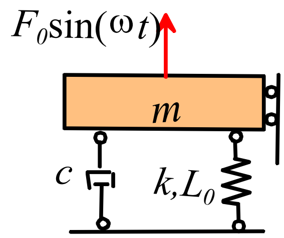
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Vibrations
Due Friday April 8 2022

1. The spring-mass system shown is a simple idealization of a building subjected to wind loading. The mass is 250x10^6 kg, spring stiffness is 2.5 GN/m and the dashpot coefficient is 160MNs/m. The building is observed to vibrate at a frequency of 1Hz with an amplitude of 3mm. Find the amplitude of the force.



We can use the formulas. Start by calculating the natural frequency and damping ratio

$$\omega_n = \sqrt{k/m} = 3.16$$

$$\zeta = c/(2\sqrt{km}) = 0.1012$$

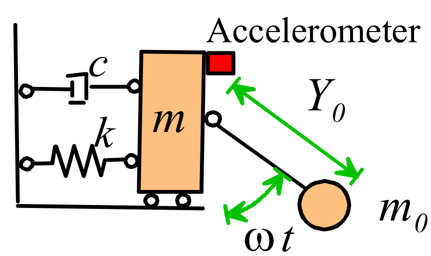
Then find the magnification (don't forget to use $\omega = 2\pi f$ with $f=1$Hz)

$$M = \frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}} = 0.336$$

And finally use the formula for vibration amplitude $X_0 = KMF_0$ with K=1/k. This gives $F_0 = 22MN$.

[3 POINTS]

2. An unbalanced rotating fan is mounted on isolation pads giving a spring mass system with natural frequency $\omega_n = 10$ rad/s, damping factor $\zeta = 0.2$, total mass $m + m_0 = 10kg$, and mass unbalance $Y_0 m_0 = 0.1kgm$. The system is at rest for time $t<0$. The fan is then turned on and spins with angular speed $20rad/s$. Plot a graph showing the acceleration of the mass as a function of time, for a time interval 0<t<4s. (Be careful when calculating the phase of the steady state response, the phase must be in the range $-\pi < \phi < 0$. When $\omega > \omega_n$ the phase should be less than $-\pi/2$). It is best to do the calculation and plot in MATLAB. You need not submit your MATLAB code, however – just explain how you did the calculation and submit your plot.



This is just a question of using the formulas to calculate the steady state and transient solutions. The solution is

$$x(t) = C + x_h(t) + x_p(t)$$

where the steady state solution is

$$x_p(t) = X_0 \sin(\omega t + \phi) \qquad X_0 = KY_0 M(\omega/\omega_n, \zeta)$$

$$M(\omega/\omega_n, \zeta) = \frac{\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi = \tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2} \qquad (-\pi < \phi < 0)$$

And (since the system is underdamped) the transient solution is

$$x_h(t) = \exp(-\varsigma\omega_n t)\left\{x_0^h \cos\omega_d t + \frac{v_0^h + \varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

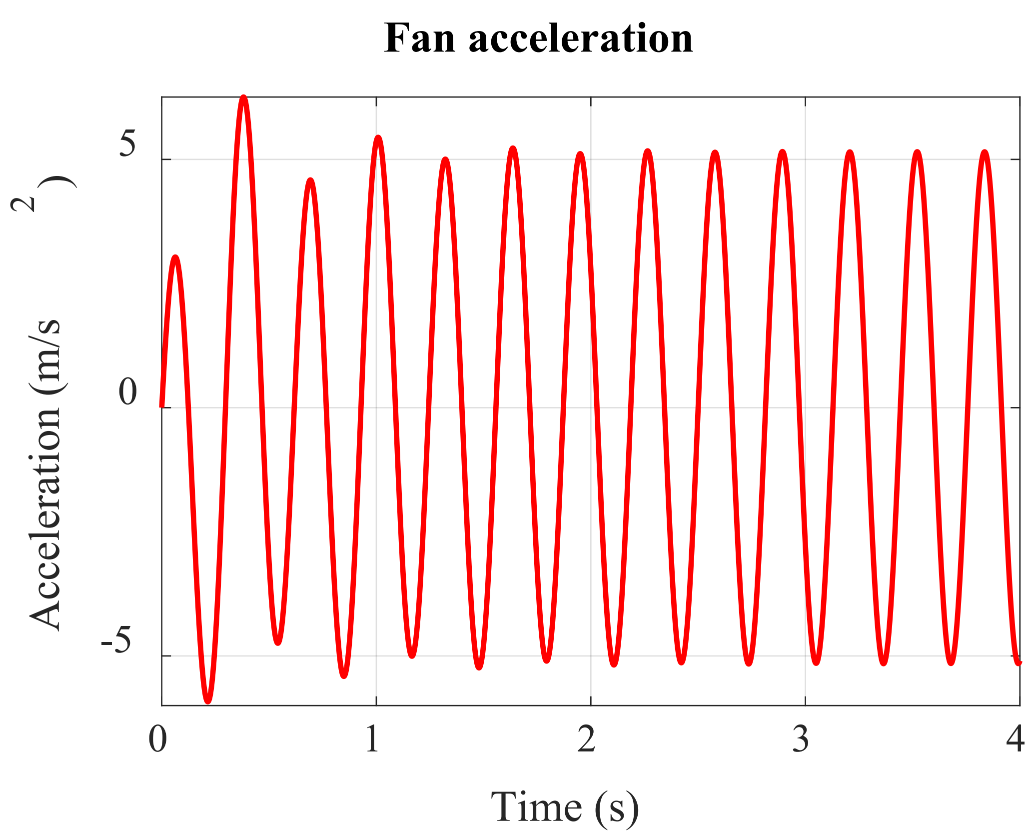
where $\omega_d = \omega_n\sqrt{1-\varsigma^2}$

In all three preceding cases, we have set

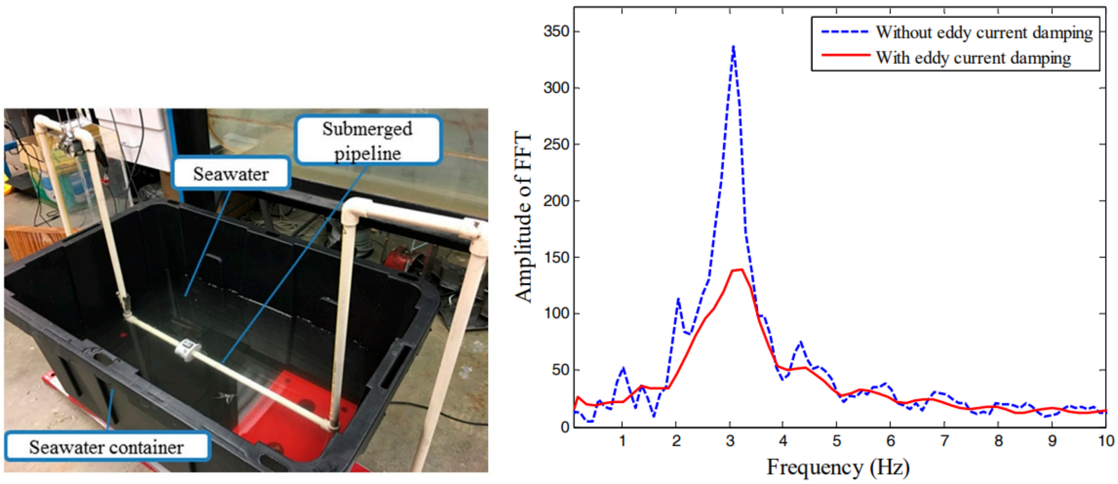
$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin\phi$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0\omega\cos\phi$$

with $C = 0,\ x_0 = 0,\ v_0 = 0$. Once the displacement has been found, the acceleration follows by differentiating the displacement with respect to time twice….
The plot is shown below.



[5 POINTS]



3. The figure (from this publication) shows the measured vibration amplitude of a submerged pipeline with, and without, and eddy current damper installed, as a function of frequency.

3.1 What (approximately) is the undamped natural frequency of vibration of the system ω_n ?

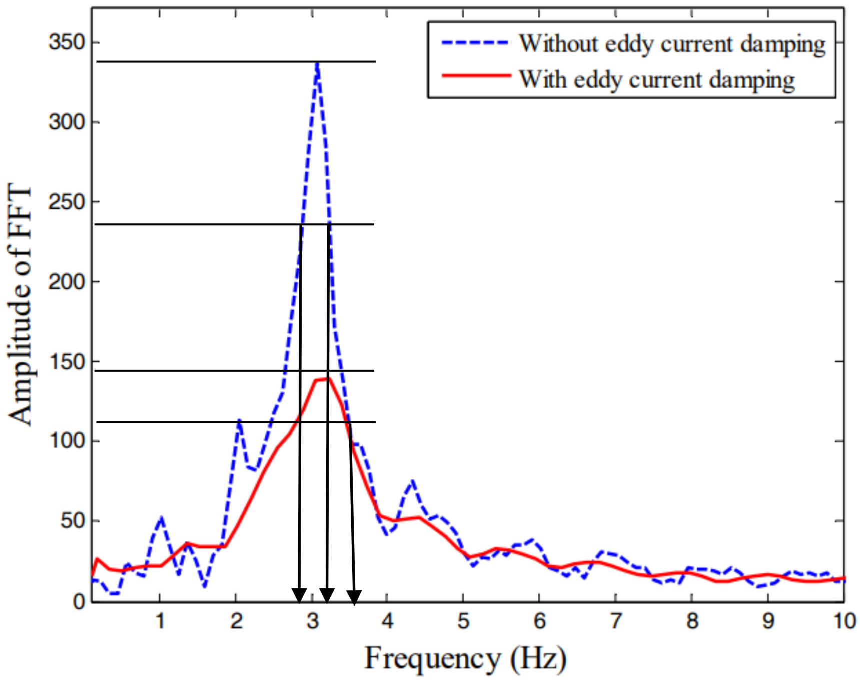
The frequency is the peak of the curve – the undamped curve will give a more accurate measure of the undamped natural frequency – which is about 3Hz, or 6π rad/s.

[1 POINT]

3.2 What is the damping factor ζ , before and after the damper is added?

We can use the recipe – divide the peak by $\sqrt{2}$ and find the bandwidth (the difference between the two frequencies where the amplitude is equal to $X_{\max} / \sqrt{2}$), find the Q factor $Q = \omega_{\max} / (\omega_2 - \omega_1)$ and the damping factor follows as $1/(2Q)$.

The figure shows this construction – for the undamped system $Q \approx 3/0.38 = 7.9 \Rightarrow \zeta \approx 0.06$
For the damped system $Q \approx 3.1/0.8 = 3.8 \Rightarrow \zeta \approx 0.13$

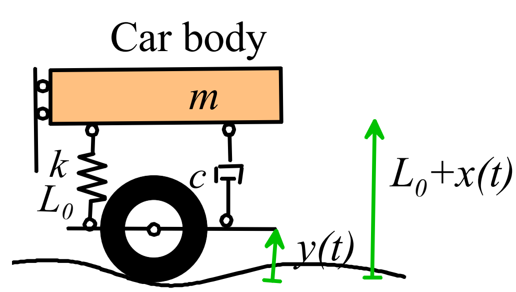


[3 POINTS]

4. The re-designed vehicle suspension in [Section 5.6.9 of the lecture notes] had the following specification:

- Natural frequency $\omega_n = 5\pi / 2.1$ rad/s
- Damping factor $\zeta = 0.38$

Suppose that the car drives over a road with roughness amplitude 5cm and wavelength 8m.



4.1 Show that if the mass of the wheel can be neglected, the normal force exerted by the wheel on the ground is given by

$$N = kY_0 \left[\frac{g}{Y_0 \omega_n^2} - M \frac{\omega^2}{\omega_n^2} \sin(\omega t + \phi) \right]$$

where Y_0 is the amplitude of the roughness, $\omega = 2\pi V / L$ is the excitation frequency, ω_n is the natural frequency of the suspension system and M is the magnification.

Newton's law for the wheel and car body as a combined system gives

$$N - mg = m \frac{d^2 x}{dt^2}$$

We know that $x = X_0 \sin(\omega t + \phi)$, where $X_0 = KY_0 M$ (and *K=1* for the base excited system). Hence

$$N = mg - m\omega^2 X_0 \sin(\omega t + \phi) = kY_0 \left[\frac{g}{Y_0 k / m} - \frac{\omega^2}{k/m} M \sin(\omega t + \phi) \right]$$

which rearranges to the result given

[3 POINTS]

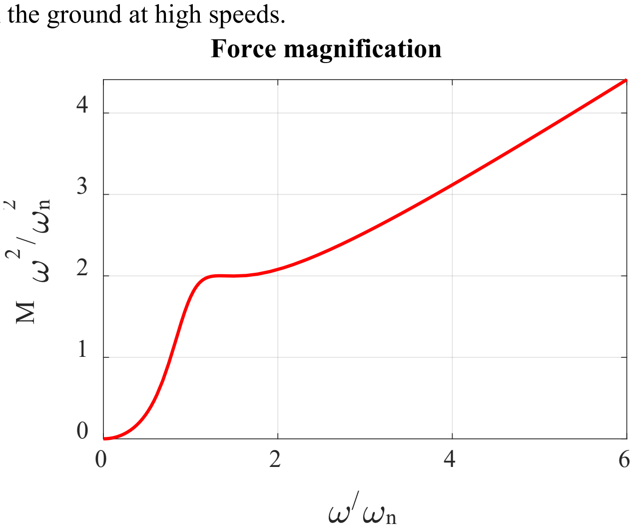
4.2 Hence, find the car speed where the wheel will lose contact with the ground. You might find it helpful to plot a graph of $(\omega / \omega_n)^2 M$ as a function of ω / ω_n

Substituting numbers, we find that $g / Y_0 \omega_n^2 = 3.5067$
The car loses contact with the ground if *N<0*. The minimum value of *N* occurs when $\sin(\omega t + \phi) = 1$, so the car loses contact with the ground if $(\omega / \omega_n)^2 M > 3.5067$

Recall that

$$M = \frac{\sqrt{1 + (2\zeta\omega / \omega_n)^2}}{\sqrt{\left\{ (1 - \omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2 \right\}}}$$

A plot of $(\omega / \omega_n)^2 M$ as a function of ω / ω_n for $\zeta = 0.38$ is shown below, which shows that the car will lose contact with the ground at high speeds.



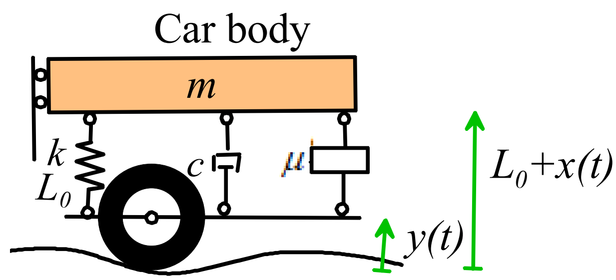
MATLAB solves the equation $(\omega / \omega_n)^2 M = 3.5067$ without trouble, giving $\omega / \omega_n = 4.2351$.
So $\omega > 31.68$ rad/s. The car speed follows as $V = \omega L / (2\pi) = 40.33 m / s$

[3 POINTS]

5. Formula 1 racecars have been using 'inerters' in their suspensions for over 10 years. An 'inerter' is a mechanical element that, like a spring or dashpot, can be stretched by a force. In an inerter, the force is proportional to the relative acceleration of its ends (the second time derivative of its length)

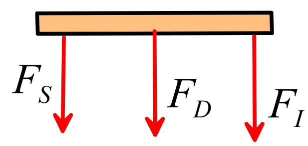
$$F_I = \mu \frac{d^2 L}{dt^2}$$

The goal of this problem is to investigate the effects of adding an inerter to a suspension.



Before attempting this problem you might find it helpful to review the analysis of a conventional spring-mass-damper suspension system discussed in class and in the e-notes.

5.1 Draw a free body diagram showing the forces acting on the mass (the car body). Neglect gravity.



[2 POINTS]

5.2 Hence, show that the displacement of the wheel y, and the displacement of the mass are related by an equation of the form

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = \frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y$$

and give formulas for ω_n, λ, ζ in terms of c, k, μ

Newton's law, together with the formulas for forces in springs, dampers and inerters, gives

$$m\frac{d^2(x+L_0)}{dt^2} = -\mu\frac{d^2(x+L_0-y)}{dt^2} - c\frac{d(x+L_0-y)}{dt} - k(x-y)$$

Rearranging and noting that L_0 is a constant

$$\frac{(m+\mu)}{k}\frac{d^2x}{dt^2} + \frac{c}{k}\frac{dx}{dt} + x = \frac{\mu}{k}\frac{d^2y}{dt^2} + \frac{c}{k}\frac{dy}{dt} + y$$

Defining

$$\omega_n = \sqrt{\frac{k}{m+\mu}} \qquad \zeta = \frac{c}{2\sqrt{k(m+\mu)}} \qquad \lambda = \sqrt{\frac{\mu}{m+\mu}}$$

This reduces to

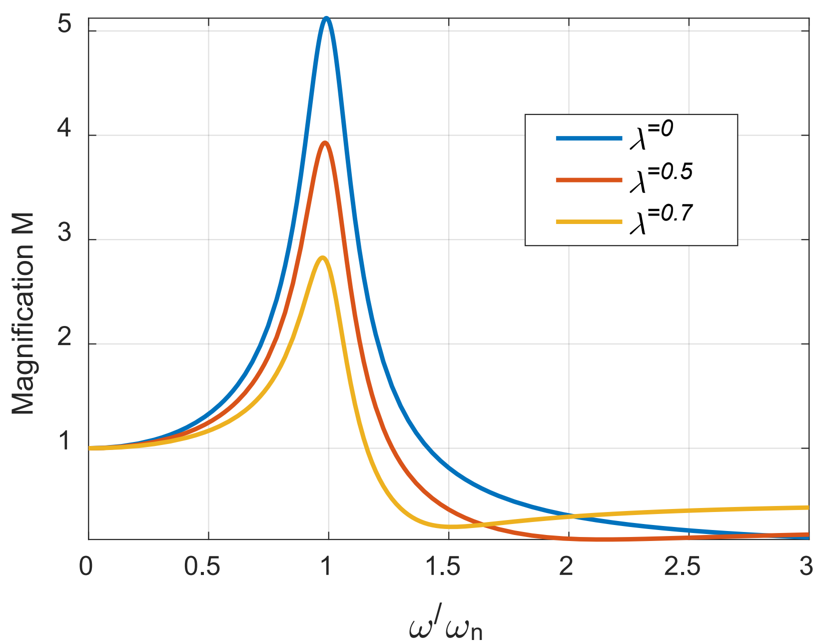
$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = \frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y$$

[3 POINTS]

5.3 The solution to the EOM in part 4.2 can be found in the tabulated solutions to differential equation for common vibration problems. Note that (like the conventional suspension system) the amplitude of vibration of the car body is given by

$$X_0 = KM\left(\omega/\omega_n, \zeta, \lambda\right)Y_0$$

where K=1. To understand the effects of the inerter, plot (on the same axes) a graph of the magnification M as a function of ω/ω_n for $0<\omega/\omega_n<3$, $\zeta=0.1$, and $\lambda=0$ (a conventional suspension), $\lambda=0.5$, and $\lambda=0.7$ (representing the addition of a weak and a strong inerter). You will find that the inerter gives the suspension a so-called 'anti-resonance' – a magic frequency where M has a minimum value.



[3 POINTS]

5.4 What is the frequency corresponding to the anti-resonance (the minimum value of M), in terms of λ, ω_n (give an approximate solution for $\zeta \ll 1$)? What is (approximately) the smallest vibration amplitude (in terms of λ, ζ)?

The magnification (from the handout) is $M(\omega/\omega_n, \zeta) = \dfrac{\left\{\left(1-\lambda^2\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}$

The minimum will occur (approximately) when $\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)=0 \Rightarrow \omega=\omega_n/\lambda$. The corresponding vibration amplitude is

$$M=\frac{\left(\frac{2\zeta}{\lambda}\right)}{\sqrt{\left(1-\frac{1}{\lambda^2}\right)^2+\left(\frac{2\zeta}{\lambda}\right)^2}}$$

If λ is not close to 1, then

$$M\approx\frac{2\zeta}{\left|\left(\lambda-\frac{1}{\lambda}\right)\right|}$$

[2 POINTS]

5.5 For what range of frequency (in terms of λ, ω_n) does the antiresonant system give better performance than the simpler spring-mass-damper system?

The magnification for the anti-resonant isolator is equal to that of the conventional system when

$$\sqrt{\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}=\sqrt{1+\left(\frac{2\zeta\omega}{\omega_n}\right)^2}$$

This gives

$$\left(1-\frac{\lambda^2\omega^2}{\omega_n^2}\right)^2=1$$

$$\Rightarrow \omega=0 \qquad \text{or} \qquad \frac{\lambda^2\omega^2}{\omega_n^2}=2\Rightarrow\omega=\frac{\sqrt{2}}{\lambda}\omega_n$$

The anti-resonant system is better than the conventional system for ω below this value.

[2 POINTS]

5.6 Re-visit the suspension system that was re-designed in Section 5.6.9 of the lecture notes. Re-design the system (with an inerter) to meet the following specifications:

(1) The vehicle is to be designed to drive over a roadway with roughness wavelength 10m, and amplitude is 20cm.
(2) The suspension should give the minimum vibration amplitude of the car body at 50 mph
(3) The suspension should be effective (i.e. the car should vibrate with amplitude less than that of the roadway) for all vehicle speeds exceeding 20 mph
(4) The amplitude of vibration of the car's body should not exceed 35cm at any speed
(5) The static deflection of the suspension spring should be as small as possible, while still satisfying (1-4)

For your design, recommend values of ω_n, ζ, λ , and k, c, μ . You can use small ζ approximations to simplify calculations (it's possible to get a solution without this approximation in MATLAB but it's quite tricky to set up –

try it if you would like a challenge. It changes the numbers you get a bit, but doesn't have an appreciable effect on the performance of the design at the end).

Constraint (1) (from the notes) tells us that the vibration frequency of the vehicle is $\omega = 2\pi V / L$, which corresponds to $\omega_{50} = 14.04\, rad/s$ at 50 mph and $\omega_{20} = 5.62\, rad/s$ at 10 mph

Constraint (2) requires (roughly) that $\omega_{50} = \omega_n / \lambda$

Constraint (3) requires that $M < 1$ for $\omega > \omega_{10}$. This requires

$$1 > \frac{\left\{ \left(1-\lambda^2 \omega_{20}^2 / \omega_n^2\right)^2 + \left(2\varsigma\omega_{20} / \omega_n\right)^2 \right\}^{1/2}}{\left\{ \left(1-\omega_{20}^2 / \omega_n^2\right)^2 + \left(2\varsigma\omega_{20} / \omega_n\right)^2 \right\}^{1/2}}$$

$$\Rightarrow \left(1-\lambda^2 \omega_{20}^2 / \omega_n^2\right)^2 + \left(2\varsigma\omega_{20} / \omega_n\right)^2 < \left(1-\omega_{20}^2 / \omega_n^2\right)^2 + \left(2\varsigma\omega_{20} / \omega_n\right)^2$$

$$\Rightarrow \left(1-\omega_{20}^2 / \omega_{50}^2\right)^2 < \left(1-\omega_{20}^2 / \omega_n^2\right)^2$$

$$\Rightarrow 1-\omega_{20}^2 / \omega_{50}^2 < \omega_{20}^2 / \omega_n^2 - 1$$

$$\Rightarrow \omega_{20}^2 / \omega_n^2 > 2 - \frac{20^2}{50^2}$$

$$\Rightarrow \omega_n < \frac{\omega_{20}}{\sqrt{2-\frac{4}{25}}}$$

Since we want to minimize the static deflection of the spring, we want the highest possible spring stiffness, and therefore want to choose the highest possible value of ω_n. This gives

$$\omega_n = \omega_{20} / \sqrt{(2-4/25)} = 4.14\, rad/s$$

$$\lambda = \omega_n / \omega_{50} = 0.295$$

Finally, we can estimate ζ by assuming the maximum value of M occurs at $\omega = \omega_n$, giving

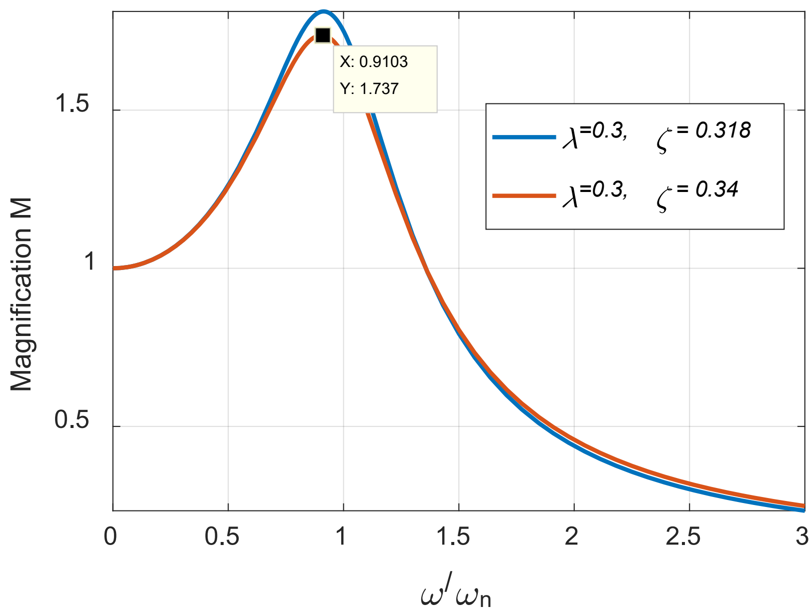
$$M_{\max} = \frac{\left\{ \left(1-\lambda^2\right)^2 + \left(2\varsigma\right)^2 \right\}^{1/2}}{\left(2\varsigma\right)}$$

Constraint (4) requires $M_{\max} < 35/20$. We want to select the smallest allowable value of ζ to make the suspension as effective as possible at speeds above 20 mph. Therefore

$$1.75^2 \left(2\varsigma\right)^2 = \left(1-\lambda^2\right)^2 + \left(2\varsigma\right)^2$$

$$\Rightarrow \zeta = \frac{\left(1-\lambda^2\right)}{2\sqrt{1.75^2 - 1}} = 0.32$$

We can check our design by plotting the magnification as a function of frequency



Taking $\zeta = 0.318$ slightly exceeds the allowable value of M (this is because we used an approximation for very small damping to estimate it, and it turned out the damping is not as small as we hoped). Increasing ζ to 0.34 fixes the problem.

You don't have to do this graphically, of course – if you want a really accurate answer you can have MATLAB maximize M for you and then solve for the necessary value of ζ - but this is quite complicated. Here's a solution that works with my MATLAB in 2020 – you might have to tweak it if the roots of the equations come out in a different order

```
% 'Exact' calculation
syms lambda zeta omega_n
wwatmaxormin = solve(diff(M(ww,lambda,zeta))==0,ww) % Find stationary points
M1 = simplify(M(wwatmaxormin(2),lambda,zeta)) % Value of M at first stationary point with positive value
M2 = simplify(M(wwatmaxormin(4),lambda,zeta)) % Value of M at 2nd stationary point with positive value
simplify(subs(M1^2-M2^2,[zeta,lambda],[0.1,0.1])) % Check whether M1 or M2 is larger with some sensible numbers
eq = M(omega20/omega_n,lambda,zeta)^2==1  % The equation stating M=1 at 20 mph
omeganfunc = solve(eq,omega_n) % We can solve this for omega_n in terms of zeta and lambda
eq1 = wwatmaxormin(2)==omega50/omeganfunc(2) % The equation saying the minimum occurs at 50 mph
zetafunc = solve(eq1,zeta) % We can solve the second equation for zeta in terms of lambda
eq2 = subs(M2,zeta,zetafunc(2))==1.75 % The equation saying the max value of M is 1.75
lambdaval = vpasolve(eq2,lambda,0.3) % We con solve this for lambda
zetaval = subs(zetafunc(2),lambda,lambdaval) % Substitute back for zeta
omeganval = subs(omeganfunc(2),[lambda,zeta],[lambdaval,zetaval]) % Substitute back for omega_n
```

lambdaval = $0.55165488318169340495128576885578$

zetaval = $0.26203112855770118043634923980 78$

omeganval = $1.4440557911519642675399390762221\,\pi$

[5 POINTS]

Finally we can calculate values for the properties of the suspension elements. Here's a matlab solution (but it's easy to do by hand as well)

```
syms mu m k c
lbtokg = 0.45359;
m = 3000*lbtokg;
eq1 = lambdaval == sqrt(mu/(m+mu));
muval = double(solve(eq1,mu))
eq2 = omeganval == sqrt(k/(m+muval));
kval = solve(eq2,k)
eq3 = zetaval == c/(2*sqrt(kval*(m+muval)));
cval = solve(eq3,c)
staticdef = m*lbtokg*9.81/kval
```

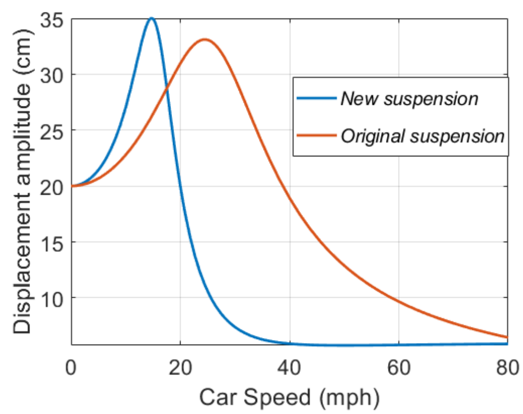
```
muval = 595.2674

kval = 40257.318175249441297809295656966

cval = 4650.4382051281526848514926602592

staticdef = 0.15040849468471784637792129389299
```

Graders – people may get different numbers if they use different values for λ, ζ, ω_n so please check the method used rather than the answer…

[2 POINTS]

5.7 Compare your new design to the one in the notes by plotting graphs (on the same figure) of the predicted vibration amplitude for the two designs (use the system designed at the end of Section 5.6.9 as the 'original' design, and plot your new design on the same graph) as a function of car speed in the range $0<V<80$ mph.

The performance of the system (with the 'exact' parameters) is compared to the old design below

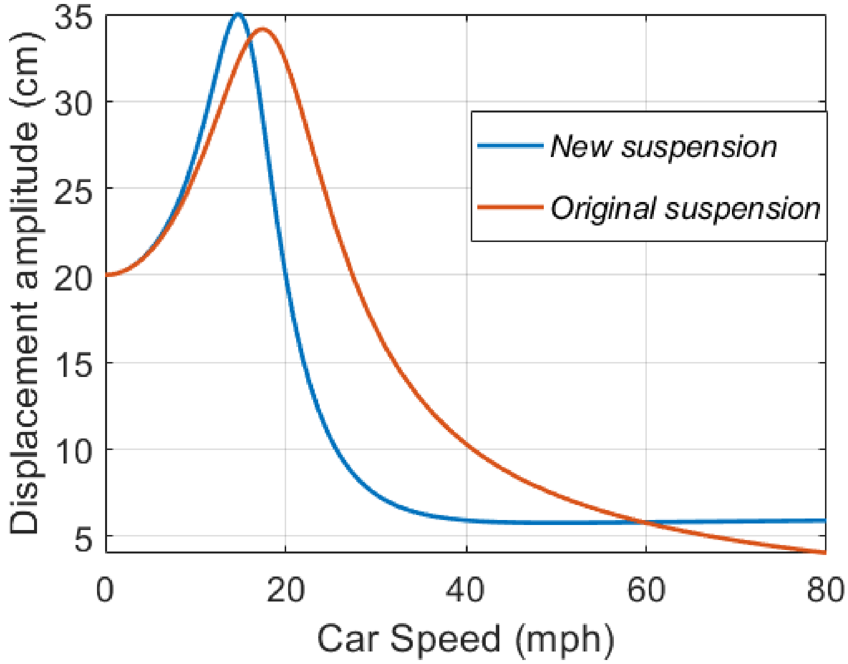


[3 POINTS]

A few final remarks:

(1) The graph gives the impression that the inerter greatly improves the performance of the suspension. This is partly true, but the comparison is a bit misleading because the two designs have different natural frequencies (they were designed with different constraints). The original suspension could be improved if its natural frequency were smaller. A better comparison is to use the same spring stiffness in both designs (or to use the same natural frequency – but the original suspension would need softer spring than the new

design, which is undesirable). The graph below compares a conventional suspension with the same stiffness to the suspension with an inerter. The inerter-based design is better than the original over a range of frequencies, but worse at high frequency.



(2) The static deflection of the suspension is 15cm – this is quite large.

(3) There is another consideration in suspension design that was not addressed here – the 'road holding' performance. We could do a rough comparison of road holding between the two suspensions by following the procedure in problem 4 – which tells us that the new suspension will be better than the old one as long as its magnification is lower. So the suspension will improve road holding below 60mph. But to analyze road-holding properly, we really need to consider the mass of the wheel, which turns the suspension into a 2DOF system, which is beyond the scope of engn40.