School of Engineering
Brown University

EN40: Dynamics and Vibrations

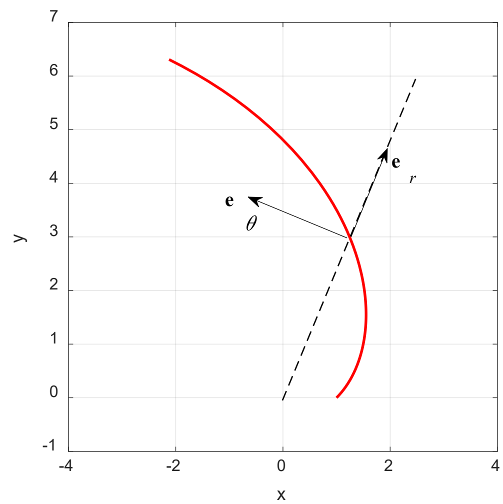
Homework 3: Kinematics and Dynamics of Particles
Due Friday Feb 25 2022

Please submit your solutions to the MATLAB coding problems 4, 5 by uploading a single file, with a .m extension, to Canvas.

1. Polar Coordinates: A particle travels at constant speed V along a curve described by the parametric polar coordinates

$$r = r_0 e^{\theta}$$

where r_0 is a constant



1.1 Find a formula for the velocity vector of the particle, in terms of $r, \theta, \frac{d\theta}{dt}$ (differentiate *r* with respect to time using the chain rule)

Using the polar coordinate formulas:

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$
$$= r_0 e^{\theta}\frac{d\theta}{dt}\mathbf{e}_r + r_0 e^{\theta}\frac{d\theta}{dt}\mathbf{e}_\theta$$

[2 POINTS]

1.2 Use the solution to 1.1 and the fact that the speed of the particle is known to find a differential equation for θ. Solve the differential equation (by hand – separate variables and integrate) to find θ as a function of time, and hence deduce a formula for *r* as a function of time (and V, r_0).

We are told that the speed is constant, which means that (taking the magnitude of the velocity in 1.1)

$$V = \sqrt{2} r_0 e^{\theta}\frac{d\theta}{dt}$$

Separate variables and integrate:

$$\frac{V}{\sqrt{2} r_0} t = \int_0^{\theta} e^{\theta} d\theta = e^{\theta} - 1$$

Hence $\theta = \log\left(1 + \frac{Vt}{\sqrt{2} r_0}\right)$. We also see that $r = r_0 + Vt/\sqrt{2}$

[3 POINTS]

1.3 Hence, find the acceleration of the particle as a function of time, in the polar coordinate basis.

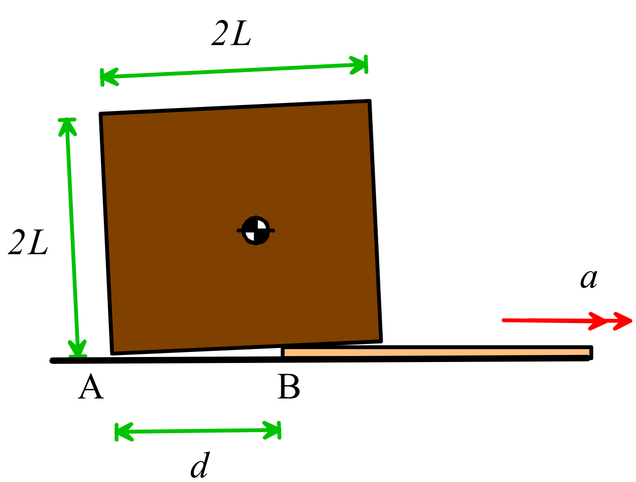
The polar coordinate formula for acceleration
$\mathbf{a} = \left\{ (d^2 r / dt^2) - r(d\theta / dt)^2 \right\} \mathbf{e}_r + \left\{ r d^2\theta / dt^2 + 2(dr/dt)(d\theta/dt) \right\} \mathbf{e}_\theta$ gives

$$\mathbf{a} = -\left(r_0 + \frac{Vt}{\sqrt{2}} \right) \left(\frac{V/(\sqrt{2} r_0)}{1 + Vt/(\sqrt{2} r_0)} \right)^2 \mathbf{e}_r + \left\{ -\left(r_0 + \frac{Vt}{\sqrt{2}} \right) \frac{V^2/(2r_0^2)}{\left(1 + Vt/(\sqrt{2} r_0)\right)^2} + 2\frac{V}{\sqrt{2}} \left(\frac{V/(\sqrt{2} r_0)}{1 + V/(\sqrt{2} r_0)} \right) \right\} \mathbf{e}_\theta$$

$$= \frac{V^2}{2\left(r_0 + Vt/\sqrt{2} \right)} \left(-\mathbf{e}_r + \mathbf{e}_\theta \right)$$

[2 POINTS]

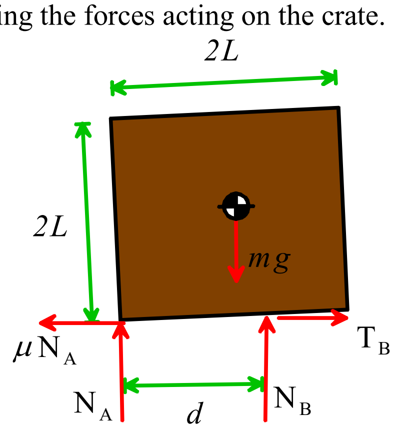
2. Newton's law problem. A cubic crate with side length $2L$ rests on a rug, as shown in the figure. The left edge of the rug is a distance $d>L$ from the left hand edge of the crate. The crate contacts the ground at its left edge, and contacts the rug at the left hand edge of the rug. Both contacts have the same friction coefficient μ. The goal of this problem is to calculate the critical acceleration necessary to pull the rug from beneath the crate.



As discussed in class, the procedure is to assume that the rug does not slip (so the crate and rug move with the same acceleration, with no slip between them), and then calculate the critical acceleration necessary for slip to begin at the rug/crate contact.

The small tilt of the crate can be neglected.

2.1 Draw a free body diagram showing the forces acting on the crate. Assume slip at A and no slip at B.



(Its OK to draw Tb acting left as well since there is no slip at B. The force actually acts right, of course, so with the latter choice Tb will turn out to be negative.

[3 POINTS]

2.2 Write down Newton's law and the moment balance equation for the crate (remember to take moments about the COM).

Newton's law is $(T_B - \mu N_A)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma\mathbf{i}$

The moment balance equation is $(T_B - \mu N_A)L + N_B(d - L) - N_A L = 0$
(Sign of the Tb term will change if Tb was drawn acting left in both these eqs)

[2 POINTS]

2.3 Hence, calculate the reaction forces at the two contacts.

The equations in 2.2 show that

$$\begin{aligned} &T_B - \mu N_A = ma \\ &N_B(d - L) - N_A L = -maL \\ &N_A + N_B = mg \\ &\Rightarrow N_B d = mL(g - a) \qquad N_A d = mg(d - L) + maL \end{aligned}$$

And hence

$$N_B = m(g - a)L/d \qquad T_B = ma + \mu N_A = ma + \mu m\left(aL/d + g(1 - L/d)\right)$$

$$N_A = mg(d - L)/d + maL/d \qquad T_A = \mu N_A$$

(Tb will be negative if drawn acting left)

[4 POINTS]

2.4 Finally, find a formula for the critical acceleration, in terms of μ, g, L, d

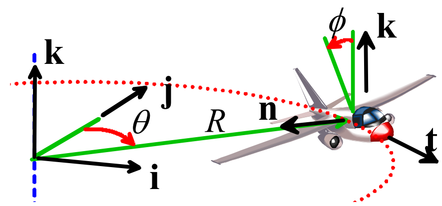
Slip will occur if

$$\begin{aligned} &|T_B| > \mu N_B \\ &\Rightarrow ma + \mu m\left(aL/d + g(1 - L/d)\right) > \mu m(g - a)L/d \\ &\Rightarrow a(1 + 2\mu L/d) > \mu g(2L/d - 1) \\ &\Rightarrow a > \mu g\frac{(2L/d - 1)}{(1 + 2\mu L/d)} \end{aligned}$$

(If Tb was drawn left the – sign will disappear when the | | is taken)

[2 POINTS]

3. "Steep turns" are part of the "Private Pilot – Airplane Airperson Certification Standards" They are 360 degree turns flown with a 45 degree bank angle, at a (constant) speed specified by the airplane manufacturer. In a C-172 the airspeed is 90 knots.

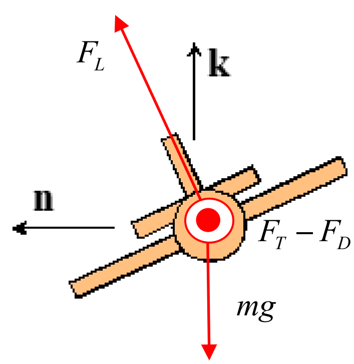


3.1 Write down the acceleration vector of an aircraft in a steep turn (use **n,t** coordinates) in terms of its speed and the radius of the turn.

This is just the circular motion formula $\mathbf{a} = (V^2 / R)\mathbf{n}$

[1 POINT]

3.2 Draw a free body diagram showing the forces acting on the aircraft (assume that the longitudinal axis of the aircraft is tangent to the path).



[2 POINTS]

3.3 Write down **F**=m**a** for the aircraft using **n,t** coordinates.

$$(F_L \cos 45 - mg)\mathbf{k} + F_L \sin 45\mathbf{n} + (F_T - F_D)\mathbf{t} = m(V^2 / R)\mathbf{n}$$

[2 POINTS]

3.4 Hence (for a C-172), calculate the radius of the turn, the load factor (the ratio of the lift force on the aircraft to its weight), and the time it takes to complete a 360 degree turn.

The component of Newton's law yields

$$F_L \cos(45) = mg \Rightarrow F_L / mg = \sqrt{2} \approx 1.41$$

You can compare this with the value cited in the video. The video's explanation for the increase in load factor is 'centrifugal force' but the load factor would increase if the aircraft were in a slip (banking without turning, so there is no centripetal acceleration) - it's really because the vertical component of lift has to balance the aircraft's weight, and if the lift acts at an angle its magnitude must increase.

The horizontal component gives

$$mV^2 / R = F_L / \sqrt{2} \Rightarrow mV^2 / R = mg$$
$$\Rightarrow R = V^2 / g = 46.3^2 / 9.81 = 218.5m$$

The time for a complete turn is the distance around the circumference/speed = $2\pi R / V = 29.7s$

[3 POINTS]

3.5 This private pilot course states that during steep turns "pull back on the yoke will increase the rate of turn but does not allow the aircraft to climb"

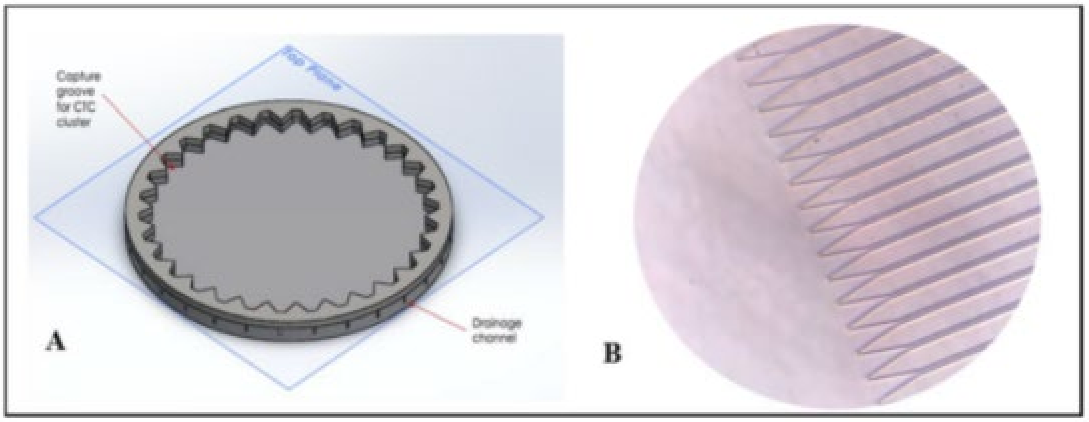
Pulling back on the yoke during level flight pitches the airplane's nose upwards, increasing the angle of attack of the wings. At constant speed, this increases both the lift force and drag force on the aircraft. If the power setting is fixed, the aircraft climbs briefly because lift exceeds weight, but since the drag also exceeds the thrust, the aircraft slows down until thrust and drag balance. Since lift and drag are both proportional to V^2, once the thrust and drag are equal, the lift and weight balance again, so the aircraft stops climbing, and travels at a lower speed.

Why does pulling back the yoke in a turn increase the turn rate?

Pulling back on the yoke in a turn will increase the angle of attack, and, just as in level flight, after a brief climb the aircraft will slow down. The turn rate is $\omega = V / R = V / (V^2 / g) = g / V$. Decreasing V therefore increases the turn rate (and also reduces the radius of the turn). It also means that a slow aircraft will complete a 360 degree turn in a shorter time than a fast one!

[2 POINTS]

Figure 2. (**A**) CAD drawing of microfluidic centrifugal device (MCD) for circulating tumor cells (CTC) cluster capture and analysis. Note: Figure not to scale. The real device has a radius of 36.5 mm and a total of 1000 wells along the circumference. (**B**) Micrograph (20X) of individual traps and corresponding channels. Scale: 30 μm is the distance from one capture well to another.



4. [This publication] describes a centrifugal microfluidic device for extracting single cells from a fluid suspension. It consists of a shallow cylinder with an array of wells around its circumference. Cells are suspended in a viscous fluid inside the cylinder, which spins about its axis at an angular speed between 50 and 2000 rpm. The cells have a higher density than the fluid. Cells drift towards the circumference of the disk as a result of its rotational motion, and are trapped in the channels.

The goal of this problem is to calculate the trajectory of a cell inside the device. To simplify calculations, we will assume that the fluid spins with the same (constant) angular speed Ω as the cylinder

4.1 The publication describes some calculations of the cell trajectories but several equations in the paper contain errors. Explain what is wrong with equation (2) in the paper.

> The equation appears to be an attempt to write down the radial component of Newton's law, but it includes the Coriolis acceleration, which should be in the hoop direction, not the radial direction.

[2 POINTS]

4.2 Assume that a cell in the fluid is subjected to:

(i) A radial buoyancy force (which is caused by a radial pressure gradient that develops in the spinning fluid) $\mathbf{F}_B = -(4\pi R_c^3/3)\rho_f \Omega^2 r \mathbf{e}_r$, where R_c is the radius of the cell (idealized as a sphere); ρ_f is the mass density of the fluid; and r is the radial position of the cell.

(ii) A drag force $\mathbf{F}_D = 6\pi R_c \eta \left(r\Omega \mathbf{e}_\theta - \mathbf{v}_c \right)$, where η is the viscosity of the fluid.

Write down Newton's law for the cell (express the acceleration and velocity of the cell in polar coordiantes, in terms of its radial-polar coordinates (r,θ))

$$\frac{4\pi R_c^3}{3}\rho_c\left\{\left(\frac{d^2r}{dt^2}-r\left[\frac{d\theta}{dt}\right]^2\right)\mathbf{e}_r+\left(2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta\right\}=6\pi R_c\eta\left\{-\frac{dr}{dt}\mathbf{e}_r+\left(r\Omega-r\frac{d\theta}{dt}\right)\mathbf{e}_\theta\right\}-\frac{4\pi R_c^3}{3}\rho_f\Omega^2 r\mathbf{e}_r$$

[2 POINTS]

4.3 Hence, show that the equations for (r,θ) can be re-arranged into four first order differential equations

$$\frac{d}{dt}\begin{bmatrix} r \\ \theta \\ v_r \\ \omega \end{bmatrix}=\begin{bmatrix} v_r \\ \omega \\ r\omega^2-C_B\Omega^2 r-C_D v_r \\ C_D(\Omega-\omega)-2v_r\omega/r \end{bmatrix} \qquad C_B=\frac{\rho_f}{\rho_c} \qquad C_D=\frac{9}{2}\frac{\eta}{\rho_c R_c^2}$$

We introduce the time derivatives of (r,θ) as additional unknowns, which satisfy $v_r = dr/dt \qquad \omega = d\theta/dt$. This gives the first two equations stated. The equations for the time derivatives of v_r,ω come from the radial and hoop components of Newton's law

$$\frac{4\pi R_c^3}{3}\rho_c\frac{dv_r}{dt}=\frac{4\pi R_c^3}{3}\rho_c r\omega^2-6\pi R_c\eta v_r-\frac{4\pi R_c^3}{3}\rho_f\Omega^2 r$$

$$\frac{4\pi R_c^3}{3}\rho_c r\frac{d\omega}{dt}=-\frac{4\pi R_c^3}{3}\rho_c 2v_r\omega+6\pi R_c\eta r(\Omega-\omega)$$

These can be re-arranged into the form stated in the problem.

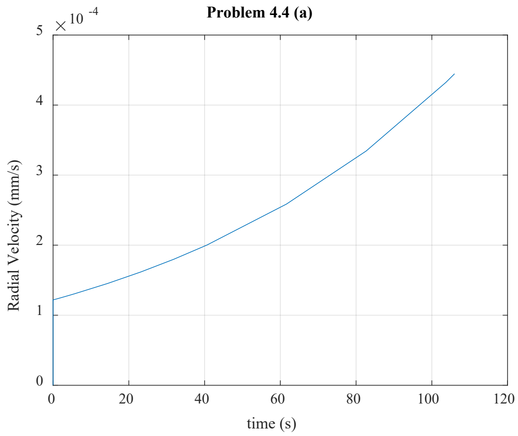
[2 POINTS]

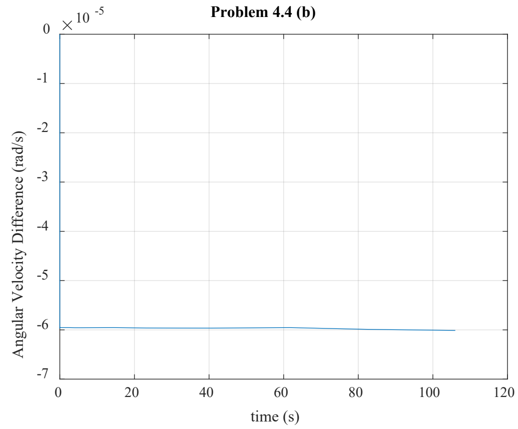
4.4 Solve the equations in 4.3 using ode45, using the following values for parameters:

- $\eta = 0.001 Ns/m^2$
- $\rho_c = 1050\ kg/m^3 \quad \rho_f = 1000\ kg/m^3$
- $R_c = 10\times10^{-6} m$

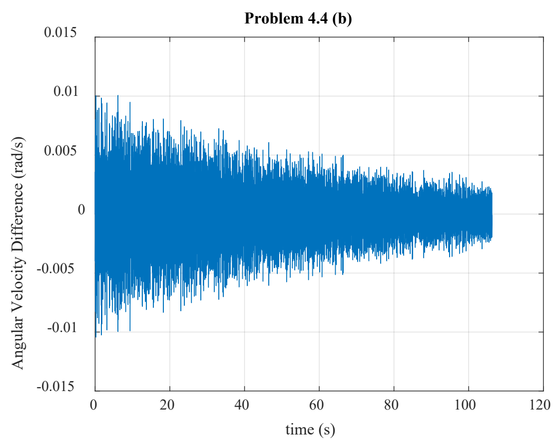
Add an 'event' function that will stop the calculation when the cell reaches the circumference of the cylinder, which has a radius of 36.5mm.

Plot (on two separate plots) (a) the variation of v_r with time, (b) the variation of $\omega-\Omega$ with time, for $\Omega = 1000 rpm$ (you will need to convert this to rad/s) and initial conditions $r = 10mm,\ \theta = 0,\ v_r = 0,\ \omega = \Omega$. The calculation might be a bit slow because the equation of motion is 'stiff' (because the cell mass is very small, so a small change in the forces acting on the cell causes a very large change in acceleration – ode45 needs to take very small time steps to solve this kind of problem). If it's too slow with ode45 on your computer you can switch to ode15s (an special solver for stiff differential equations)





These were produced with ode15s. ode45 gives a very noisy prediction for the second plot (you can fix this by increasing the accuracy of the solver using the 'RelTol' argument as described in prob 5, but that's not required).



[4 POINTS]

4.5 The results of 4.4 suggest that for the conditions of the experiment $\omega \approx \Omega$ and dv_r / dt is small. Show that, with these approximations the radial speed of the cell can be approximated as

$$v_r = \frac{dr}{dt} = \frac{(1-C_B)\Omega^2}{C_D} r$$

Find a formula for *r* as a function of time, with initial condition $r = R_0$ at time *t=0.*

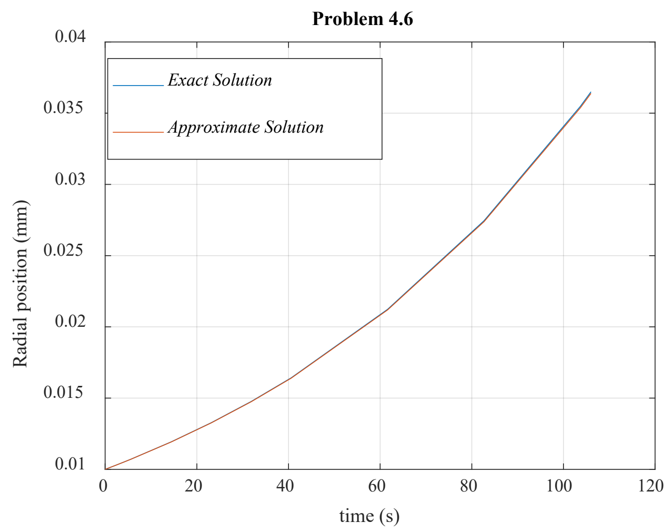
Separate variables and integrate

$$\int_{R_0}^{r} \frac{dr}{r} = \frac{(1-C_B)\Omega^2}{C_D}\int_0^t dt \Rightarrow \log(r/R_0) = \frac{(1-C_B)\Omega^2}{C_D}t$$

$$\Rightarrow r = R_0 \exp\left\{\frac{(1-C_B)\Omega^2}{C_D}t\right\}$$

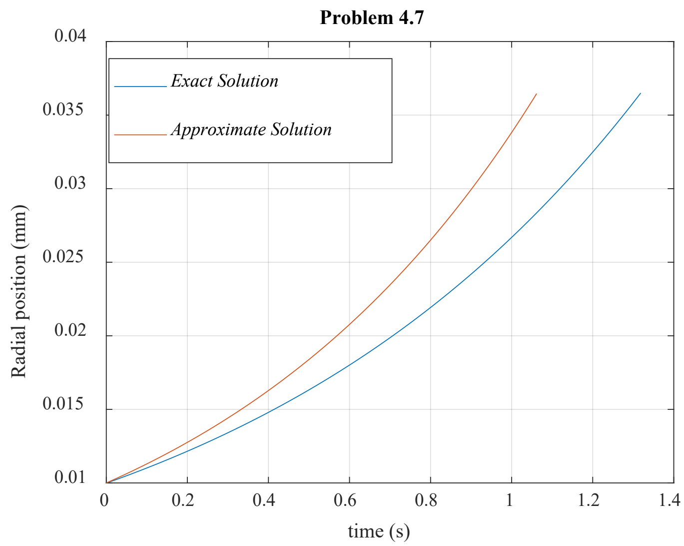
[2 POINTS]

4.6 For the conditions in 4.4, plot a graph of r as a function of time as predicted by ode45, together with the prediction in part 4.5 (on the same plot).



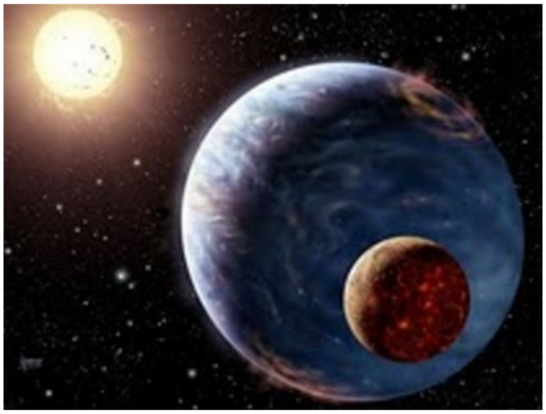
[2 POINTS]

4.7 Repeat problem 4.6 with a cell diameter $R_c = 100\times10^{-6}\,m$



[2 POINTS]

5. The "three-body problem" in classical mechanics is to predict the motion of three massive planets that exert gravitational forces on one another. There is no known general analytical solution to the problem (special solutions do exist, of course), but it can be solved numerically. That is the goal of this question. To keep things simple, we will assume that all three planets move in the (*x,y*) plane.



The gravitational force acting on a body with mass m_1 at position vector $\mathbf{r}_1$ by a second body with mass m_2 located at position vector $\mathbf{r}_2$ can be expressed as

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{R_{12}^3}\left(\mathbf{r}_2 - \mathbf{r}_1\right)$$

where $\mathbf{R}_{12} = \left|\mathbf{r}_1 - \mathbf{r}_2\right|$ is the distance between the two particles and G=6.67x10^{-11} m^{3}kg^{-1}s^{-2} is the gravitational constant

5.1 The goal of the problem is to calculate the x, y coordinates and the v_x, v_y components of velocity of the three particles. The unknowns can be stored in a MATLAB vector as

$$\mathbf{w} = \left[x_1, y_1, x_2, y_2, x_3, y_3, v_{x1}, v_{y1}, v_{x2}, v_{y2}, v_{x3}, v_{y3}\right]$$

Find a formula for the time derivative of **w**, in terms of the components of **w** as well as G, m_1, m_2, m_3

The EOM can be expressed as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ v_{x3} \\ v_{y3} \end{bmatrix} = \begin{bmatrix} v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ v_{x3} \\ v_{y3} \\ F_{x1}/m_1 \\ F_{y1}/m_1 \\ F_{x2}/m_2 \\ F_{y2}/m_2 \\ F_{x3}/m_3 \\ F_{y3}/m_3 \end{bmatrix}$$

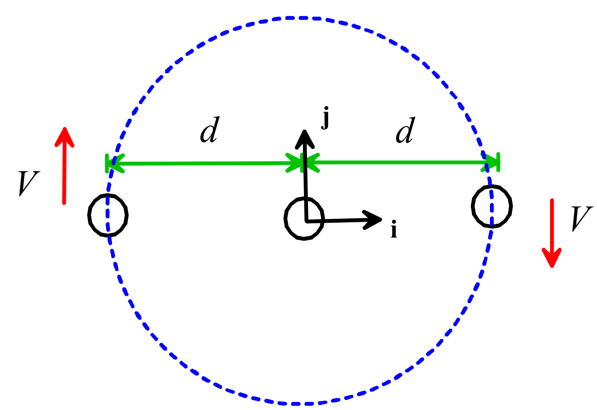
$$\mathbf{F}_1 = \frac{Gm_1m_2}{R_{12}^3}\left[(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}\right]+\frac{Gm_1m_3}{R_{13}^3}\left[(x_3-x_1)\mathbf{i}+(y_3-y_1)\mathbf{j}\right]$$

$$\mathbf{F}_2 = \frac{Gm_1m_2}{R_{12}^3}\left[(x_1-x_2)\mathbf{i}+(y_1-y_2)\mathbf{j}\right]+\frac{Gm_2m_3}{R_{23}^3}\left[(x_3-x_2)\mathbf{i}+(y_3-y_2)\mathbf{j}\right]$$

$$\mathbf{F}_3 = \frac{Gm_3m_1}{R_{13}^3}\left[(x_1-x_3)\mathbf{i}+(y_1-y_3)\mathbf{j}\right]+\frac{Gm_3m_2}{R_{23}^3}\left[(x_2-x_3)\mathbf{i}+(y_2-y_3)\mathbf{j}\right]$$

[2 POINTS]

5.2 The figure shows a configuration with the central planet stationary, and the other two planets in circular orbit with radius *d* around it. The three planets have identical mass *m*. Find formulas for the speed of the two moving planets, and the time for one orbit, in terms of the orbit radius *d*, as well as *G* and *m*.



Newton's law for either of the moving planets (in normal-tangential coordinates) gives

$$\frac{mV^2}{d}\mathbf{n} = \frac{Gm^2}{d^2}\mathbf{n} + \frac{Gm^2}{(2d)^2}\mathbf{n}$$

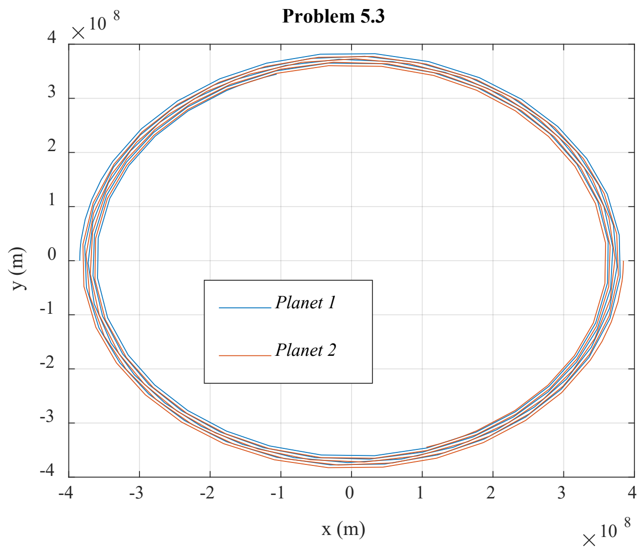
$$\Rightarrow V = \sqrt{\frac{5Gm}{4d}}$$

The time for an orbit follows as

$$T = \frac{2\pi d}{V} = \frac{4\pi d^{3/2}}{\sqrt{5Gm}}$$

[2 POINTS]

5.3 Write a MATLAB script that solves the equations in 5.1 using ode45. Use the following value for the masses of the planets: $m_1 = m_2 = m_3 = 7.348 \times 10^{22}\, kg$ (The mass of the moon), and use the configuration in 5.2 with $d = 384.4 \times 10^6\, m$ (the average distance between the moon and the earth) as initial conditions. Run the simulation for a time that will allow the two moving planets to complete 4 orbits. Plot a graph showing the trajectories of the two moving planets. Upload your MATLAB code as a solution to this problem and include an image of the plot in your pdf upload.

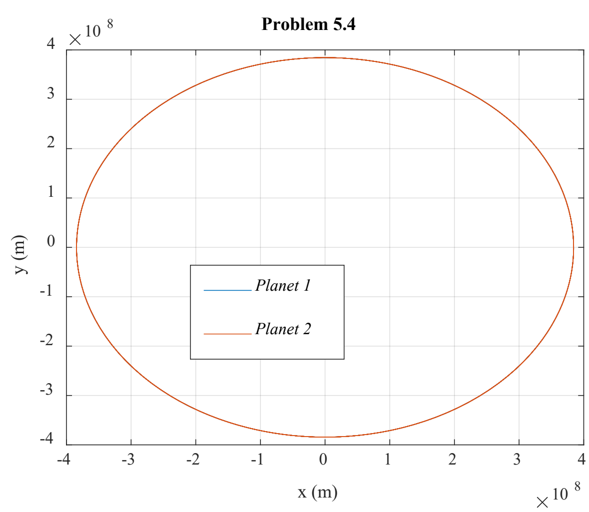


[4 POINTS]

5.4 The solution to 5.3 is clearly wrong – MATLAB is not predicting the correct circular orbit. Correct the problem by improving the accuracy of ode45. You can do this by setting the 'RelTol' parameter for the ode solver using

```
options = odeset('RelTol',0.00001);
```

and then providing the 'options' variable as the last argument in the call to ode45. Upload your MATLAB code as a solution to this problem and include an image of the plot in your pdf upload.



[2 POINTS]

5.5 Finally, run a simulation with the same mass for each planet and with the tolerance in 5.4, but with the following initial conditions

$$\mathbf{r}_1 = -(d/2)\mathbf{i} - d\mathbf{j} \qquad \mathbf{v}_1 = 0$$

$$\mathbf{r}_2 = (d/10)\mathbf{i} + d\mathbf{j} \qquad \mathbf{v}_2 = 0$$

$$\mathbf{r}_3 = d\mathbf{i} - (d/2)\mathbf{j} \qquad \mathbf{v}_3 = 0$$

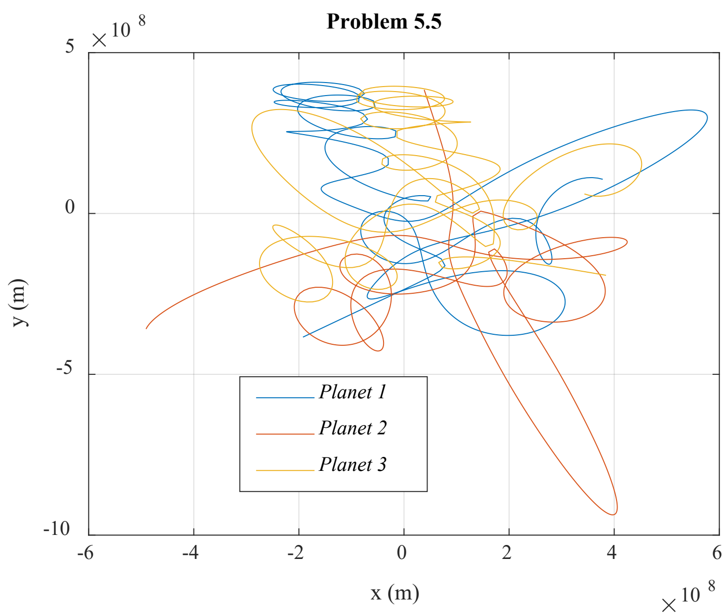
Where d is the orbit radius calculated in 5.2. Run your simulation for a time interval of 4 times the orbit period calculated in 5.2. Plot the trajectory of all three planets.

If you would like to visualize the motion, you can download the function called 'animate_planets' from the web site and save the file in the same directory as the function that runs your homework. You will need to set up ode45 to calculate the solution at equally spaced time intervals, eg with

```
time_int = [0:T/200:4*T];
[times,sols] = ode45(@(t,w) threebody(t,w,G,m1,m2,m3),time_int,initial_w,options);
animate_planets(times,sols)
```

Here, T is the orbit period calculated in 5.2.

Upload your MATLAB code as a solution to this problem and include an image of the plot in your pdf upload.



[2 POINTS]