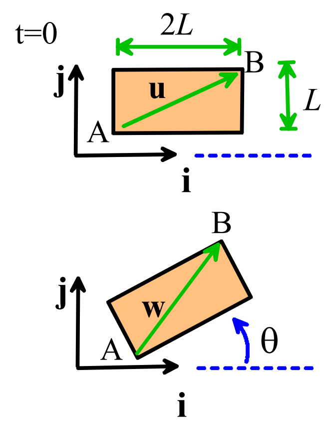
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 7: Rigid Body Kinematics, Inertial properties of rigid bodies
Due Friday April 22 2022



1 The rectangle shown in the figure starts with $\theta = 0$ at time $t=0$ and rotates with an angular velocity vector $\boldsymbol{\omega} = 4t^3\mathbf{k}$. At a time $t=2$sec

1.1 Find the angular acceleration vector

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = 12t^2\mathbf{k} = 48\mathbf{k} rad/s^2$$

[1 POINT]

1.2 Find the rotation matrix

The angle follows as $\theta = \int_0^t 4t^3 dt = t^4 = 16 rad$

Then

$$\mathbf{R} = \begin{bmatrix} \cos t^4 & -\sin t^4 & 0 \\ \sin t^4 & \cos t^4 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 16 & -\sin 16 & 0 \\ \sin 16 & \cos 16 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.9577 & 0.2879 & 0 \\ -0.2879 & -0.9577 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[2 POINTS]

1.3 Find the spin tensor

The spin tensor can be found from the angular velocity,

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -32 & 0 \\ 32 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Alternatively

$$\mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T$$

(easily calculated using MATLAB) gives the same answer

[2 POINTS]

1.4 Find the time derivative of the vector **w**.

We know that $\mathbf{w} = \mathbf{Ru}$ and so

$$\frac{d\mathbf{w}}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{u} = \mathbf{WRu} = \begin{bmatrix} 0 & -32 & 0 \\ 32 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.9577 & 0.2879 & 0 \\ -0.2879 & -0.9577 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2L \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} 49.07L \\ -52.08L \\ 0 \end{bmatrix}$$

[4 POINTS]

2. Let $\mathbf{R}_x(\theta_x), \mathbf{R}_y(\theta_y), \mathbf{R}_z(\theta_z)$ denote rotations through angles $(\theta_x, \theta_y, \theta_z)$ about the $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ axes (without RH screw convention), respectively. This publication shows that a sequence of 6 rotations of the form

$$\mathbf{R} = \mathbf{R}_x(\theta_x)\mathbf{R}_y(\theta_y)\mathbf{R}_z(\theta_z)\mathbf{R}_x(-\theta_x)\mathbf{R}_z(-\theta_z)\mathbf{R}_y(-\theta_y)$$

can rotate an object to any desired orientation, without any net rotation about the $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ axes (this is helpful in robots whose actuators can only turn through a finite angle).

2.1 Find the rotation matrix caused by this sequence of rotations with $\theta_x = \pi/2$, $\theta_y = -\pi/2$, $\theta_z = -\pi/2$. (you can use MATLAB to do the matrix multiplications, if you wish)

MATLAB gives $\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

[2 POINTS]

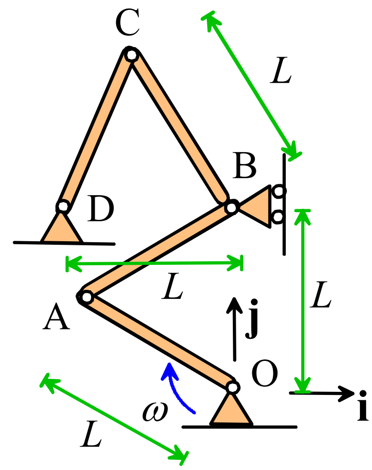
2.2 Find the axis and angle of the rotation matrix you found in 3.1.

Using the formulas $1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz} = 0 \Rightarrow \theta = 120^0$

$$\mathbf{n} = \frac{1}{2\sin\theta}\left[\left(R_{zy} - R_{yz}\right)\mathbf{i} + \left(R_{xz} - R_{zx}\right)\mathbf{j} + \left(R_{yx} - R_{xy}\right)\mathbf{k}\right]$$
$$= \frac{1}{2\sqrt{3}/2}\left[\mathbf{i} + \mathbf{j} + \mathbf{k}\right]$$

[2 POINTS]

3. The figure (from this publication) shows a candidate design for the 6 bar chain mechanism in an electrically actuated clamp. Member OA rotates clockwise with constant angular speed ω For the configuration shown in the figure (note that OAB and BCD are equilateral triangles):



3.1 calculate the velocity of point B, along with the angular speeds of members AB, BC and BD.

$$\mathbf{v}_A - \mathbf{v}_O = -\omega\mathbf{k}\times(\mathbf{r}_A - \mathbf{r}_O)$$
$$= -\omega\mathbf{k}\times L\left(-(\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right) = (\omega L/2)\mathbf{i} + (\sqrt{3}L\omega/2)\mathbf{j}$$
$$\mathbf{v}_B - \mathbf{v}_A = \omega_{AB}\mathbf{k}\times(\mathbf{r}_B - \mathbf{r}_A)$$
$$= \omega_{AB}\mathbf{k}\times L\left((\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right) = -(\omega_{AB}L/2)\mathbf{i} + (\sqrt{3}L\omega_{AB}/2)\mathbf{j}$$
$$\mathbf{v}_C - \mathbf{v}_B = \omega_{BC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_B)$$
$$= \omega_{BC}\mathbf{k}\times L\left(-(1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j}\right) = -(\sqrt{3}L\omega_{BC}/2)\mathbf{i} - (\omega_{BC}L/2)\mathbf{j}$$
$$\mathbf{v}_D - \mathbf{v}_C = \omega_{CD}\mathbf{k}\times(\mathbf{r}_D - \mathbf{r}_C)$$
$$= \omega_{CD}\mathbf{k}\times L\left(-(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}\right) = (\sqrt{3}L\omega_{CD}/2)\mathbf{i} - (\omega_{CD}L/2)\mathbf{j}$$

Add the first two equations to see that

$$\mathbf{v}_B - \mathbf{v}_O = (\omega L/2)\mathbf{i} + (\sqrt{3}L\omega/2)\mathbf{j} - (\omega_{AB}L/2)\mathbf{i} + (\sqrt{3}L\omega_{AB}/2)\mathbf{j}$$

Since B can only move vertically (so the **i** component is zero) it follows that $\omega_{AB} = \omega$ and $\mathbf{v}_B = \sqrt{3}L\omega\mathbf{j}$
Add the last two equations to get

$$\mathbf{v}_D - \mathbf{v}_B = -(\sqrt{3}L\omega_{BC}/2)\mathbf{i} - (\omega_{BC}L/2)\mathbf{j} + (\sqrt{3}L\omega_{CD}/2)\mathbf{i} - (\omega_{CD}L/2)\mathbf{j}$$

D is stationary, so both the **i,j** components of its velocity are zero. Therefore the **i** component gives $\omega_{CD} = \omega_{BC}$ and the **j** component gives

$$\sqrt{3}L\omega - \omega_{BC}L/2 - \omega_{CD}L/2 = 0 \Rightarrow \omega_{BC} = \sqrt{3}\omega \quad \omega_{CD} = \sqrt{3}\omega$$

[4 POINTS]

3.2 calculate the acceleration of point B, along with the angular accelerations of members AB, BC and BD.

$$\mathbf{a}_A - \mathbf{a}_O = -\omega^2 (\mathbf{r}_A - \mathbf{r}_O) = -\omega^2 L\left(-(\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right)$$

$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{AB}\mathbf{k}\times(\mathbf{r}_B - \mathbf{r}_A) - \omega_{AB}^2 (\mathbf{r}_B - \mathbf{r}_A)$$

$$= \alpha_{AB}\mathbf{k}\times L\left((\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right) - \omega^2 L\left((\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right)$$

$$= -(\alpha_{AB}L/2)\mathbf{i} + (\sqrt{3}L\alpha_{AB}/2)\mathbf{j} - \omega^2 L\left((\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right)$$

$$\mathbf{a}_C - \mathbf{a}_B = \alpha_{BC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_B) - \omega_{BC}^2 (\mathbf{r}_C - \mathbf{r}_B)$$

$$= \alpha_{BC}\mathbf{k}\times L\left(-(1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j}\right) - 3\omega^2 L\left(-(1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j}\right)$$

$$= -(\sqrt{3}L\alpha_{BC}/2)\mathbf{i} - (\alpha_{BC}L/2)\mathbf{j} - 3\omega^2 L\left(-(1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j}\right)$$

$$\mathbf{a}_D - \mathbf{a}_C = \alpha_{CD}\mathbf{k}\times(\mathbf{r}_D - \mathbf{r}_C) - \omega_{CD}^2 (\mathbf{r}_D - \mathbf{r}_C)$$

$$= \alpha_{CD}\mathbf{k}\times L\left(-(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}\right) - 3\omega^2 L\left(-(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}\right)$$

$$= (\sqrt{3}L\alpha_{CD}/2)\mathbf{i} - (\alpha_{CD}L/2)\mathbf{j} - 3\omega^2 L\left(-(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}\right)$$

Add the first two equations to see that

$$\mathbf{a}_B - \mathbf{a}_O = -\omega^2 L\left(-(\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right) - (\alpha_{AB}L/2)\mathbf{i} + (\sqrt{3}L\alpha_{AB}/2)\mathbf{j} - \omega^2 L\left((\sqrt{3}/2)\mathbf{i} + (1/2)\mathbf{j}\right)$$

$$= +(\alpha_{AB}L/2)\mathbf{i} - (\omega^2 L + \sqrt{3}L\alpha_{AB}/2)\mathbf{j}$$

B has no acceleration in the **i** direction, so $\alpha_{AB} = 0$ and $\mathbf{a}_B = -\omega^2 L\mathbf{j}$

Add the last two equations to get

$$\mathbf{a}_D - \mathbf{a}_B = -(\sqrt{3}L\alpha_{BC}/2)\mathbf{i} - (\alpha_{BC}L/2)\mathbf{j} - 3\omega^2 L\left(-(1/2)\mathbf{i} + (\sqrt{3}/2)\mathbf{j}\right)$$

$$+ (\sqrt{3}L\alpha_{CD}/2)\mathbf{i} - (\alpha_{CD}L/2)\mathbf{j} - 3\omega^2 L\left(-(1/2)\mathbf{i} - (\sqrt{3}/2)\mathbf{j}\right)$$

$$= (\sqrt{3}L(\alpha_{CD} - \alpha_{BC})/2)\mathbf{i} - ((\alpha_{BC} + \alpha_{CD})L/2)\mathbf{j} + 3\omega^2 L\mathbf{i}$$

$$\Rightarrow \mathbf{a}_D = (\sqrt{3}L(\alpha_{CD} - \alpha_{BC})/2)\mathbf{i} - ((\alpha_{BC} + \alpha_{CD})L/2)\mathbf{j} + 3\omega^2 L\mathbf{i} - L\omega^2\mathbf{j}$$

The acceleration of D is zero,

$$(\sqrt{3}L(\alpha_{CD} - \alpha_{BC})/2) = -3L\omega^2$$

$$((\alpha_{BC} + \alpha_{CD})L/2) = -L\omega^2$$

$$\Rightarrow \alpha_{CD} = -\left(1+\sqrt{3}\right)\omega^2 \qquad \alpha_{BC} = -\left(1-\sqrt{3}\right)\omega^2$$

[4 POINTS]

4. The figure shows a portal lift gear. Find the ratio of the gearbox (count the teeth on the gears). What is the purpose of the two smaller 'idler' gears? Why are there two of them?

There are 43 teeth on the large gear (connected to the output shaft) and 17 on each of the three smaller gears. They are a standard set of spur gears – all three of the small gears rotate at the same speed, and the output gear rotates at speed $\omega_0 = (17/43)\omega_i$

The gearbox is an after-market part used to lower a car's wheels (or equivalently to raise up the car body) for off-road driving (there's a picture in the link provided). The output shaft connects to the wheel; the input shaft connects to the axle (connected to the wheel in the original vehicle). The input and output shafts must rotate in the same direction, otherwise adding the gearbox will make the car go backwards! The two 'idler' gears are in the gearbox for this purpose - they don't change the gear ratio, but reverse the direction of rotation of the output shaft. There are two of them to reduce the forces on the bearings and gear teeth (some portal lift gears have only a single idler).

[3 POINTS]

5. This video shows an epicyclic gear clock in which the sun gear is connected to the minute hand, the planet carrier is connected to the hour hand and the ring gear is stationary. Calculate the number of teeth on the sun, ring, and planet gear (of course the solution is not unique – recommend something sensible!)

The sun must complete 60 revolutions in the time that it takes the planet carrier to complete 1 revolution. The formulas for gear ratios from the notes are

$$\frac{\omega_{zR} - \omega_{zPC}}{\omega_{zs} - \omega_{zPC}} = -\frac{N_S}{N_R} \qquad 2N_P = N_R - N_S$$

We know that $\omega_{zR} = 0 \quad \omega_{zS} = 12\omega_{zPC}$ hence

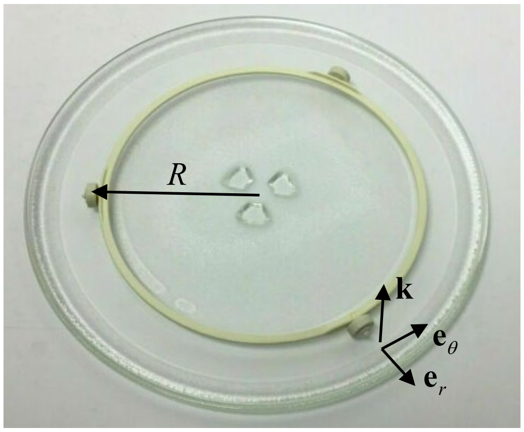
$$\frac{0 - \omega_{zPC}}{12\omega_{zPC} - \omega_{zPC}} = -\frac{N_S}{N_R} \Rightarrow \frac{N_R}{N_S} = 11$$

We could try, eg $N_S = 7 \quad N_R = 77 \quad N_P = 35$.

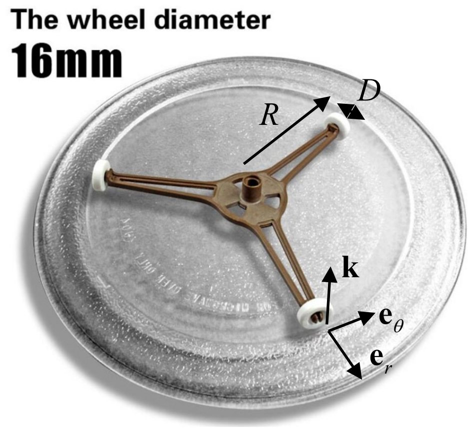
Any numbers in the same ratio are fine.

As a curiosity – there is a famous alternative design for an epicyclic clock called the 'Strutt' clock after its inventor. In Strutt's design the ring is driven by the escapement, the planet carrier is connected to the minute hand and the sun is connected to the hour hand. An ingenious stationary second gear outside the sun drives the sun at the correct angular speed. The geometry of Strutt's epicyclic is non-standard so you'd have to re-derive the formulas for the gear ratios to figure out how the design works.

[3 POINTS]



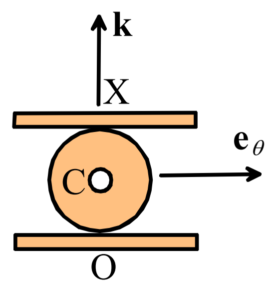
(a)



(b)

6. The figure shows two designs for a microwave turntable. In (a) the motor drives the glass plate directly, while in (b) it drives the plastic support.

If the motor turns with angular velocity $\omega\mathbf{k}$, find formulas for the angular velocity vectors of the support, the turntable, and the white rollers for each design. Express your answer in a basis that rotates with the support, as shown in the pictures.



For (a) the turntable rotates with the same angular velocity as the motor, since they are directly connected. The white roller has two components of angular velocity: it rotates with the plastic support with an angular velocity (to be determined $\omega_S\mathbf{k}$), and rotates also about the $\mathbf{e}_r$ axis at angular rate ω_r.

- The point on the roller that touches the turntable has velocity vector $\mathbf{v}_X = -D\omega_r\mathbf{e}_\theta$ (you can verify these with the rigid body kinematics formula)
- The point X on the turntable that touches the roller is in circular motion and has velocity vector $\omega R\mathbf{e}_\theta$.
- The roller and turntable have the same velocity at the point where they touch, therefore $-D\omega_r = \omega R \Rightarrow \omega_r = -\omega\dfrac{R}{D}$
- Similarly, the point on the support that is the roller's axle is in circular motion and therefore has velocity vector $R\omega_S\mathbf{e}_\theta$.
- The rolling wheel formula shows that the axle of the white roller has velocity $\mathbf{v}_C = -\dfrac{D}{2}\omega_r\mathbf{e}_\theta$.
- Matching these gives $-\dfrac{D}{2}\omega_r = R\omega_S \Rightarrow \omega_S = -\dfrac{D}{2R}\omega_r = \dfrac{\omega}{2}$

Putting everything together gives

$$\boldsymbol{\omega}_{\text{turntable}} = \omega\mathbf{k}$$

$$\boldsymbol{\omega}_{\text{support}} = \frac{1}{2}\omega\mathbf{k}$$

$$\boldsymbol{\omega}_{\text{roller}} = \frac{1}{2}\omega\mathbf{k} - \omega\frac{R}{D}\mathbf{e}_r$$

For (b) we can use the same argument, but the support must now turn with twice the angular speed of (a), since it turns with the same speed as the motor. The turntable must therefore rotate twice as fast as the motor.

Hence

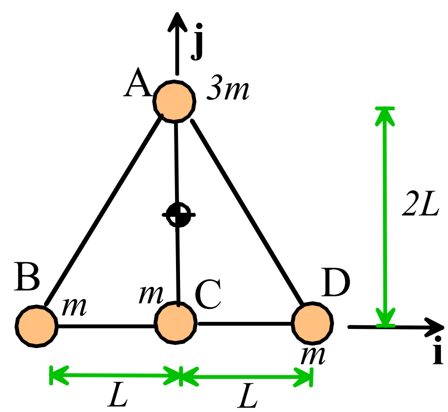
$$\boldsymbol{\omega}_{\text{turntable}} = 2\omega\mathbf{k}$$

$$\boldsymbol{\omega}_{\text{support}} = \omega\mathbf{k}$$

$$\boldsymbol{\omega}_{\text{roller}} = \omega\mathbf{k} - 2\omega\frac{R}{D}\mathbf{e}_r$$

[5 POINTS]

7. The figure shows four particles connected by rigid massless links. The particle at A has mass $3m$; those at B, C and D have mass m. The assembly rotates at constant angular speed ω about an axis parallel to **k** passing through the center of mass. The point of this problem is to demonstrate that the rigid body formula for the kinetic energy of the system gives the same answer as calculating the kinetic energy of each mass separately, and summing them. The rigid body formulas for angular momentum and kinetic energy are just fast ways of summing the total angular momentum and KE of a system of particles.



7.1 Calculate the position of the center of mass of the assembly

Use the formula $\mathbf{r}_G = \frac{1}{\sum_i m_i}\sum_i m_i\mathbf{r}_i = \frac{1}{6m}\left(-mL\mathbf{i} + \mathbf{0} + mL\mathbf{i} + 3m.2L\mathbf{j}\right) = L\mathbf{j}$

[1 POINT]

7.2 Calculate the 2D mass moment of inertia of the system about the center of mass

$$I_{Gzz} = \sum_i m_i\left(d_{xi}^2 + d_{yi}^2\right)$$

where $\mathbf{d}_i = d_{xi}\mathbf{i} + d_{yi}\mathbf{j} = \mathbf{r}_i - \mathbf{r}_G$ is the position vector of the ith particle with respect to the center of mass.

The formula gives

$$I_{Gzz} = \sum_i m_i\left(d_{xi}^2 + d_{yi}^2\right)$$
$$= m(L^2 + L^2) + mL^2 + m(L^2 + L^2) + 3mL^2 = 8mL^2$$

[2 POINTS]

7.3 Suppose that the assembly rotates about its center of mass with angular velocity $\omega\mathbf{k}$ (the center of mass is stationary). What are the speeds of the particles A,B and C?

The masses are all in circular motion about G, so we can use the circular motion formula $V = R\omega$ to find their speeds. This gives

$$V_A = V_C = \omega L \qquad V_B = V_D = \sqrt{2}\omega L$$

7.4 Calculate the total kinetic energy of the system (a) using your answer to 7.2; and (b) using your answer to 7.3. (The point of this problem is to demonstrate that the rigid body formula $(1/2)I\omega^2$ is just a quick way of summing the kinetic energies of the 4 masses. For the simple 2D system here it is quite simple to prove the equivalence for any arrangement of masses. For 3D the derivation is more complicated, but the idea is the same.)

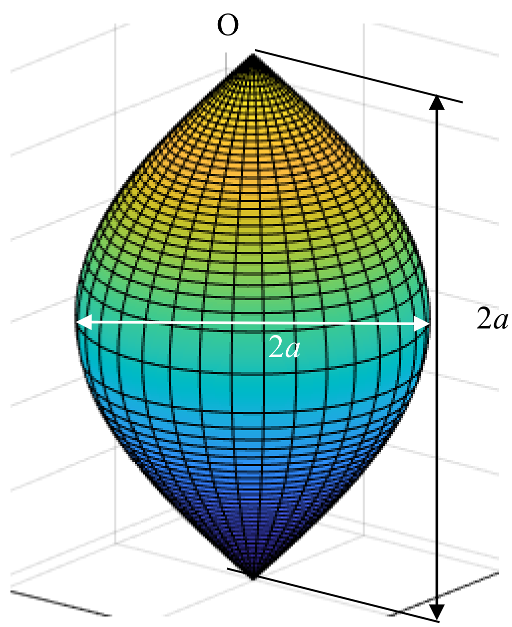
The rigid body formula gives $T = \frac{1}{2} I_{Gzz}\omega^2 = 4mL^2\omega^2$

Direct summation of the individual KEs gives

$$T = \sum \frac{1}{2} m_i V_i^2 = \frac{1}{2} m(L\omega)^2 + \frac{1}{2} 3m(L\omega)^2 + \frac{1}{2} m(\sqrt{2}L\omega)^2 + \frac{1}{2} m(\sqrt{2}L\omega)^2 = 4mL^2\omega^2$$

[3 POINTS]

8 The figure shows a solid of revolution with base radius a, height $2a$, and profile $r = a\cos(\pi z/(2a))$. It has uniform mass density ρ. Using a Matlab 'Live Script', calculate



8.1 The total mass M (you will need to do the relevant integrals using cylindrical-polar coordinates)

MATLAB gives $M = \pi a^3 \rho$

[1 POINT]

8.2 The inertia tensor (matrix) about the center of mass (which must be at the origin, by symmetry, but you could verify this by doing the integrals if you wish), in the basis shown

$$\mathbf{I}_G = Ma^2 \begin{bmatrix} \frac{25}{48} - \frac{2}{\pi^2} & 0 & 0 \\ 0 & \frac{25}{48} - \frac{2}{\pi^2} & 0 \\ 0 & 0 & \frac{3}{8} \end{bmatrix}$$

[3 POINTS]

8.3 Using the parallel axis theorem, calculate the mass moment of inertia about the tip O.

MATLAB gives

$$\mathbf{I}_O = Ma^2 \begin{bmatrix} \frac{73}{48} - \frac{2}{\pi^2} & 0 & 0 \\ 0 & \frac{73}{48} - \frac{2}{\pi^2} & 0 \\ 0 & 0 & \frac{3}{8} \end{bmatrix}$$

[2 POINTS]