculated using the constant speed circular motion formula

$$a_n = \frac{V^2}{R} = 0.23 \text{ m/s}^2$$

[2 POINTS]

5. Harder circular motion problem. A particle travels around a circular path with radius R . The particle starts at rest and has tangential acceleration

$$a_t = a_0(1 - V^2 / v_0^2)$$

where V is the particle's speed, and a_0, v_0 are two constants.

5.1 Find a formula for the particle's speed as a function of distance (arc length) travelled s .

We have

$$\begin{aligned} a_t &= V \frac{dV}{ds} = a_0 \left(1 - \frac{V^2}{v_0^2}\right) \\ \Rightarrow \int_0^V \frac{V dV}{(1 - V^2 / v_0^2)} &= \int_0^s a_0 ds \Rightarrow -\frac{v_0^2}{2} \log(1 - V^2 / v_0^2) = a_0 s \\ \Rightarrow V &= v_0 \sqrt{1 - \exp(-2a_0 s / v_0^2)} \end{aligned}$$

[2 POINTS]

5.2 Find formulas for the angular speed ω and angular acceleration α as functions of distance traveled.

Using the formulas:

$$V = \omega R \Rightarrow \omega = V / R = (v_0 / R) \sqrt{1 - \exp(-2a_0 s / v_0^2)}$$

$$a_t = \alpha R \Rightarrow \alpha = a_t / R = (a_0 / R) (1 - V^2 / v_0^2) = (a_0 / R) \exp(-2a_0 s / v_0^2)$$

[1 POINT]

5.3 Find a formula for the magnitude of the acceleration as functions of distance traveled.

$$|\mathbf{a}| = \sqrt{a_t^2 + a_n^2} = \sqrt{a_t^2 + V^4 / R^2} = \sqrt{(a_0 \exp(-2a_0 s / v_0^2))^2 + \frac{v_0^4}{R^2} (1 - \exp(-2a_0 s / v_0^2))^2}$$

[2 POINTS]

6. Normal-Tangential Coordinates. The x, z coordinates (x is horizontal, z is vertical) as a function of time of an aircraft performing an aerobatic loop are given by

$$x = 150 \sin \{0.487t - 0.921 - 0.933 \tanh[(t - 10) / 4]\}$$

$$z = 150 \left(1 - \cos \{0.487t - 0.921 - 0.933 \tanh[(t - 10) / 4]\}\right)$$

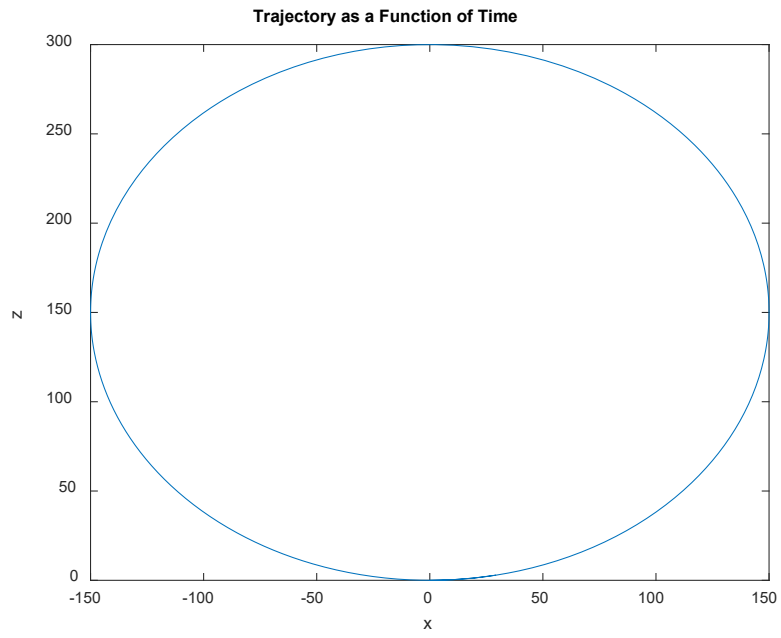
with x and z in meters, and $0 < t < 17$ in seconds. ($\tanh(x)$ is a 'hyperbolic tangent' defined as

$$\tanh(x) = [\exp(x) - \exp(-x)] / [\exp(x) + \exp(-x)]$$

- it is a built-in function in MATLAB). Use a MATLAB 'Live Script' to answer the following questions (as a solution to this problem please (i) include screen shots of the graphs in your pdf upload; and (ii) upload your 'Live Script' to CANVAS.

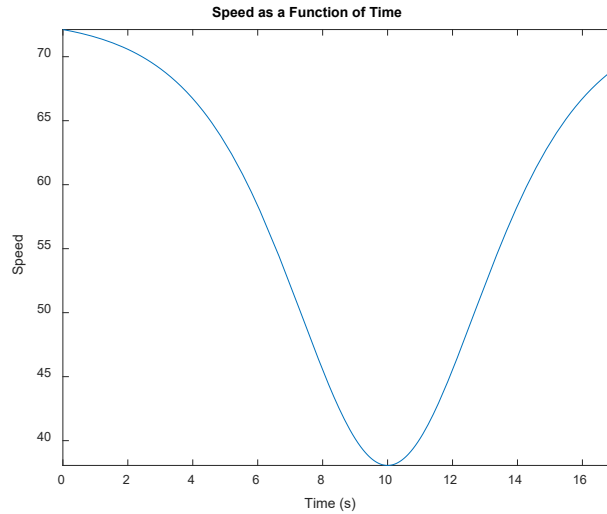


6.1 Plot the trajectory (z vs x) for a time interval $0 < t < 17$ s



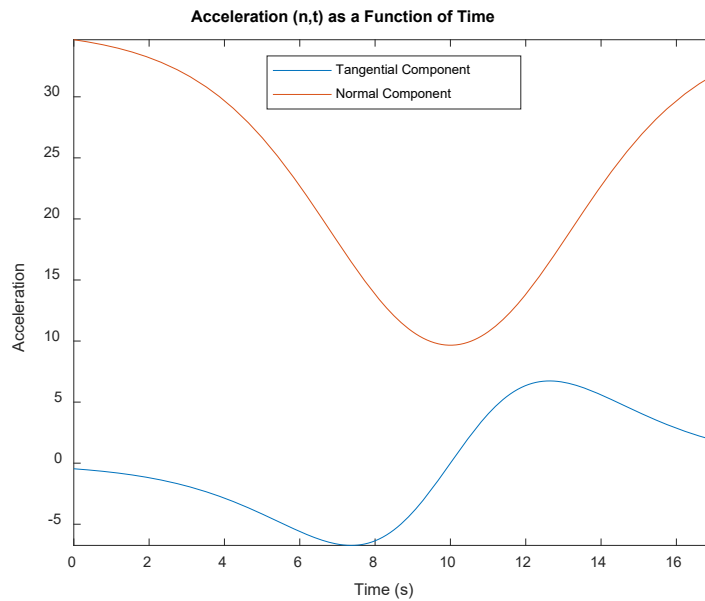
[1 POINT]

6.2 Plot the speed as a function of time, for a time interval $0 < t < 17s$



[2 POINTS]

6.3 Let \mathbf{n}, \mathbf{t} be unit vectors normal and tangent to the path (take \mathbf{n} to be the vector that gives a positive normal acceleration). Plot the normal and tangential components of acceleration (on the same plot), for a time interval $0 < t < 17$ s.



[2 POINTS]

6.4 The lift force on the aircraft acts in the \mathbf{n} direction, the drag force and propulsive forces acts in the \mathbf{t} direction (note the negative sign!), and gravity acts vertically. Use Newton's law to find and plot the

F_L / mg) as a function of time.

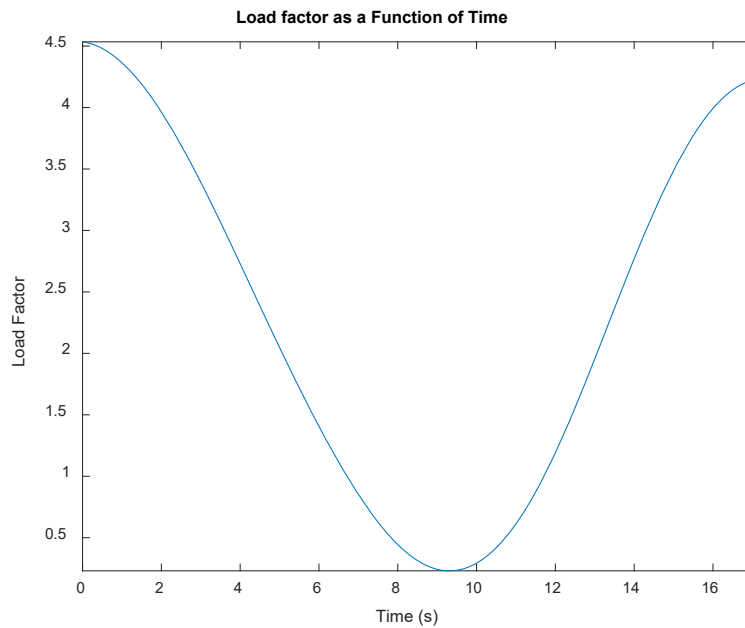
Newton's law gives $F_L \mathbf{n} + (F_T - F_D) \mathbf{t} - mg \mathbf{k} = m(a_n \mathbf{n} + a_t \mathbf{t})$

To find the lift force, dot both sides of this equation with \mathbf{n} and not $\mathbf{n} \cdot \mathbf{n} = 1$ $\mathbf{n} \cdot \mathbf{t} = 0$ so

$$F = mg \mathbf{k} \cdot \mathbf{n} + ma_n$$

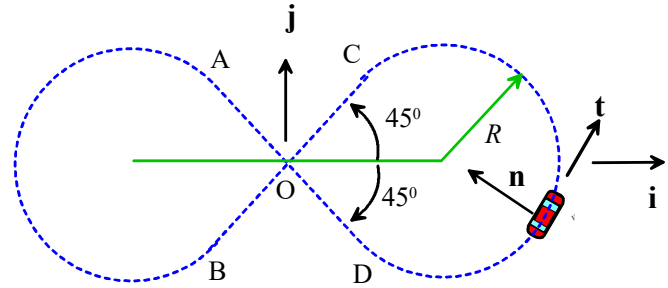
Finally note that $\mathbf{k} \cdot \mathbf{n}$ is the z component of the normal vector. Therefore

$$F_L / mg = n_z + a_n / g$$



[2 POINTS]

7. A vehicle travels around the figure-8 path shown in the figure. To avoid skidding, the magnitude of its acceleration cannot exceed a_{\max} . Find a formula for the shortest possible time to complete a full lap of the track, in terms of R and a_{\max} (you can assume travels with constant speed around the circular parts of the track, and has constant acceleration/deceleration on the straight portions of the track).



This problem can be solved using a combination of circular motion formulas and straight line motion formulas.

1. The acceleration around the circular arcs is $R\omega^2 = a_{\max} \Rightarrow \omega = \sqrt{a_{\max}/R}$. The vehicle turns through an angle $3\pi/2$ around each circular arc, which takes a time $t_1 = 3\pi / (2\omega) = (3\pi/2)\sqrt{R/a_{\max}}$
2. Note that the straight portions of the track (eg AOD) have length R
3. Along AOD the vehicle first accelerates from speed $v_A = R\omega$ to $v_O = \sqrt{v_A^2 + 2a_{\max}R} = \sqrt{3a_{\max}R}$ and then decelerates again to $v_D = R\omega$.

4. The average speed between A and O is therefore $(v_A + v_O)/2 = \frac{(1 + \sqrt{3})}{2} \sqrt{Ra_{\max}}$

5. It therefore takes a time $t_2 = 2R / \left(\frac{(1 + \sqrt{3})}{2} \sqrt{Ra_{\max}} \right) = \frac{2}{(1 + \sqrt{3})} \sqrt{\frac{R}{a_{\max}}}$ to travel from A to O.

By symmetry, the time to travel from O to D, O to B and C to O is the same.

6. The time to complete a full lap is therefore

$$2t_1 + 4t_2 = \left(3\pi + \frac{8}{1 + \sqrt{3}} \right) \sqrt{\frac{R}{a_{\max}}} \approx 12.35 \sqrt{\frac{R}{a_{\max}}}$$

[8 POINTS]