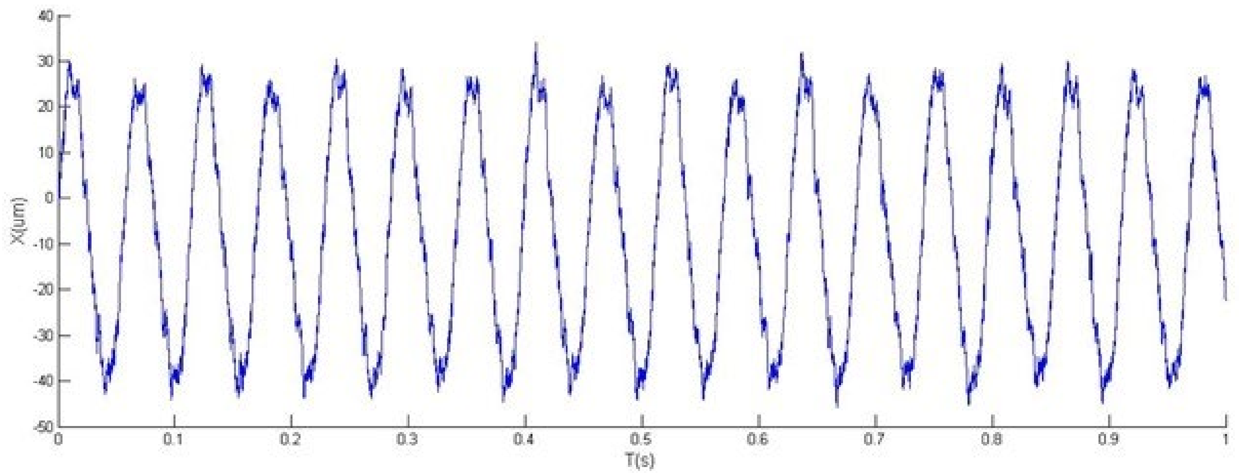
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Vibrations
Due Friday March 25 2022



1. The figure (from this publication) shows a displacement measurement from a test on the vibration characteristics of a magnetic bearing (the units for the displacement are microns).

1.1 The amplitude of the displacement

From the graph, $X_0 \approx (25+40)/2 = 32.5\mu m$

[1 POINT]

1.2 The period of the vibration

14 cycles takes 0.8 secs so T=0.057s.

[1 POINT]

1.3 The frequency (in Hertz) and angular frequency (in rad/s)

The frequency is 1/T=17.5Hz, or $\frac{2\pi}{0.057} = 110 rad/s$ rad/s

[1 POINT]

1.4 The amplitude of the velocity

The simple harmonic motion formulas give $V_0 = \omega X_0 \Rightarrow V_0 = 3.58 mm/s$

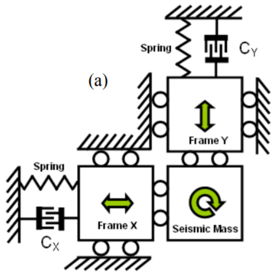
[1 POINT]

1.5 The amplitude of the acceleration

The simple harmonic motion formulas give $A_0 = \omega V_0 \Rightarrow A_0 = 393 mm/s^2$

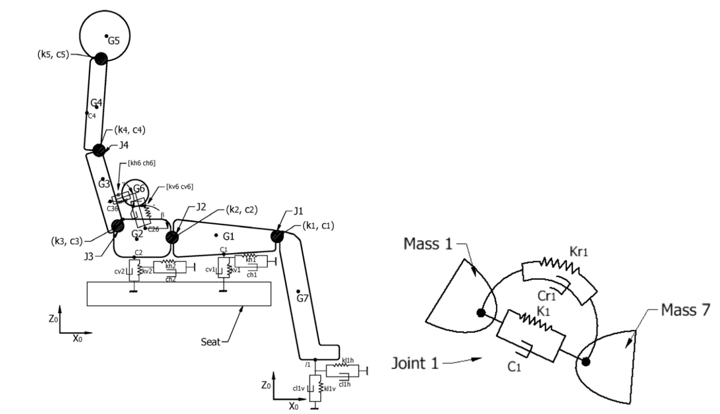
[1 POINT]

2. Find the number of degrees of freedom and vibration modes for each of the systems shown in the figures (you may need to consult the publications to understand the system)

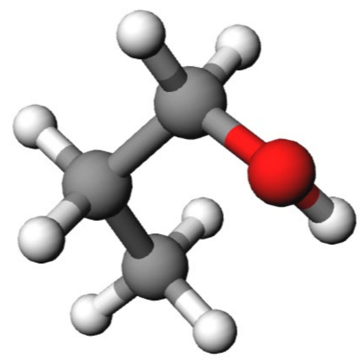


(a)2D model of an energy harvesting system

(b) Motion simulation platform (the joints connecting the platform to the members beneath it are spherical joints. The rest are all pin joints)



(c) Human body on a seat (masses 1,2,3,4,5 and 7 are rigid bodies; mass 6, representing the viscera, is a particle. Note that the joints are flexible, and permit relative rotation and motion)



(d) 1-Propanol molecule

For (a): the figure shows 3 masses that can be idealized as particles; the rollers introduce one constraint each, so 4 constraints. #DOF = 2p-c=2. If the masses are idealized as rigid bodies instead, the rollers introduce 3 constraints each (preventing rotation/relative rotation) so we get the same answer. The 2DOF are clearly horizontal and vertical motion of the 'Seismic mass.' The constraints prevent constant speed motion/rotation in all directions so no rigid body modes => #vibration modes = 2.

For (b) take everything including the black actuators to be outside the system. There are then 7 rigid bodies in the system. There are 6 pin joints with 5 constraints each, plus 3 spherical joints with 3 constraints each so c = 39. #DOF = 6r-c=3. Or we can read the title of the paper "Inverse Kinematics of a 3 DOF Parallel Manipulator: A Conformal Geometric Algebra Approach" The base is fixed, so there are no rigid body modes=> #vibration modes = 3

For (c) there are 6 rigid bodies and 1 particle. The joint all permit relative rotation and motion of the two points they connect, so there are no constraints. That gives #DOF=3x6 + 2x1 = 20. The joints don't allow translation or rotation at constant speed in any direction so there are no rigid body modes; # vibration modes = 20.

For (d) There are 12 atoms (all particles), which gives #DOF = 3*12=36; the molecule can translate in 3 direction and rotate about 3 axes at constant speed so 6 rigid body modes. # vibration modes = 30.

[8 POINTS]

3. Solve the following differential equations (please solve them by hand, using the tabulated solutions to differential equation – you can check the answers with matlab if you like)

3.1 $\frac{d^2y}{dt^2} + 144y = 24 \qquad y = 1 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

3.2 $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 25y = 0 \qquad y = 1 \qquad \frac{dy}{dt} = 0 \qquad t = 0$

3.3 $\frac{d^2y}{dt^2} + 20\frac{dy}{dt} + 100y = 200\sin 10t \qquad y = -2 \qquad \frac{dy}{dt} = 5\sqrt{3} \qquad t = 0$

3.1 Rearrange in standard form

$$\frac{1}{12^2}\frac{d^2y}{dt^2} + y = \frac{1}{6} \qquad y = 1 \qquad \frac{dy}{dt} = 0 \qquad t = 0$$

This is a Case I equation - compare with the standard form to see that $\omega_n = 12 \qquad C = 1/6$

The solution is

$$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$$

We are given $x_0 = 1 \quad v_0 = 0$ so

$$y(t) = \frac{1}{6}(1 + 5\cos 12t)$$

[3 POINTS]

3.2 Rearrange in standard form

$$\frac{1}{5^2}\frac{d^2y}{dt^2} + \frac{2\times 0.5}{5}\frac{dy}{dt} + y = 0 \qquad y = 1 \qquad \frac{dy}{dt} = 0 \qquad t = 0$$

This is a Case III equation - compare with the standard form to see that $\omega_n = 5, \ \zeta = 0.5 \ C = 0$. The solution is

$$x(t) = C + \exp(-\varsigma\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \varsigma\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$$

where $\omega_d = \omega_n\sqrt{1-\varsigma^2} = 5\sqrt{1-1/2^2} = 5\sqrt{3}/2$

$$y(t) = \exp(-2.5t)\left\{\cos\left(5\sqrt{3}t/2\right) + \frac{1}{\sqrt{3}}\sin\left(5\sqrt{3}t/2\right)\right\}$$

[3 POINTS]

3.3 $\dfrac{d^2y}{dt^2} + 20\dfrac{dy}{dt} + 100y = 200\sin 10t \qquad y = -2 \qquad \dfrac{dy}{dt} = 5\sqrt{3} \qquad t = 0$

We can rearrange this as a Case 4 equation

$$\frac{1}{10^2}\frac{d^2y}{dt^2} + \frac{2\times 1}{10}\frac{dy}{dt} + y = 2\sin 10t$$

It appears that $\omega = 10,\ KF_0 = 2,\ \omega_n = 10,\ \zeta = 1\ C = 0$.

The steady-state solution follows as

$$x_p(t) = X_0\sin\left(\omega t + \phi\right) \qquad X_0 = KF_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n,\zeta) = \frac{1}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi = \tan^{-1}\left(\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \quad (-\pi < \phi < 0)$$

The homogeneous solution is

$$x_h(t) = \left\{x_0^h + \left[v_0^h + \omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

where

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0\sin\phi$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0\omega\cos\phi$$

Substituting numbers gives

$$M(\omega/\omega_n,\zeta) = \frac{1}{\left\{0+(2)^2\right\}^{1/2}} = \frac{1}{2} \qquad \phi = \tan^{-1}\left(\infty\right) \quad = -\pi/2$$

$$X_0 = 1$$

$$x_0^h = -1 \qquad v_0^h = 5\sqrt{3}$$

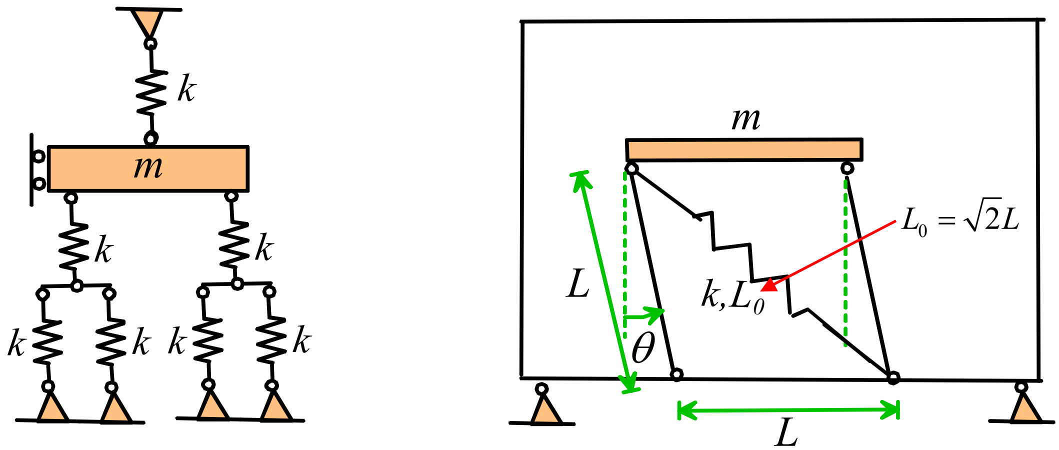
The total solution is therefore

$$y(t) = \left\{-1 + \left[5\sqrt{3} - 10\right]t\right\}\exp(-10t) + \sin(10t - \pi/2) = \left\{-1 + \left[5\sqrt{3} - 10\right]t\right\}\exp(-10t) - \cos(10t)$$

We can check that this is correct by substituting it into the differential equation, and by substituting *t=0* into *y* and *dy/dt* and checking that initial conditions are satisfied.

[3 POINTS]

4. Find formulas for the natural frequency of vibration for the systems shown in the figure. For the system on the right, the unstretched length of the spring $L_0 = \sqrt{2}L$



For the first system, we can replace the springs with an equivalent single spring. On the bottom we have two sets of two springs in parallel, which together are in series with a single spring. The effective stiffness of this combination is

$$\frac{1}{k_{eff}} = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{2k}\right) \Rightarrow k_{eff} = 4k/3$$

This equivalent spring is in parallel with a spring on top, so the effective stiffness of the entire assembly is $7k/3$. The formula for natural frequency gives $\omega = \sqrt{7k/(3m)}$

[2 POINTS]

We can get an EOM for the second system using the energy method. The platform is in circular motion, so its speed (from the circular motion formula) is

$$v = L\left(\frac{d\theta}{dt}\right)$$

and therefore the KE is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$

The PE includes gravity and the energy of the springs. Geometry shows that the spring length is $2L\sin(\pi/4 + \theta/2)$ (to see this draw a perpendicular to the spring through the pivot at the base of the left most support, which bisects and angle $\pi/2 + \theta$; then use Pythagoras on the right angle triangle. You can also use the law of cosines, but that gives a rather more messy formula) so

$$U = mgL\cos\theta + \frac{1}{2}k\left(2L\sin(\pi/4 + \theta/2) - \sqrt{2}L\right)^2$$

$$T+U=const \Rightarrow \frac{d}{dt}(T+U)=0$$

$$mL^2\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right)-mgL\sin\theta\frac{d\theta}{dt}+kL\cos\left(\pi/4+\theta/2\right)\left(2L\sin(\pi/4+\theta/2)-\sqrt{2}L\right)\frac{d\theta}{dt}=0$$

$$mL^2\left(\frac{d^2\theta}{dt^2}\right)-mgL\sin\theta+kL^2\sin(\pi/2+\theta)-\sqrt{2}kL^2\cos(\pi/4+\theta/2)=0$$

(here we used the formula $2\sin\theta\cos\theta=\sin 2\theta$ to make finding the small angle approximation easier but its fine to leave this term as just $2\sin\theta\cos\theta$)

To linearize the equation can use $\sin\theta\approx\theta$ and then expand the other two trig terms with Taylor series

$$\sin(\pi/2+\theta)\approx\sin(\pi/2)+\cos\left(\pi/2\right)\theta\approx 1$$

$$\cos(\pi/4+\theta/2)\approx\cos(\pi/4)-\sin(\pi/4)\theta/2\approx\frac{1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\theta$$

, which gives

$$mL^2\left(\frac{d^2\theta}{dt^2}\right)-mgL\theta+kL^2-\sqrt{2}kL^2\left(\frac{1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\theta\right)=0$$

$$\Rightarrow\frac{2mL^2}{\left(kL^2-2mgL\right)}\left(\frac{d^2\theta}{dt^2}\right)+\theta=0$$

and compare to the standard case I EOM to see that

$$\omega_n=\sqrt{\frac{kL^2-2mgL}{2mL^2}}$$

Remarks:

(1) This arrangement is sometimes used to design a vibration isolation system. You can tune the spring stiffness to make the natural frequency as low as you like.

(2) It's worth thinking about what happens if $2mgL>kL^2$. If that happens our formula predicts that the natural frequency is complex – what does that mean, exactly? We can figure this out in two different ways. One idea is to recognize that our equation of motion

$$2mL^2\left(\frac{d^2\theta}{dt^2}\right)+\left(kL^2-2mgL\right)\theta=0$$

has a problem in this limit, because the coefficient of θ is negative. So it's not a 'Case I' equation. But we can turn it into a 'Case II' equation if we want

$$2mL^2\left(\frac{d^2\theta}{dt^2}\right)-\left(2mgL-kL^2\right)\theta=0$$

$$\frac{2mL^2}{\left(mgL-kL^2\right)}\left(\frac{d^2\theta}{dt^2}\right)-\theta=0$$

Which has the form

$$\frac{1}{\alpha^2}\left(\frac{d^2\theta}{dt^2}\right)-\theta=0$$

The solution this equation has the form $\theta = Ae^{\alpha t} + Be^{-\alpha t}$, where A, B are constants (they depend on the initial conditions). So the math tells us that this system won't vibrate, but instead θ will increase exponentially. This looks a bit weird too, because that means the platform would zip around in a circle at progressively increasing speed, but of course our EOM is approximate, and only accurate when θ is small. So what we are really learning is that the system is unstable, and will collapse, if the springs are too soft. That makes sense.

There's another way to look at the math if you happen to be familiar with trig functions of complex numbers. Math will actually let us work with complex valued natural frequencies – if we have an equation of the form

$$\frac{1}{\omega_n^2}\left(\frac{d^2\theta}{dt^2}\right)+\theta=0$$

where $\omega_n = i\alpha$ is a complex number, we can use the 'Case I'solution $\theta = A\sin i\alpha t + B\cos i\alpha t$. Euler's formula for complex numbers tells us that trig functions of complex numbers are actually exponentials – for example

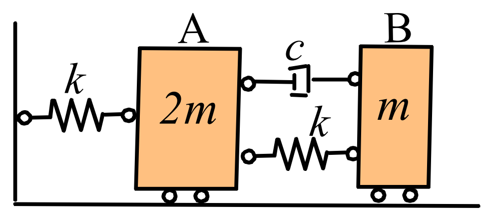
$$e^{i\beta} = \cos\beta + i\sin\beta \Rightarrow \cos\beta = (e^{i\beta t} + e^{-i\beta t})/2$$

$$\cos i\alpha t = (e^{-\alpha t} + e^{\alpha t})/2$$

So again, the math is telling us that the motion is not harmonic vibrations any more, but exponential growth, i.e. collapse.

[3 POINTS]

5. When mass A is held fixed and mass B vibrates, the system shown in the figure has natural frequency ω_n and damping factor ζ. Find the natural frequency and damping factor when mass B is held fixed and mass A vibrates.



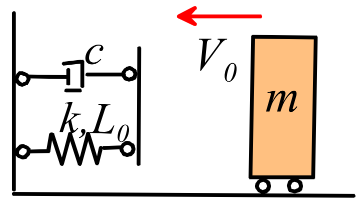
For A fixed/B vibrating we know that $\omega_{nB} = \sqrt{k/m}$ $\qquad \zeta_B = c/(2\sqrt{km})$

For B fixed/A vibrating the two springs are in parallel and have effective stiffness 2k. Therefore

$$\omega_{nA} = \sqrt{\frac{2k}{2m}} = \omega_{nB} \qquad \zeta_A = \frac{c}{2\sqrt{2k.2m}} = \zeta_B/2$$

[3 POINTS]

6. A shock absorber (consisting of a spring and dashpot) is to be designed for installation at the end of a linear conveyor. Its purpose is to bring packages to rest at the end of the moving conveyor. It must meet the following specifications:



(i) Packages have mass 5kg and strike the shock absorber at speed 2 m/s
(ii) The maximum acceleration of the package must not exceed 2g
(iii) The shock absorber should recover to its stretched length as quickly as possible, and the package should not rebound off the absorber.

6.1 Find the spring stiffness and dashpot coefficient that will meet the specification.

Condition (iii) means the spring-mass system should be critically damped. Its deflection (i.e. the change in position of the mass after it just strikes the shock absorber) as a function of time is therefore

$$x(t) = C + \left\{(x_0 - C) + \left[v_0 + \omega_n (x_0 - C)\right] t\right\} \exp(-\omega_n t)$$

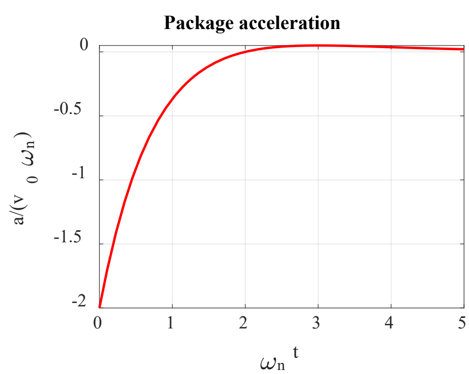
The EOM for x has $C=0$ and $x_0 = 0$, which gives

$$x(t) = v_0 t \exp(-\omega_n t)$$

The acceleration follows as

$$a(t) = \frac{d^2 x}{dt^2} = -v_0 \omega_n (2 - \omega_n t) \exp(-\omega_n t)$$

The maximum acceleration occurs at time $t=0$ (to see this, plot the acceleration as a function of time, as shown below)



The maximum acceleration is therefore $\left|-2v_0\omega_n\right|$, so to meet the second constraint we require $\omega_n < 2g/(2v_0) = 9.81/2$. To make the absorber rebound as quickly as possible we need the highest possible value of ω_n.

We can now calculate the spring stiffness and dashpot coefficient (noting that the system is critically damped)

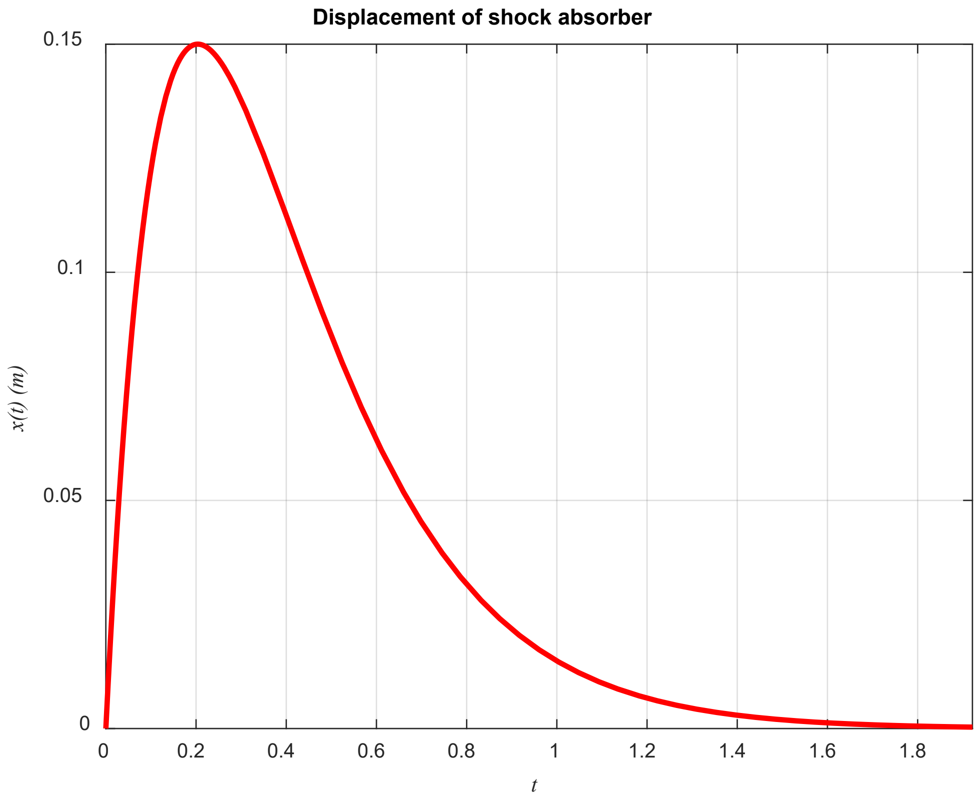
$$k = m\omega_n^2 = 120 N/m$$

$$c = 2\zeta\sqrt{km} = 49 Ns/m$$

[4 POINTS]

6.2 Find the maximum deflection of the absorber after it is struck, and the time required for the shock absorber's deflection to recover to below 1% of the max deflection.

It helps to plot the deflection as a function of time



The maximum deflection occurs when

$$dx/dt = 0 \Rightarrow v_0(1 - t\omega_n)\exp(-\omega_n t) = 0 \Rightarrow t = 1/\omega_n$$

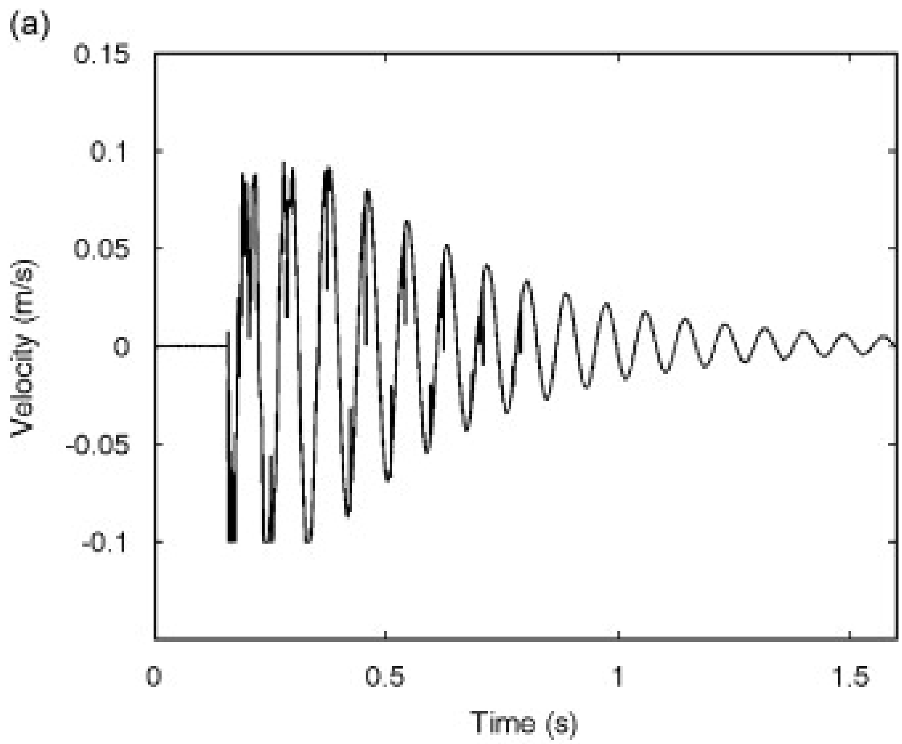
$$\Rightarrow x_{\max} = (v_0/\omega_n)\exp(-1) = 15cm$$

To find the time needed to return to 1% of the max deflection, solve

$$x = v_0 t \exp(-\omega_n t) = 0.01 x_{\max}$$

using MATLAB vpasolve (you have to specify the range for the root of interest, because there are two solutions – we want the higher of the two values). This gives t=1.5sec.

[3 POINTS]



7. The figure (from this publication) shows the measured velocity at the tip of a cantilever beam as a function of time.

7.1 Find the period and log decrement of the signal

Picking the fourth peak at (*t=0.458s, v=0.07* m/s) and 10th peak at (*t=0.977, v=0.0209* m/s) we find the period T=(0.977-0.458)/6=0.0865s, and the log decrement $\delta = \frac{1}{6}\log(0.0768/0.0209) = 0.2169$

[2 POINTS]

7.2 Hence, calculate the natural frequency and damping factor for the beam.

Using the formulas

$$\omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T} = 72.7 rad/s$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.0345$$

[2 POINTS]